Vertex-minors of graphs: A survey

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Abstract

For a vertex v of a graph, the local complementation at v is an operation that replaces the neighborhood of v by its complement graph. Two graphs are locally equivalent if one is obtained from the other by a sequence of local complementations. A graph H is a vertex-minor of a graph G if H is an induced subgraph of a graph locally equivalent to G. Although this concept was introduced in the 1980s, it was not widely known and except for the survey paper of Bouchet published in 1990, there is no comprehensive survey listing all the new developments. We survey classic and recent theorems and conjectures on vertex-minors and related concepts such as circle graphs, cut-rank functions, rank-width, and interlace polynomials.

1 Introduction

We aim to survey known results and conjectures for vertex-minors of graphs. In this paper, all graphs are simple, meaning that neither loops nor parallel edges are allowed. For a vertex v of a graph G, we write $G - v$ to denote the graph obtained by deleting the vertex v and all edges incident with v.

Let us start with their definitions. Vertex-minors are defined in terms of two graph operations, local complementations and vertex deletions. The *local complementation* of a graph G at a vertex v is an operation to obtain a new graph denoted by $G * v$ from G by 'toggling' the adjacencies between all pairs of neighbors of v, see Figure 1. In other words, two distinct vertices x and y are adjacent in $G * v$ if and only if exactly one of the following holds.

(a) x and y are adjacent in G.

(b) Both x and y are neighbors of v in G.

Two graphs are *locally equivalent* if one is obtained from the other by a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph locally equivalent to G. The name 'vertex-minor' first appeared in Oum [88] but it appeared previously under the various names such as l-reduction [22] and i-minor [14]. According to Bouchet [19], local complementations were introduced by Kotzig [75, 76].

For two adjacent vertices x and y, we write $G \wedge xy$ to denote $G * x * y * x$. It is easy to check that $G * x * y * x = G * y * x * y$ and so this operation is well defined and is called the *pivot* operation. We note that $G \wedge xy$ could be obtained from G by toggling the adjacency between every pair of vertices in two different sets among $N_G(x) - (N_G(y) \cup \{y\})$, $N_G(x) \cap N_G(y)$, and

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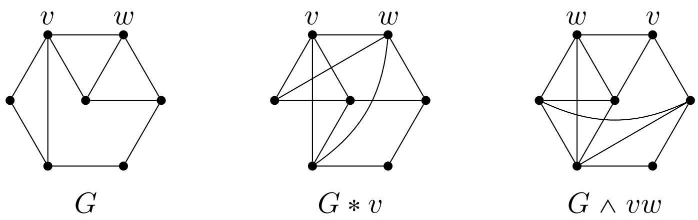


Figure 1: Examples of local complementations and pivotings.

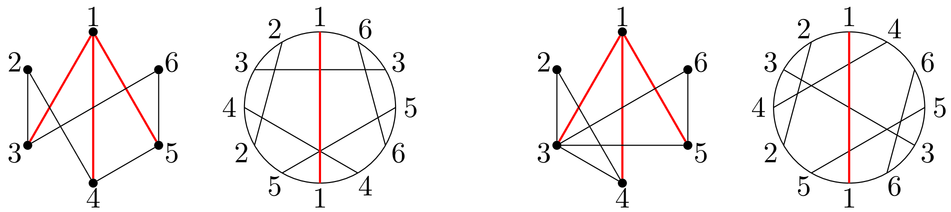


Figure 2: Two circle graphs G, $G * 1$ and their chord diagrams. The chord diagram for $G * 1$ is obtained by flipping one side of the circle divided by the chord representing 1.

$N_G(y) - (N_G(x) \cup \{x\})$ and then switching labels of x and y, see Figure 1. Another graphical description was given in Oum [88] along with the proof of the following well-known fact:

$$G \wedge xy \wedge yz = G \wedge xz.$$

Two graphs are *pivot-equivalent* if one is obtained from the other by a sequence of pivotings. A graph H is a *pivot-minor* of a graph G if H is an induced subgraph of a graph pivot-equivalent to G.

In Section 2, we motivate the vertex-minor theory by exploring the concepts related to circle graphs. In Sections 3 and 4, we review some basic properties of local equivalences and vertex-minors. In Sections 5–8, we survey the cut-rank function of a graph, which is a connectivity function for vertex-minors. In Sections 9 and 10, we give structural theorems for the vertex-minor, and in Section 11, we present recent progress on χ-boundedness for vertex-minor-closed classes of graphs. Section 12 provides some algorithmic results on vertex-minors, and in Section 13, we review interlace polynomials. Section 14 concludes this survey with several conjectures motivated by the Graph Minors Project of Robertson and Seymour. We decided not to include isotropic systems introduced by Bouchet [12] even though they are closely related to vertex-minors [15].

2 Circle graphs and vertex-minors

Circle graphs are one of the major examples of classes of graphs closed under taking vertex-minors. A *circle graph* is the intersection graph of chords in a circle. In other words, a circle graph is represented by a chord diagram where vertices are chords and two vertices are adjacent if and only if they are intersecting. It is straightforward to see that if G is a circle graph, then so is $G * v$ for every vertex v of G; this can be achieved by taking the chord represented by v in the chord diagram representing G and reversing one side of the circle to obtain the chord diagram of $G * v$, see Figure 2. Since deleting a chord in a chord diagram corresponds to deleting a vertex in the associated circle graph, every vertex-minor of a circle graph is also a circle graph.

Bouchet [22] proved the following analog of Kuratowski's theorem on planar graphs for circle graphs. Lee [82] presented an alternative proof.

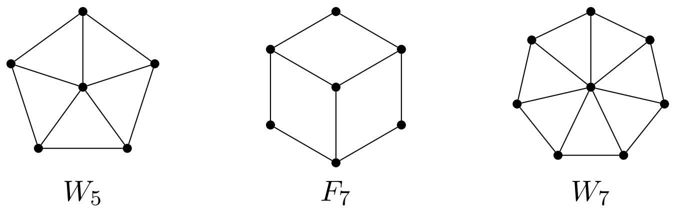


Figure 3: Vertex-minor obstructions for the class of circle graphs.

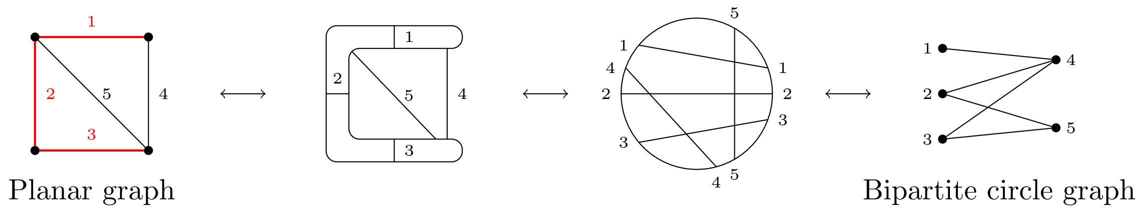


Figure 4: Planar graphs and bipartite circle graphs.

Theorem 2.1 (Bouchet [22]). *A graph is a circle graph if and only if it has no vertex-minor isomorphic to* W_5, F_7, *or* W_7 *in Figure 3.*

Robertson and Seymour [98] proved one of the central theorems in the graph minor theory, that is, the class of graphs having no minor isomorphic to a graph H has bounded tree-width if and only if H is planar. Geelen, Kwon, McCarty, and Wollan [60] proved an analogous theorem for vertex-minors, rank-width, and circle graphs. The definition of the rank-width will be reviewed in Section 5.

Theorem 2.2 (Geelen, Kwon, McCarty, and Wollan [60]). *The class of graphs having no vertex-minor isomorphic to a graph* H *has bounded rank-width if and only if* H *is a circle graph.*

As planar graphs play an essential role in the graph minor theory, circle graphs play a similar role in the vertex-minor theory, as witnessed by the previous theorem. This is not a coincidence because there is a connection between planar graphs and bipartite circle graphs as follows. The *fundamental graph* of a graph G with respect to a maximal acyclic subgraph T of G is a bipartite graph H with a bipartition $(E(T), E(G) - E(T))$ such that $x \in E(T)$ and $y \in E(G) - E(T)$ are adjacent in H if and only if x is in the unique cycle of $T + y$.

Theorem 2.3 (de Fraysseix [45]). *A bipartite graph is a circle graph if and only if it is a fundamental graph of a planar graph.*

Figure 4 illustrates a proof of Theorem 2.3. Every pivot-minor of a bipartite circle graph G is a bipartite circle graph associated with a minor of a planar graph corresponding to G. Let H be a bipartite circle graph that is the fundamental graph of a planar graph G with respect to a maximal acyclic subgraph T. Then for each $x \in E(T)$ and $y \in E(G) - E(T)$,

(a) $H - x$ is the fundamental graph of G/x with respect to T/x,[1]

(b) $H - y$ is the fundamental graph of $G - y$ with respect to T, and

(c) if $xy \in E(H)$, then $H \wedge xy$ is the fundamental graph of G with respect to $T - x + y$.

More generally, pivot-minors of bipartite graphs are associated with minors of binary matroids, see Oum [88].

[1] For an edge x of a graph G, we write G/x to denote the graph obtained from G by contracting the edge x.

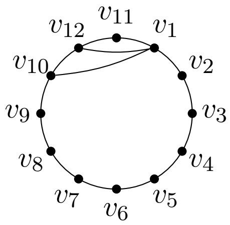


Figure 5: The cycle graph C_n on n vertices has at least $2^{\lfloor (n-4)/2 \rfloor}$ pairwise non-isomorphic vertex-minors locally equivalent to C_{n-1}. This can be seen from $C_{n-1} * v_{n-2} * v_{n-1}$ by applying local complementations at some of vertices in $\{v_2, v_4, v_6, \ldots, v_{2\lfloor (n-4)/2 \rfloor}\}$.

Conjectures and theorems on pivot-minors sometimes imply their counterpart on minors of graphs and minors of binary matroids. For instance, Geelen and Oum [61] proved a pivot-minor analog of Theorem 2.1 and their result implies Kuratowski's theorem. They showed that there is a list L of 15 graphs such that a graph is a circle graph if and only if it has no pivot-minor isomorphic to any graph in L. In L, there are exactly three bipartite graphs F_7, H_1, and H_2, where H_1 and H_2 are fundamental graphs of $K_{3,3}$ and K_5, respectively. Now we explain how this implies Kuratowski's theorem. Let G be a non-planar graph. Then its fundamental graph H is a non-circle bipartite graph by Theorem 2.3. Hence H has a pivot-minor isomorphic to F_7, H_1, or H_2 by Geelen and Oum [61]. Equivalently, G has a minor isomorphic to a graph G' that has F_7, H_1, or H_2 as a fundamental graph. Note that F_7 is a fundamental graph of the Fano matroid, and a connected binary matroid is determined by its fundamental graph up to duality. If G' has a fundamental graph isomorphic to F_7, then the cycle matroid $M(G')$ of G' is isomorphic to the Fano matroid or its dual, contradicting the well-known fact that the Fano matroid is not regular. Hence G' is isomorphic to H_1 or H_2. Then $M(G')$ is isomorphic to one of $M(K_{3,3})$, $M(K_5)$, and their duals. Because neither $M^*(K_{3,3})$ nor $M^*(K_5)$ is graphic, $M(G')$ is isomorphic to $M(K_{3,3})$ or $M(K_5)$. As every 3-connected graph is determined by its cycle matroid [96, Section 5.3], G' is isomorphic to either $K_{3,3}$ or K_5. Therefore, G contains a minor isomorphic to $K_{3,3}$ or K_5, implying Kuratowski's theorem.

3 Vertex-minors having one fewer vertex

For a graph G and a vertex v, there can be exponentially many vertex-minors of G on $V(G) - \{v\}$, even if we count vertex-minors up to isomorphisms. For instance, the cycle graph on n vertices has at least $2^{\Omega(n)}$ non-isomorphic vertex-minors on $n-1$ vertices; see Figure 5. However, the following theorem, proved by Bouchet and Fon-Der-Flaass independently, ensures that essentially there are only three ways to remove one vertex to have a vertex-minor.

Theorem 3.1 (Bouchet [15, (9.2)] and Fon-Der-Flaass [49, Corollary 4.3])**.** *Let H be a vertex-minor of a graph G and v be a vertex in $V(G) - V(H)$. Then H is a vertex-minor of one of $G - v$, $G * v - v$, and $G \wedge vw - v$ for some neighbor w of v.*

We note that the choice of w in Theorem 3.1 does not change the outcome up to pivot-equivalence, as $(G \wedge vw) - v = ((G \wedge vw') - v) \wedge ww'$ for distinct neighbors w and w' of v in G. Bouchet [15] proved Theorem 3.1 using isotropic systems and Fon-Der-Flaass [49] proved it in graph-theoretic notions. Sometimes it is convenient to use the following form presented in Geelen and Oum [61, Lemma 3.1].

Lemma 3.2 (Geelen and Oum [61, Lemma 3.1])**.** *Let G be a graph and v and w be vertices.*

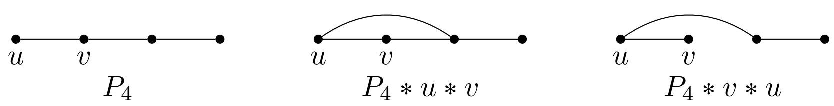


Figure 6: Local complementations are not commutable.

(i) *If $v \neq w$ and v is non-adjacent to w, then $(G*w)-v$, $(G*w*v)-v$, and $(G*w)\wedge vv'-v$ for some neighbor v' of v in $G*w$ are locally equivalent to $G-v$, $(G*v)-v$, and $G\wedge vv''-v$ for some neighbor v'' of v in G, respectively.*

(ii) *If $v \neq w$ and v is adjacent to w, then $(G*w)-v$, $(G*w*v)-v$, and $(G*w)\wedge vv'-v$ for some neighbor v' of v in $G*w$ are locally equivalent to $G-v$, $G\wedge vv''-v$, and $(G*v)-v$ for some neighbor v'' of v in G, respectively.*

(iii) *If $v = w$, then $(G*w)-v$, $(G*w*v)-v$, and $(G*w)\wedge vv'-v$ for some neighbor v' of v are locally equivalent to $(G*v)-v$, $G-v$, and $G\wedge vv''-v$ for some neighbor v'' of v, respectively.*

Lemma 3.2 immediately implies Theorem 3.1. If H is a vertex-minor of a graph G and $v \in V(G)-V(H)$, then there is a graph G' locally equivalent to G such that H is an induced subgraph of $G'-v$. By successively applying Lemma 3.2, we deduce that $G'-v$ is locally equivalent to one of $G-v$, $G*v-v$, and $G\wedge vw-v$ for some neighbor w of v.

4 Various properties about local equivalence of graphs

Local complementations do not commute; see Figure 6. Hence when we enumerate a sequence of local complementations, there might have many vertices appearing more than once. Fon-Der-Flaass [49] showed that for every pair of local equivalent graphs, there is a good sequence of local complementations and pivotings avoiding redundancy.

Theorem 4.1 (Fon-Der-Flaass [49, Corollary 4.2])**.** *Let G and H be locally equivalent graphs. Then there is a sequence of local complementations and pivotings such that all vertices used for local complementations and all ends of edges used for pivotings appear only once.*

Pivot-equivalent graphs can be explained by a matrix operation originated from Tucker [107]. For a $V\times V$ matrix A and a subset $X\subseteq V$, we write $A[X]$ to denote its $X\times X$ principal submatrix. Let

$$A = \begin{array}{c c} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}.$$

If $A[X]=\alpha$ is nonsingular, then let

$$A*X = \begin{array}{c c} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta-\gamma\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

This operation is called the *pivoting* or the *principal pivoting*. It is known that if $A[X]$ and $(A*X)[Y]$ are nonsingular, then $(A*X)*Y = A*(X\triangle Y)$, where $X\triangle Y = (X-Y)\cup(Y-X)$, see [56, Theorem 2.8].

Let us write $A(G)$ to denote the adjacency matrix of a graph G over the binary field. It turns out that

$$A(G\wedge uv) = A(G)*\{u,v\}.$$

Thus we observe the following.

Lemma 4.2. *Two graphs G and H on the vertex set V are pivot-equivalent if and only if $A(G) = A(H) * X$ for some $X \subseteq V$ such that $A(H)[X]$ is nonsingular.*

This means that if $H = G \wedge e_1 \wedge e_2 \cdots \wedge e_k$, then $V(e_1)\triangle V(e_2)\triangle \cdots \triangle V(e_k)$ determines H and conversely if H and G are pivot-equivalent, then we can choose edges e_1, e_2, $\ldots$, e_k such that $H = G \wedge e_1 \wedge e_2 \cdots \wedge e_k$ and $V(e_i) \cap V(e_j) = \emptyset$ for all $i \neq j$, because for any nonsingular non-trivial skew-symmetric matrix X, one can find a 2×2 nonsingular principal submatrix to apply pivoting.

Also, note that if $A(G) = A(H) * X$, then $A(G)[Y]$ is nonsingular if and only if $A(H)[X\triangle Y]$ is nonsingular by Tucker [107], see [56, Theorem 2.7]. So we deduce the following.

Lemma 4.3. *Let G and H be graphs on the vertex set V. Then G and H are pivot-equivalent if and only if there exists $X \subseteq V$ such that*

$$\{Y : Y \subseteq V \text{ and } \det A(G)[Y] \neq 0\} = \{Y \triangle X : Y \subseteq V \text{ and } \det A(H)[Y] \neq 0\}.$$

The above lemma is the essence of *even binary delta-matroids*. For a graph G on V, the pair $(V, \mathcal{F})$ where $\mathcal{F} = \{X \subseteq V : \det A(G)[X] \neq 0\}$ defines even binary delta-matroids introduced by Bouchet [16]. Bouchet introduced delta-matroids more generally [11] and pivot-minors of graphs can be defined as minors of even binary delta-matroids. This relation is useful when studying pivot-minors of graphs.

Here is another theorem on two locally equivalent graphs.

Theorem 4.4 (Fon-Der-Flaass [50, 51])**.** *Let G and H be locally equivalent graphs. Then there are vertices $v_1, \ldots, v_k$ and edges $e_1, \ldots, e_\ell, e'_1, \ldots, e'_m$ such that*

(i) $e_i \in E(G \wedge e_1 \wedge \cdots \wedge e_{i-1})$ *for* $1 \leqslant i \leqslant \ell$,

(ii) $\{v_1, \ldots, v_k\}$ *is an independent set in* $G \wedge e_1 \wedge \cdots \wedge e_\ell$,

(iii) $e'_j \in E(G \wedge e_1 \wedge \cdots \wedge e_\ell * v_1 * \cdots * v_k \wedge e'_1 \wedge \cdots \wedge e'_{j-1})$ *for* $1 \leqslant j \leqslant m$, *and*

(iv) $H = G \wedge e_1 \wedge \cdots \wedge e_\ell * v_1 * \cdots * v_k \wedge e'_1 \wedge \cdots \wedge e'_m$.

Fon-Der-Flaass published his proof in Russian. A paper by the authors [73] includes a proof based on isotropic systems. As an easy consequence of Theorem 4.4, Fon-Der-Flaass [51] proved that two locally equivalent bipartite graphs are pivot-equivalent. This was a conjecture of Bouchet [19, Conjecture 3.5] and is also implied by Theorem 5.3.

Theorem 4.5 (Fon-Der-Flaass [49, Theorem 2.2])**.** *If a graph G is locally equivalent to a tree T, then G has a subgraph isomorphic to T.*

Theorem 4.5 implies the following theorem, which was initially conjectured by Mulder at the Oberwolfach meeting in 1986. Bouchet also solved this conjecture independently.

Theorem 4.6 (Bouchet [17] and Fon-Der-Flaass [49])**.** *If two trees are locally equivalent, then they are isomorphic.*

Jeong, Kwon, and Oum [69] used the idea of the above theorem to prove the following theorem on block graphs. A *block graph* is a graph in which every 2-connected induced subgraph is a complete graph. Equivalently, a block graph is a graph that can be obtained from the disjoint union of complete graphs by repeatedly identifying a pair of vertices in distinct components. A vertex is *simplicial* if its neighbors are pairwise adjacent.

Theorem 4.7 (Jeong, Kwon, and Oum [69, Theorem 1.4])**.** *If two block graphs without simplicial vertices of degree 2 are locally equivalent, then they are isomorphic.*

Theorem 4.8 (Fon-Der-Flaass [49, Theorem 5.1]). *Let G be a graph and $n \geqslant 5$. If G is locally equivalent to C_n, then G has a subgraph isomorphic to C_n.*

Theorem 4.8 implies that if an n-vertex bipartite graph is locally equivalent to a cycle, then n is even, proved by Allys [4, Lemma 5.2] later by using isotropic systems. Another proof of this fact is presented in the appendix of [73].

5 Cut-rank functions

For a graph G, the cut-rank function ρ_G is defined over the subsets X of $V(G)$ such that $\rho_G(X)$ is the rank of the $X \times (V(G) - X)$ matrix over the binary field whose entry is 1 if and only if two vertices representing the column and the row are adjacent in G. Cut-rank functions play an important role in the study of vertex-minors because they are invariant under taking local complementations. As elementary row operations do not change the rank and $1 + 1 = 0$ in the binary field, it is easy to observe the following theorem.

Theorem 5.1 (Bouchet [18, Corollary 2]; see Oum [88]). *If G and H are locally equivalent, then they have an identical cut-rank function.*

Since deleting a row or a column does not increase the rank of a matrix, we deduce the following corollary.

Corollary 5.2. *If H is a vertex-minor of a graph G, then $\rho_H(X \cap V(H)) \leqslant \rho_G(X)$ for all $X \subseteq V(G)$.*

Bouchet [19] conjectured that the converse of Theorem 5.1 is true but Fon-Der-Flaass disproved the conjecture. A counterexample on 10 vertices is presented in [51]. According to a computer search, the smallest example is a pair of two graphs on 9 vertices that are not locally equivalent but have an identical cut-rank function.

The converse of Theorem 5.1 is true for bipartite graphs in the stronger sense that we only need pivoting. This follows from the result of Seymour [105].

Theorem 5.3. *Let G_1, G_2 be bipartite graphs on the same set V of vertices. If*

$$\rho_{G_1}(X) = \rho_{G_2}(X) \text{ for all subsets } X \subseteq V,$$

then G_1 is pivot-equivalent to G_2.

Proof. We may assume that both G_1 and G_2 are connected because the cut-rank function of a graph determines components. For $i = 1, 2$, let (S_i, T_i) be a bipartition of $V(G_i)$, let A_i be an $S_i \times V$ matrix over the binary field such that for $v \in S_i$ and $w \in V$,

$$A_i(v, w) = \begin{cases} 1 & \text{if } v = w, \\ 1 & \text{if } w \in T_i \text{ and } v \text{ and } w \text{ are adjacent in } G_i, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 1, 2$, let M_i be the binary matroid represented by A_i and let λ_i be the matroid connectivity function of M_i, that is $\lambda_i(X) = r_i(X) + r_i(V - X) - r_i(V)$ for the rank function r_i of M_i. This construction is to make G_i the fundamental graph of M_i with respect to S_i. Then, it is well known that the matroid connectivity function coincides with the cut-rank function of its fundamental graph, that is $\rho_{G_i}(X) = \lambda_i(X)$, see [88].

Seymour [105] showed that if two connected binary matroids M_1 and M_2 have the same matroid connectivity function, then $M_1 = M_2$ or $M_1 = M_2^*$. We may assume that $M_1 = M_2$ because we may swap S_2 and T_2 to replace M_2 with M_2^*. Any two fundamental graphs of a binary matroid are pivot-equivalent, see Oum [88, Corollary 3.5] and therefore G_1 and G_2 are pivot-equivalent. □

Motivated by Corollary 5.2, we can define width parameters of graphs in terms of cut-rank functions so that their values do not increase when we take vertex-minors. The most well-known example is the rank-width, introduced by Oum and Seymour [95], as a dense analog of tree-width. A *rank-decomposition* of a graph G is a pair (T, L) of a tree T of maximum degree at most 3 and a bijection L from $V(G)$ to the set of leaves of T. For every edge e, $T - e$ gives a partition (A_e, B_e) of the leaves of T and the *width* of an edge e in T is defined as $\rho_G(L^{-1}(A_e))$. The *width* of (T, L) is defined as the maximum width of all edges of T. The *rank-width* of a graph G is the minimum width of all rank-decompositions of G. If $|V(G)| < 2$, then it admits no rank-decompositions and we define the rank-width to be 0. By Corollary 5.2, if H is a vertex-minor of G, then the rank-width of H is less than or equal to that of G.

One can also define a dense analog of path-width. The *linear rank-width* of a graph G is the minimum k such that there is an ordering $v_1, v_2, \ldots, v_n$ of all vertices of G with the property that

$$\rho_G(\{v_1, \ldots, v_i\}) \leqslant k$$

for all $i = 1, 2, \ldots, n := |V(G)|$. Again, if H is a vertex-minor of G, then the linear rank-width of H is less than or equal to that of G.

We also have a dense analog of tree-depth, called the *rank-depth*, introduced by DeVos, Kwon, and Oum [46]. A *decomposition* of a graph G is a pair (T, σ) of a tree T having at least one internal node and a bijection σ from $V(G)$ to the set of leaves of T. Note that a rank-decomposition is a decomposition in which every node in the tree has degree at most 3. The *radius* of a decomposition (T, σ) is the radius of the tree T. Each internal node v of T induces a partition $\mathcal{P}_v$ of $V(G)$ by taking vertices of G mapped to the same component of $T - v$ by σ as one part. The *ρ_G-width* of a partition $\mathcal{P}_v$ is defined as $\max\{\rho_G(\bigcup_{X \in \mathcal{Q}} X) : \varnothing \neq \mathcal{Q} \subseteq \mathcal{P}_v\}$. The *width* of a decomposition (T, σ) is the maximum width of $\mathcal{P}_v$ among all internal nodes v of T. The *rank-depth* of a graph G [46] is the minimum integer k such that G has a decomposition of radius at most k and width at most k. Again, if H is a vertex-minor of G, then the rank-depth of H is less than or equal to that of G.

From definitions, the rank-width is less than or equal to the linear rank-width and the rank-depth. The linear rank-width is at most the square of the rank-depth, which is explained below. Let G be a graph of rank-depth k. Then G admits a decomposition (T, σ) of radius at most k and width at most k. By the depth-first search on T, we can order the leaves of T and so the vertices $v_1, \ldots, v_n$ of G. Since T has radius at most k, for each $1 \leqslant m \leqslant n$, there are at most k internal nodes $x_1, \ldots, x_{k'}$ of T such that $\{v_1, \ldots, v_m\}$ is the union of $X_1, \ldots, X_{k'}$ for some $X_i \in \mathcal{P}_{x_i}$. Thus, $\rho_G(\{v_1, \ldots, v_m\}) \leqslant \sum_{i=1}^{k'} \rho_G(X_i) \leqslant k^2$.

The rank-width and the linear rank-width have equivalent graph parameters defined earlier, which are the *clique-width* [30, 31] and the *linear clique-width* [63, 83], respectively. For the rank-depth, there is also an equivalent concept, called the *shrub-depth* [54, 55], which is only defined for classes of graphs. We omit definitions of the clique-width, linear clique-width, and shrub-depth in this survey.

Theorem 5.4. *Let* rw*,* lrw*,* cw*, and* lcw *be the rank-width, linear rank-width, clique-width, and linear clique-width of a graph G. Then*

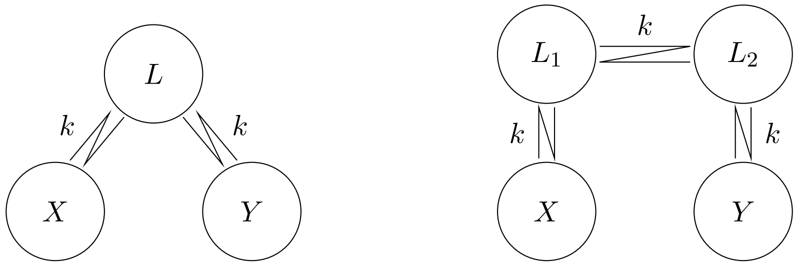
(i) $\mathrm{rw} \leqslant \mathrm{cw} \leqslant 2^{\mathrm{rw}+1} - 1$ *[95],*

(ii) $\mathrm{lrw} \leqslant \mathrm{lcw} \leqslant 2^{\mathrm{lrw}} + 1$ *[93], and*

(iii) *a class of graphs has bounded shrub-depth if and only if it has bounded rank-depth [46].*

6 Vertex-minors certifying large cut-rank

By Corollary 5.2, if G has a vertex-minor H having a cut of large cut-rank, then all cuts of G inducing the same cut on H will have large cut-rank. The following theorem shows



(i) (ii)

Figure 7: Illustrations of Theorem 6.2.

that the converse holds. This is an analog of Tutte's linking theorem [108] on matroids; see Oxley [96, Section 8.5].

Theorem 6.1 (Oum [88, Theorem 6.1])**.** *Let G be a graph and X and Y be disjoint subsets of $V(G)$. The following are equivalent.*

(i) *For every vertex set Z, if $X \subseteq Z \subseteq V(G) - Y$, then $\rho_G(Z) \geqslant k$.*

(ii) *G has a vertex-minor H on $X \cup Y$ such that $\rho_H(X) \geqslant k$.*

(iii) *G has a pivot-minor H on $X \cup Y$ such that $\rho_H(X) \geqslant k$.*

Theorem 6.1 allows us to find a small vertex-minor certifying that any cuts separating a pair of disjoint sets X and Y have large cut-rank. However, often it is convenient to have such a certificate while keeping the adjacency between X and Y. The following theorem provides such a small vertex-minor.

Theorem 6.2 (Geelen, Kwon, McCarty, and Wollan [60, Lemma 4.3])**.** *There is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $k > 0$, if G is a graph and X and Y are disjoint subsets of $V(G)$ such that $\rho_{G[X\cup Y]}(X) < k$ and*

$$\rho_G(Z) \geqslant f(k) \text{ whenever } X \subseteq Z \subseteq V(G) - Y,$$

then there exists a graph H locally equivalent to G such that $H[X \cup Y] = G[X \cup Y]$ and either

(i) *there is a set $L \subseteq V(H) - (X \cup Y)$ of size k such that $\rho_{H[X\cup L]}(L) = \rho_{H[Y\cup L]}(L) = k$, or*

(ii) *there are disjoint subsets $L_1, L_2 \subseteq V(H) - (X \cup Y)$ of size k such that $\rho_{H[X\cup L_1]}(L_1) = \rho_{H[L_1\cup L_2]}(L_1) = \rho_{H[Y\cup L_2]}(L_2) = k$, all vertices in L_1 have the same set of neighbors in Y, and all vertices in L_2 have the same set of neighbors in X.*

See Figure 7 for illustrations of Theorem 6.2. The pair of a graph H and a set L or $L_1 \cup L_2$ obtained from Theorem 6.2 certifies that any cuts separating X and Y have large cut-rank as shown by the following proposition. This is also described in [60] as a motivation of Theorem 6.2. For an $X \times Y$ matrix A, if $X' \subseteq X$ and $Y' \subseteq Y$, then we denote by $A[X', Y']$ the $X' \times Y'$ submatrix of A.

Proposition 6.3. *Let H be a graph and let X and Y be disjoint subsets of $V(H)$.*

(i) *If there is a set $L \subseteq V(H) - (X \cup Y)$ such that $\rho_{H[X\cup L]}(L) = \rho_{H[Y\cup L]}(L) = |L|$, then $\rho_H(Z) \geqslant |L|/2$ for every $X \subseteq Z \subseteq V(H) - Y$.*

(ii) *If there are disjoint subsets $L_1, L_2 \subseteq V(G) - (X\cup Y)$ such that $\rho_{H[X\cup L_1]}(L_1) = \rho_{H[L_1\cup L_2]}(L_1) = \rho_{H[Y\cup L_2]}(L_2) = |L_1| = |L_2|$, then $\rho_H(Z) \geqslant |L_1|/3$ for every $X \subseteq Z \subseteq V(H) - Y$.*

Proof. Let A be the adjacency matrix of H over the binary field. Let Z be a set such that $X \subseteq Z \subseteq V(H) - Y$.

(i) We have that $\rho_H(Z) = \operatorname{rank}(A[Z, V(H) - Z]) \geqslant \operatorname{rank}(A[X, L - Z]) \geqslant \operatorname{rank}(A[X, L]) - |L \cap Z| = \rho_{H[X \cup L]}(L) - |L \cap Z| = |L - Z|$. Similarly, $\rho_H(Z) \geqslant \operatorname{rank}(A[L \cap Z, Y]) \geqslant |L \cap Z|$. Therefore, $2\rho_H(Z) \geqslant |L|$.

(ii) Note that $\rho_H(Z) \geqslant \operatorname{rank}(A[X, L_1 - Z]) \geqslant |L_1 - Z|$ and $\rho_H(Z) \geqslant \operatorname{rank}(A[L_2 \cap Z, Y]) \geqslant |L_2 \cap Z|$. Observe that $\rho_H(Z) \geqslant \operatorname{rank}(A[L_1 \cap Z, L_2 - Z]) \geqslant \operatorname{rank}(A[L_1, L_2]) - |L_1 - Z| - |L_2 \cap Z| = |L_1| - |L_1 - Z| - |L_2 \cap Z|$. Hence $3\rho_H(Z) \geqslant |L_1|$. $\square$

Lee and Oum [81] proved the following extension of Theorem 6.1.

Theorem 6.4 (Lee and Oum [81]). *Let k, ℓ be non-negative integers. Let G be a graph and Q, R, S, T be subsets of $V(G)$ such that $Q \cap R = S \cap T = \varnothing$,*

$$\rho_G(Z_1) \geqslant k \text{ whenever } Q \subseteq Z_1 \subseteq V(G) - R,$$

and

$$\rho_G(Z_2) \geqslant \ell \text{ whenever } S \subseteq Z_2 \subseteq V(G) - T.$$

Then there is a pivot-minor H of G such that H contains $Q \cup R \cup S \cup T$, $|V(H)| < |Q \cup R \cup S \cup T| + (2\ell + 1)2^{2k}$,

$$\rho_H(Z_1) \geqslant k \text{ whenever } Q \subseteq Z_1 \subseteq V(H) - R,$$

and

$$\rho_H(Z_2) \geqslant \ell \text{ whenever } S \subseteq Z_2 \subseteq V(H) - T.$$

7 Split decompositions and prime graphs

A *split* of a graph G is a partition (X, Y) of the vertex set of G such that $|X|, |Y| \geqslant 2$ and $\rho_G(X) \leqslant 1$. Note that $\rho_G(X) \leqslant 1$ if there are subsets $X' \subseteq X$ and $Y' \subseteq V(G) - X$ for which the set of adjacent pairs of a vertex $a \in X$ and a vertex $b \in V(G) - X$ is precisely $X' \times Y'$. We say a graph is *prime* if it has no splits. By Theorem 5.1, locally equivalent graphs have the same splits and therefore if a graph is locally equivalent to a prime graph then it is prime. We will discuss prime graphs and their vertex-minors in Section 8.

If a graph admits a split, it can be built by the 1-join of two graphs. The *1-join* of two graphs G_1 and G_2 with marker vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ is defined to be a graph obtained from the disjoint union of $G - v_1$ and $G - v_2$ by adding an edge between every neighbor of v_1 in G_1 and every neighbor of v_2 in G_2. Whenever a graph G admits a split (X, Y), we choose $v_1 \in Y$ and $v_2 \in X$ so that v_1 have neighbors in X and v_2 have neighbors in Y if there is at least one edge between X and Y. Then G is the 1-join of $G[X \cup \{v_1\}]$ and $G[Y \cup \{v_2\}]$ with marker vertices v_1, v_2. Thus if G is not prime, it can be decomposed into two smaller graphs on at least three vertices by the 1-join operation.

Now starting from a set $\{G\}$ consisting of a single graph, we recursively pick a graph H in the set having a split and replace it with two smaller graphs so that their 1-join is H as long as H is not a complete graph or a star, until no further replacement is possible. We can associate a tree by having one node for each graph in the resulting set and adding an edge between two nodes if the corresponding pairs have marker vertices that are used when making the 1-join at some point. This decomposition is called the *split decomposition* of a graph. Cunningham [33] proved that a connected graph has a unique split decomposition and therefore this is sometimes called a *canonincal decomposition.* Note that complete graphs and stars have exponentially many splits and for the sake of having the unique split decomposition, the recursive process stops at those graphs. If we do not care much about the uniqueness, then we could decompose a connected graph into prime graphs and build any graph from prime graphs by repeatedly taking 1-join.

There are linear-time algorithms to find a split decomposition [27, 39].

Prime graphs are important building blocks of graphs. As the planarity of a graph can be determined by investigating the planarity of 3-connected topological minors, there are properties of graphs that can be determined by the properties of prime induced subgraphs.

Here is a theorem on the unique representation of circle graphs up to local equivalence. This is analogous to the theorem that every 3-connected planar graph has a unique embedding on the sphere, see [86, Section 2.5].

Theorem 7.1 (Bouchet [13]). *Prime circle graphs on at least 5 vertices have a unique representation up to cyclic equivalence.*

Furthermore, if G_1 and G_2 are circle graphs, then so is their 1-join [13, (4.2)]. We can rewrite as follows because any prime induced subgraph is an induced subgraph of some graph in the canonical decomposition.

Theorem 7.2 (Bouchet [13]). *A graph is a circle graph if and only if all of its prime induced subgraphs are circle graphs.*

So in order to decide whether a graph is a circle graph, it is enough to check whether all graphs in its canonical decomposition are circle graphs.

A similar property of Theorem 7.2 holds for graphs of rank-width at most k. As the rank-width of the 1-join of two graphs is equal to the maximum rank-width of G or H, it is easy to deduce the following.

Theorem 7.3 (Hliněný, Oum, Seese, and Gottlob [68, Theorem 4.3]). *The rank-width of a graph is the maximum rank-width of all its prime induced subgraphs.*

The class of graphs of rank-width at most one is precisely the class of graphs without prime induced graphs having at least four vertices. This class is also known as the class of distance-hereditary graphs, and we will discuss it in Section 9.2.

8 Chain theorems

Graph theory employs chain theorems as a means to reduce the size of a graph while maintaining some notion of connectedness. A detailed overview of these theorems can be found in Sections 7.2 and 7.3 of Chapter 2 in [62], which offer a comprehensive survey on chain theorems of graph minors. This survey describes variants of Tutte's chain theorem and examines their applications.

A significant chain theorem for vertex-minors was proven by Bouchet [14]. This theorem serves as a critical tool used by Bouchet for recognizing circle graphs in polynomial time.

Theorem 8.1 (Bouchet [14]). *Let G be a prime graph on at least 6 vertices. Then G has a prime vertex-minor H such that $|V(H)| = |V(G)| - 1$.*

Note that no graph on four vertices is prime and all prime graphs on five vertices are locally equivalent to C_5 [13, Lemma 3.1]. Gabor, Hsu, and Supowit [52] provide an $O(|V(G)| \times |E(G)|)$-time algorithm recognizing a circle graph for an input graph G, using the next result.

Theorem 8.2 (Gabor, Hsu, and Supowit [52]). *Every prime graph on at least 5 vertices contains an induced subgraph isomorphic to a cycle of length at least 5 or a graph in Figure 8.*

Allys proved a strengthening of Theorem 8.1 by using isotropic systems.

Theorem 8.3 (Allys [4]). *Let G be a prime graph on at least 6 vertices. Then G has a vertex v such that $G - v$ or $G * v - v$ is prime.*

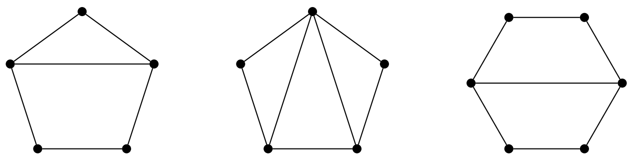


Figure 8: Unavoidable induced subgraphs in prime graphs that are not cycles.

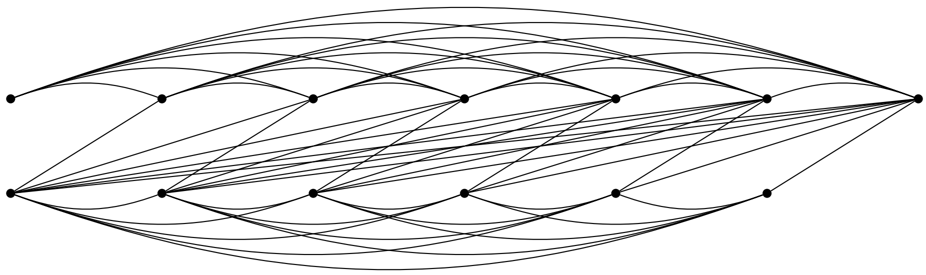


Figure 9: The graph H_{13}. No vertex v other than the unique vertex adjacent to all other vertices has the property that $H_{13} - v$ or $H_{13} * v - v$ is prime.

Oum [94] and Lee and Oum [80] proved related results for variants of primeness. Kim and Oum extended the theorem of Allys (Theorem 8.3) as follows. Let H_n be a graph on $\{v_1, v_2, \ldots, v_n\}$ such that for $1 \leqslant i < j \leqslant n$, two vertices v_i and v_j are adjacent if and only if i is even or j is odd. See Figure 9 for an illustration of H_{13}.

Theorem 8.4 (Kim and Oum [73]). *Let G be a prime graph on at least 6 vertices and x be a vertex of G. Then G has a vertex $v \neq x$ such that $G - v$ or $G * v - v$ is prime, unless G is isomorphic to $H_{|V(G)|}$, x is adjacent to all other vertices of G, and $|V(G)|$ is odd.*

Theorem 8.4 is proved by using the following theorem on the number of vertices that admit at least two ways to be removed while keeping the primeness.

Theorem 8.5 (Kim and Oum [73]). *Let G be a prime graph on at least 4 vertices.*

(i) *G has at least two vertices v such that at least two of $G - v$, $G * v - v$, and $G \wedge vw - v$ for some neighbor w of v are prime, unless G is locally equivalent to C_5.*

(ii) *G has at least three vertices v such that at least two of $G - v$, $G * v - v$, and $G \wedge vw - v$ for some neighbor w of v are prime, unless G is locally equivalent to a cycle or a graph consisting of at least three internally-disjoint paths between two fixed vertices, none of which has length 2.*

Here is an easy corollary of the previous theorem.

Corollary 8.6 (Kim and Oum [73]). *Let G be a prime graph on at least 6 vertices. For every pair of vertices $x, y \in V(G)$, G has a prime vertex-minor H containing both x and y such that $|V(H)| = |V(G)| - 1$.*

The following theorem concerns the number of ways to obtain a fixed graph H as a vertex-minor.

Theorem 8.7 (Geelen and Oum [61, Theorem 1.10]). *Let H be a vertex-minor of G. If $|V(G)| \geqslant 2^{|V(H)|}$, then there is a vertex v such that at least two of $G - v$, $G * v - v$, and $G \wedge vw - v$ for some neighbor w of v have H as a vertex-minor.*

Theorem 8.7 immediately implies the following.

Theorem 8.8 (Geelen and Oum [61, Theorem 1.3]). *Let $\mathcal{G}$ be a class of graphs closed under taking vertex-minors. If every vertex-minor-minimal graph not in $\mathcal{G}$ has at most k vertices, then every pivot-minor-minimal graph not in $\mathcal{G}$ has at most $2^k - 1$ vertices. Thus, $\mathcal{G}$ has finitely many excluded vertex-minors if and only if $\mathcal{G}$ has finitely many excluded pivot-minors.*

Geelen proved a strengthening of Theorem 8.1 in his Ph.D. thesis, analogous to the splitter theorem for 3-connected matroids by Seymour [104]. The proof is based on purely graph-theoretic methods.

Theorem 8.9 (Geelen [56, Corollary 5.11]). *Let G and H be prime graphs such that $4 \leqslant |V(H)| < |V(G)|$. If H is a vertex-minor of G, then there is a sequence of prime graphs $G_1 := G, G_2, \ldots, G_m$ such that*

- *for each $i < m$, G_{i+1} is a vertex-minor of G_i where $|V(G_{i+1})| = |V(G_i)| - 1$ and*
- *G_m is isomorphic to H.*

9 Forbidden vertex-minor characterizations

Theorem 2.1 states that a graph is a circle graph if and only if it has no vertex-minor isomorphic to W_5, F_7, or W_7. In this section, we will review similar theorems on characterizing classes of graphs in terms of forbidden vertex-minors.

9.1 Graphs having no vertex-minor isomorphic to W_5

Geelen [56] described the structure of graphs having no vertex-minors isomorphic to W_5. We denote the 3-dimensional cube graph by Q_3.

Theorem 9.1 (Geelen [56, Theorem 5.14]). *A graph G has no vertex-minor isomorphic to W_5 if and only if one of the following holds.*

(i) *G is a circle graph.*

(ii) *G is locally equivalent to a graph isomorphic to W_7, F_7, or Q_3.*

(iii) *G is the 1-join of two smaller graphs G_1 and G_2, both of which are graphs having no vertex-minors isomorphic to W_5.*

9.2 Distance-hereditary graphs

A graph is *distance-hereditary* [6] if for every connected induced subgraph H and two vertices x and y of H, the distance between x and y in H is equal to the distance between x and y in G. Two vertices x and y are *twins* of a graph G if they have the same set of neighbors in $V(G) - \{x, y\}$. Bandelt and Mulder [6] showed that all distance-hereditary graphs can be built from K_1 by creating twins and adding an isolated vertex or a pendant vertex to a distance-hereditary graph. Oum [88] observed that a graph is distance-hereditary if and only if its rank-width is at most 1. Kwon and Oum [79] showed the following obtained by combining this observation with results of Bouchet [13, 17].

Theorem 9.2 (Bandelt and Mulder [6], Bouchet [13, 17], Oum [88], and Kwon and Oum [79, Theorem 4.1]). *Let G be a graph. The following are equivalent.*

(i) *G is distance-hereditary.*

(ii) *G has rank-width at most 1.*

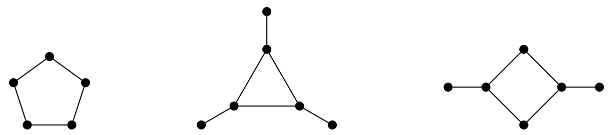


Figure 10: Vertex-minor obstructions for the class of graphs of linear rank-width 1.

(iii) *G has no vertex-minor isomorphic to C_5.*

(iv) *G is a vertex-minor of a tree.*

(v) *G admits a split decomposition into graphs on at most 3 vertices.*

9.3 Linear rank-width at most 1

It is easy to see that the rank-width of a graph is less than or equal to the linear rank-width of a graph. So, the class of graphs of linear rank-width at most 1 is a subclass of the class of distance-hereditary graphs. To describe the structure of graphs of linear rank-width at most 1, Ganian [53] defined thread graphs and proved that a graph is a thread graph if and only if it has linear rank-width at most 1.

Here is a theorem characterizing the class of graphs of linear rank-width at most 1.

Theorem 9.3 (Adler, Farley, and Proskurowski [1] and Kwon and Oum [78, Theorem 4.3])**.** *Let G be a graph. The following are equivalent.*

(i) *G has linear rank-width at most 1.*

(ii) *G has no vertex-minor isomorphic to any graphs in Figure 10.*

(iii) *G is a vertex-minor of a path.*

Here is an interesting theorem on the linear rank-width of a tree.

Theorem 9.4 (Adler and Kanté [2])**.** *For every forest T, the linear rank-width of T is equal to the path-width of T.*

9.4 Rank-width and linear rank-width

Theorem 9.5 (Oum [88])**.** *For each k, there exists a finite list of graphs each having at most $(6^{k+1}-1)/5$ vertices such that a graph has rank-width at most k if and only if it has no pivot-minor isomorphic to a graph in the list.*

Oum [94] improved $(6^{k+1}-1)/5$ of Theorem 9.5 to $(3.5\cdot 6^k-1)/5$ for $k\geqslant 2$.

Theorem 9.6 (Kanté, Kim, Kwon, and Oum [71])**.** *For each k, there exists a finite list of graphs each having at most $2^{2^{O(k^2)}}$ vertices such that a graph has linear rank-width at most k if and only if it has no pivot-minor isomorphic to a graph in the list.*

Jeong, Kwon, and Oum [69] showed that any list satisfying Theorem 9.6 contains at least $2^{\Omega(3^k)}$ graphs.

We can replace pivot-minors with vertex-minors in both Theorem 9.5 and Theorem 9.6 by the following reason. If a graph G has another graph H as a vertex-minor, then there is a graph G' locally equivalent to G such that G' has H as a pivot-minor. Therefore, if a graph class is closed under taking vertex-minors and a graph H is a pivot-minor-minimal graph not in the class, then H is also a vertex-minor-minimal graph not in the class. Note that Theorem 8.8 describes the other direction to obtain pivot-minor-minimal graphs not in a class from vertex-minor-minimal graphs not in the class.

9.5 Well-quasi-ordering

So far we have witnessed several instances of graph classes closed under taking vertex-minors that admit characterizations in terms of finitely many forbidden vertex-minors or pivot-minors. The celebrated graph minors theorem of Robertson and Seymour [102] states that every proper minor-closed class of graphs is characterized by a finite set of forbidden minors. This property can be seen easily equivalent to the following statement; Every infinite sequence G_1, G_2, ... of graphs has a pair $i < j$ such that G_i is isomorphic to a minor of G_j.

We can extend this to quasi-orders. A binary relation $\le$ on X is a *quasi-order* if $x \le x$ for all $x \in X$ and $x \le y$ and $y \le z$ implies $x \le z$. A quasi-order $\le$ on X is a *well-quasi-ordering* if every infinite sequence of $x_1, x_2, x_3, \ldots \in X$ admits a pair $i < j$ such that $x_i \le x_j$. If so, we call the set X *well-quasi-ordered* by the relation $\le$. The graph minors theorem can be equivalently stated that graphs are well-quasi-ordered by the minor relation.

Motivated by the graph minors theorem, it is very natural to propose the following conjectures.

Conjecture 9.7. *Graphs are well-quasi-ordered by the vertex-minor relation.*

Conjecture 9.8 (Oum [92, 93])**.** *Graphs are well-quasi-ordered by the pivot-minor relation.*

Oum [89] showed that graphs of bounded rank-width are well-quasi-ordered by the pivot-minor relation, which implies that such graphs are well-quasi-ordered by the vertex-minor relation.

Theorem 9.9 (Oum [89])**.** *Graphs of bounded rank-width are well-quasi-ordered by the pivot-minor relation.*

Bouchet [22] conjectured that circle graphs are well-quasi-ordered by the vertex-minor relation. This is implied by the well-quasi-ordering of 4-regular graphs by the immersion relation, proved by Robertson and Seymour [103]. The relation between the immersions on 4-regular graphs and vertex-minors of circle graphs was observed by Kotzig [75] and is described in the survey of Bouchet [19].

Theorem 9.10 (Implied by Robertson and Seymour [103]; see McCarty [85])**.** *Circle graphs are well-quasi-ordered by the vertex-minor relation.*

Pivot-minors of bipartite graphs are associated with minors of binary matroids, see Oum [88]. Geelen, Gerards, and Whittle [59] proved that for every fixed finite field $\mathbb{F}$, $\mathbb{F}$-representable matroids are well-quasi-ordered by the minor relation. This in particular means that binary matroids are well-quasi-ordered by the minor relation and so we deduce the following. As far as the authors know, its proof is still being written.

Theorem 9.11 (Implied by Geelen, Gerards, and Whittle [59])**.** *Bipartite graphs are well-quasi-ordered by the pivot-minor relation.*

The *line graph* of a graph G is a graph on $E(G)$ such that $e, f \in E(G)$ are adjacent in the line graph if and only if e and f share a common end in G. Oum [91] studied pivot-minors of line graphs via minors of *grafts*, which are pairs (G, T) of a graph G and a set $T \subseteq V(G)$. In the graft minors, contracting an edge e will make the new vertex belong to T if and only if exactly one end of e belongs to T. This can be seen as minors of group-labelled graphs where vertices have labels from the binary field. Geelen, Gerards, and Whittle announced that while proving their theorem for matroids [59], they proved that group-labelled graphs with labels from a finite field are well-quasi-ordered by the minor relation and this implies that grafts are well-quasi-ordered by the graft minor relations. Therefore, we deduce the following.

Theorem 9.12. *Pivot-minors of line graphs are well-quasi-ordered by the pivot-minor relation.*

For two graphs H_1 and H_2, let $\mathcal{C}$ be the class of graphs having no vertex-minor isomorphic to H_1 or having no vertex-minor isomorphic to H_2. Clearly, $\mathcal{C}$ is closed under taking vertex-minors. If Conjecture 9.7 holds, then there will be a list L of finitely many graphs such that a graph belongs to $\mathcal{C}$ if and only if it has no vertex-minor isomorphic to a graph in L. Each graph in L is required to have both a vertex-minor isomorphic to H_1 and a vertex-minor isomorphic to H_2. So the conjecture 9.7 implies that there are only finitely many vertex-minor-minimal graphs containing both a vertex-minor isomorphic to H_1 and a vertex-minor isomorphic to H_2. This is verified by Geelen and Oum [61] with an explicit bound by using Theorem 8.7.

Theorem 9.13 (Geelen and Oum [61, Theorem 1.11]). *For all graphs H_1 and H_2, if G is a vertex-minor-minimal graphs containing a vertex-minor isomorphic to H_1 and a vertex-minor isomorphic to H_2, then*

$$|V(G)| \leqslant 2^{|V(H_1)|} + 2^{|V(H_2)|} - 2.$$

It is open whether the analog holds for pivot-minors, which would be a consequence of Conjecture 9.8.

Conjecture 9.14 (Lee and Oum [81]). *For all graphs H_1 and H_2, there exists no infinite set of pairwise non-isomorphic pivot-minor-minimal graphs containing a pivot-minor isomorphic to H_1 and a pivot-minor isomorphic to H_2.*

10 Structural analysis of vertex-minor closed classes: Preliminary results

10.1 Unavoidable vertex-minors

Theorem 2.2 shows that for every circle graph H, there is a positive integer k such that every graph of rank-width larger than k contains a vertex-minor isomorphic to H. We review similar results for vertex-minors concerning other graph parameters. We write nK_2 for the disjoint union of n copies of K_2.

Theorem 10.1. *The following hold.*

(i) *For every positive integer n, every graph of sufficiently large rank-depth contains a vertex-minor isomorphic to P_n [77].*

(ii) *For every edgeless graph H, every graph having sufficiently many vertices contains a vertex-minor isomorphic to H [79].*

(iii) *For every complete graph H, every connected graph having sufficiently many vertices contains a vertex-minor isomorphic to H [79].*

(iv) *For every positive integer n, every graph having sufficiently many edges contains a vertex-minor isomorphic to K_n or nK_2 [79].*

All these results are the best possible. For example, we cannot replace P_n in (i) with a graph that is not a vertex-minor of a path. Theorem 10.1(i) solves a conjecture of Hliněný, Kwon, Obdržálek, and Ordyniak [67], initially stated in terms of shrub-depth. Theorem 10.1(ii)-(iv) are easy consequences of Ramsey's theorem.

Here is a conjecture on graphs of large linear rank-width. Kanté and Kwon [72] verified it for distance-hereditary graphs.

Conjecture 10.2 (Kanté and Kwon [72]). *For every tree T, every graph of sufficiently large linear rank-width contains a vertex-minor isomorphic to T.*

What happens if we replace vertex-minors with pivot-minors? Oum [91] proposed the following conjecture on pivot-minor obstructions for graphs of large rank-width. This conjecture implies Theorem 2.2 because every circle graph is a vertex-minor of a bipartite circle graph [26, Corollary 53].

Conjecture 10.3 (Oum [91])**.** *For every bipartite circle graph H, every graph G of sufficiently large rank-width contains a pivot-minor isomorphic to H.*

This conjecture is known to be true for some graphs G.

(i) It is true if G is bipartite by the theorem on binary matroids due to Geelen, Gerards, and Whittle [58], see [88, Corollary 3.9].

(ii) It is true if G is a pivot-minor of a line graph, shown by Oum [91].

(iii) It is true if G is a circle graph by the result in the Ph.D. thesis of Johnson [70, Theorem 2.5], as explained in Oum [91].

The grid minor theorem of Robertson and Seymour [98] states that for an integer $k > 0$, every graph G of sufficiently large tree-width has a minor isomorphic to a $k \times k$ grid. We remark that Conjecture 10.3 implies the grid minor theorem. Let H be a planar graph obtained from a $(k+1) \times (k+1)$ grid by adding a new vertex adjacent to all vertices of degree at most 3. Let F_G and F_H be fundamental graphs of G and H, respectively. By Theorem 2.3, F_H is a bipartite circle graph. Suppose that G has sufficiently large tree-width. Then by Robertson and Seymour [99, (5.2)], G has large branch-width, and by Hicks and McMurray [66, Theorem 4], $M(G)$ has large branch-width. By [88, Corollary 3.2], the rank-width of F_G equals to the branch-width of $M(G)$ minus one. Thus, if Conjecture 10.3 holds, then F_G has a pivot-minor isomorphic to F_H. Hence G has a minor X such that X has a fundamental graph isomorphic to F_H. As binary matroids are determined by their fundamental graphs up to duality, the cycle matroid $M(X)$ of X is isomorphic to $M(H)$ or $M^*(H) = M(H^*)$, where H^* is a dual graph of H. Note that H is 3-connected and so is H^*. Thus, by Whitney's 2-isomorphism theorem [109] (see [96, Section 8.5]), X is isomorphic to H or H^*. Hence G has a minor isomorphic to H or H^*. Because both H and H^* contain a $k \times k$ grid as an induced subgraph, we conclude that G has a minor isomorphic to a $k \times k$ grid.

For graphs of large linear rank-width, Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [35] conjectured the following. A *caterpillar* is a tree whose subgraph induced by the set of vertices of degree larger than one is a path.

Conjecture 10.4 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [35])**.** *For every caterpillar T, every graph of sufficiently large linear rank-width contains a pivot-minor isomorphic to T.*

This conjecture is verified for distance-hereditary graphs [35].

10.2 Vertex-minor closures of classes of sparse graphs

Every graph of small tree-width, path-width, and tree-depth has small rank-width, linear rank-width, and rank-depth, respectively.

Theorem 10.5. *Let G be a graph.*

(i) *If the tree-width of G is k, then the rank-width of G is at most $k+1$ [90].*

(ii) *If the path-width of G is k, then the linear rank-width of G is at most k [2].*

(iii) *If the tree-depth of G is k, then the rank-depth of G is at most k [46].*

Recall that rank-width, linear rank-width, and rank-depth do not increase by taking vertex-minors. Thus, if a class $\mathcal{C}$ of graphs has bounded tree-width, path-width, and tree-depth, then the class of vertex-minors of graphs in $\mathcal{C}$ has bounded rank-width, linear rank-width, and rank-depth, respectively. The converse holds as follows.

Theorem 10.6. *Let $\mathcal{C}$ be a class of graphs.*

(i) *If $\mathcal{C}$ has bounded rank-width, then there is a class $\mathcal{D}$ of graphs of bounded tree-width such that every graph in $\mathcal{C}$ is a vertex-minor of a graph in $\mathcal{D}$ [78].*

(ii) *If $\mathcal{C}$ has bounded linear rank-width, then there is a class $\mathcal{D}$ of graphs of bounded path-width such that every graph in $\mathcal{C}$ is a vertex-minor of a graph in $\mathcal{D}$ [78].*

(iii) *If $\mathcal{C}$ has bounded rank-depth, then there is a class $\mathcal{D}$ of graphs of bounded tree-depth such that every graph in $\mathcal{C}$ is a vertex-minor of a graph in $\mathcal{D}$ [67].*

Kwon and Oum [78] showed that one can replace vertex-minors in (i) and (ii) with pivot-minors. However, one cannot replace the vertex-minors in (iii) with pivot-minors as shown by [67]. Note that by Theorem 5.4(iii), we can state in Theorem 10.6(iii) in terms of shrub-depth, and this is how Hliněný et al. [67] wrote their theorem.

11 χ-boundedness

For a graph G, we write $\chi(G)$ to denote its chromatic number and $\omega(G)$ to denote the maximum size of a clique. A class of graphs is *χ-bounded* if there is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $\chi(G) \leqslant f(\omega(G))$ for every induced subgraph G of a graph in the class. Such a function f is called the *χ-bounding* function.

Motivated by the fact that the class of circle graphs is χ-bounded, which we will discuss in the following subsection, Geelen (see [47]) conjectured in 2009 at the DIMACS workshop held at Princeton University that every proper vertex-minor closed class of graphs is χ-bounded. This has been verified for circle graphs by Gyárfás [64, 65], graphs of bounded rank-width and graphs having no vertex-minor isomorphic to W_5 by Dvořák and Král' [47], and graphs having no vertex-minor isomorphic to a fixed wheel graph by Choi, Kwon, Oum, and Wollan [28].

James Davies resolved the conjecture as follows.

Theorem 11.1 (Davies [41])**.** *For every graph H, the class of graphs having no vertex-minor isomorphic to H is χ-bounded.*

Davies [43] also announced a strengthening of Theorem 11.1, which stated that the class of graphs having no pivot-minor isomorphic to a fixed graph is χ-bounded.

If the χ-bounding function can be taken as a polynomial, then the class is called *polynomially χ-bounded.* Kim, Kwon, Oum, and Sivaraman [74] proposed the following conjecture and proved it when H is a cycle.

Conjecture 11.2 (Kim, Kwon, Oum, and Sivaraman [74])**.** *For every graph H, the class of graphs having no vertex-minor isomorphic to H is polynomially χ-bounded.*

11.1 Circle graphs

Gyárfás [64, 65] showed that circle graphs are χ-bounded but until recently it was an open problem to decide whether or not circle graphs are polynomially χ-bounded. This is now proven by Davies and McCarty.

Theorem 11.3 (Davies and McCarty [44])**.** *If G is a circle graph, then $\chi(G) \leqslant 7\omega(G)^2$.*

Davies further improves the χ-bounding function.

Theorem 11.4 (Davies [40]). *If G is a circle graph, then $\chi(G) \leqslant O(\omega(G)\log(\omega(G)))$.*

He also improves an existing lower bound for the χ-bounding function for circle graphs.

Theorem 11.5 (Davies [40]). *For every positive integer k, there is a circle graph G such that $\omega(G) \leqslant k$ and $\chi(G) \geqslant k \log k - 2k$.*

Using Theorems 9.1, 11.3, and Theorem 1.2 of Kim, Kwon, Oum, and Sivaraman [74], we deduce the following.

Theorem 11.6. *The class of graphs having no vertex-minors isomorphic to W_5 is polynomially χ-bounded.*

11.2 Rank-width and polynomial χ-boundedness

Theorem 11.7 (Bonamy and Pilipczuk [7]). *Every class of graphs of bounded rank-width is polynomially χ-bounded.*

Now, Theorem 11.7 is implied by results on twin-width, a relatively new width parameter introduced by Bonnet, Kim, Thomassé, and Watrigant [10]. They showed that if a class of graphs has bounded rank-width, then it has bounded twin-width. Bonnet, Geniet, Kim, Thomassé, and Watrigant [8, 9] showed that every class of graphs of bounded twin-width is χ-bounded. Pilipczuk and Sokołowski [97] showed that the χ-bounding function can be taken as a quasi-polynomial. Recently, Bourneuf and Thomassé [24] showed that a class of graphs of bounded twin-width is polynomially χ-bounded. This implies Theorem 11.7.

Here is a corollary of Theorems 11.7 and 2.2 which verifies Conjecture 11.2 when H is a circle graph.

Corollary 11.8. *Let H be a circle graph. The class of graphs having no vertex-minor isomorphic to H is polynomially χ-bounded.*

Corollary 11.8 implies the result of Kim, Kwon, Oum, and Sivaraman [74] for graphs having no vertex-minor isomorphic to C_n.

The degree of a χ-bounding function for the class of graphs of rank-width at most k cannot be independent of k.

Theorem 11.9 (Bonamy and Pilipczuk [7, Lemma 5.2]). *For every k, if f is a χ-bounding polynomial for the class of graphs of rank-width at most k, then the degree of f is at least $\Omega(\log k)$.*

11.3 Linear rank-width and linear χ-boundedness

If the χ-bounded function can be taken as a linear, then the class is called *linearly χ-bounded.*

Theorem 11.10 (Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz [87]). *Every class of graphs of bounded linear rank-width is linearly χ-bounded.*

By Theorem 10.1(i), graphs having no vertex-minor isomorphic to P_n have bounded rank-depth. Since graphs of bounded rank-depth have bounded linear rank-width, we deduce the following result from Theorem 11.10. This improves the previous result by Kim, Kwon, Oum, and Sivaraman [74], who showed that a class of graphs having no vertex-minor isomorphic to P_n is polynomially χ-bounded.

Corollary 11.11. *Let n be a positive integer. Every class of graphs having no vertex-minors isomorphic to P_n is linearly χ-bounded.*

If the class of graphs having no vertex-minors isomorphic to a graph H is linearly χ-bounded, then H is a circle graph by Theorem 11.5. It will be interesting to determine graphs H such that the class of graphs having no vertex-minor isomorphic to H is linearly χ-bounded. Nešetřil et al. [87, Theorem 3.7] showed that there is a class of graphs of bounded rank-width which is not linearly χ-bounded.

11.4 Erdős-Hajnal property

A class $\mathcal{G}$ of graphs closed under taking induced subgraphs has the *Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that every graph G in $\mathcal{G}$ has an independent set or a clique of size at least $|V(G)|^\varepsilon$, where a *clique* in a graph is a set of pairwise adjacent vertices, and an *independent set* in a graph is a set of pairwise non-adjacent vertices. Erdős and Hajnal [48] conjectured that for every graph H, the class of graphs having no induced subgraph isomorphic to H has the Erdős-Hajnal property. Chudnovsky and Oum [29] proved the affirmative result for a class of graphs forbidding any graph as a vertex-minor.

Theorem 11.12 (Chudnovsky and Oum [29]). *For every graph H, the class of graphs having no vertex-minor isomorphic to H has the Erdős-Hajnal property.*

They indeed proved the following stronger theorem. In a graph, a pair of disjoint vertex sets A and B is *complete* if every vertex in A is adjacent to all vertices in B. Such a pair is *anticomplete* if there is no edge between A and B.

Theorem 11.13 (Chudnovsky and Oum [29]). *For every graph H, there exists $\varepsilon > 0$ such that for each integer $n \geqslant 2$, every n-vertex graph having no vertex-minor isomorphic to H has a complete or anticomplete pair of disjoint vertex sets A and B such that $|A|, |B| \geqslant \varepsilon n$.*

We remark that Conjecture 11.2 implies Theorem 11.12. Davies [42] proved the following strengthening of Theorem 11.13.

Theorem 11.14 (Davies [42]). *For every graph H, there exists $\varepsilon > 0$ such that for each integer $n \geqslant 2$, every n-vertex graph having no pivot-minor isomorphic to H has a complete or anticomplete pair of disjoint vertex sets A and B such that $|A|, |B| \geqslant \varepsilon n$.*

12 Testing vertex-minors

The notion of vertex-minors is interesting not only in graph theory but also in quantum information theory, because of its connection to graph states. A graph state is represented by a graph whose vertices represent qubits of the graph state. Dahlberg, Helsen, and Wehner [37] described this connection as follows:

> Graph states are ubiquitous in quantum information with diverse applications ranging from quantum network protocols to measurement based quantum computing. Here we consider the question whether one graph (source) state can be transformed into another graph (target) state, using a specific set of quantum operations (LC + LPM + CC): single-qubit Clifford operations (LC), single-qubit Pauli measurements (LPM) and classical communication (CC) between sites holding the individual qubits. … Our results make use of the insight that deciding whether a graph state $|G\rangle$ can be transformed to another graph state $|G'\rangle$ is equivalent to a known decision problem in graph theory, namely the problem of deciding whether a graph G' is a vertex-minor of a graph G.

Bouchet [20] presented a polynomial-time algorithm deciding whether two graphs are locally equivalent, by observing that this problem is equivalent to finding a solution of a system of equations over the binary field.

Theorem 12.1 (Bouchet [20]). *There is an $O(n^4)$-time algorithm to decide whether two n-vertex graphs on the same vertex set are locally equivalent.*

Dahlberg, Helsen, and Wehner [36] proved that counting locally equivalent graphs is #P-complete by reducing this problem on circle graphs to the problem of counting Eulerian circuits in a 4-regular graph.

Theorem 12.2 (Dahlberg, Helsen, and Wehner [36, Theorem V.1]). *Computing the number of graphs locally equivalent to an input graph (without isomorphism) is #P-complete, even if the input graph is a circle graph.*

Unlike the problem of deciding the local equivalence of two graphs, it is NP-complete to decide whether one graph is a vertex-minor of another graph or isomorphic to a vertex-minor of another graph.

Theorem 12.3 (Dahlberg, Helsen, and Wehner [37, Theorem 3.1]). *Deciding whether H is a vertex-minor of G for two input graphs G and H such that $V(H) \subseteq V(G)$ is NP-complete, even if H is a complete graph and G is a circle graph.*

Theorem 12.4 (Dahlberg, Helsen, and Wehner [38]). *Deciding whether H is isomorphic to a vertex-minor of G for two input graphs G and H is NP-complete, even if both G and H are circle graphs.*

It is also NP-complete for pivot-minors.

Theorem 12.5 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [34]). *Deciding whether H is isomorphic to a pivot-minor of G for two input graphs G and H is NP-complete, even if H is a star graph and G is bipartite.*

Courcelle and Oum [32] constructed a modulo-2 counting monadic second-order transduction that maps a graph into the set of all vertex-minors and this allows them to prove the following theorem.

Theorem 12.6 (Courcelle and Oum [32]). *Let H be a fixed graph and t be a constant. There is an $O(n^3)$-time algorithm to certify that an input n-vertex graph has rank-width larger than t or decide whether it contains a vertex-minor isomorphic to H.*

By Theorems 2.2 and 12.6, we deduce the following.

Theorem 12.7. *For each fixed circle graph H, there is an $O(n^3)$-time algorithm to decide whether an input n-vertex graph G contains a vertex-minor isomorphic to H.*

It is not known whether we can remove isomorphisms in the previous theorem. Dahlberg, Helsen, and Wehner [37] considered this problem when the input graph is a circle graph and H is a complete graph. By using the method of Courcelle and Oum [32], it is straightforward to deduce the following algorithm to decide whether an input graph of small rank-width contains a fixed vertex-minor on a particular vertex set.

Theorem 12.8 (Courcelle and Oum [32]). *Let H be a fixed graph and t be a constant. There is an $O(n^3)$-time algorithm to certify that an input n-vertex graph G such that $V(H) \subseteq V(G)$ has rank-width larger than t or decide whether it contains H as a vertex-minor.*

13 Interlace polynomials

We will review the interlace polynomials and the global interlace polynomials. We remark that Bouchet [21] defined the restricted Tutte-Martin polynomial and the global Tutte-Martin polynomial of an isotropic system S, which are identical to the interlace polynomial and the global interlace polynomial of a fundamental graph of S, respectively, as observed in [23]. A motivation of Bouchet was to unify Tutte polynomials of binary matroids with Martin polynomials introduced by Martin [84] for 4-regular graphs and 2-in 2-out digraphs. For more details, see [106] and a well-written survey by Brijder and Jan Hoogeboom [25].

Arratia, Bollobás, and Sorkin [5] defined (single-variable) interlace polynomials of graphs recursively but later Aigner and van der Holst [3] presented the following equivalent definition. Let A be the adjacency matrix of a graph G over the binary field. For a subset S of $V(G)$, we write $A[S]$ to denote the $S \times S$ principal submatrix of A. The (single-variable) *interlace polynomial* of G is defined as

$$q(G,x) = \sum_{S \subseteq V(G)} (x-1)^{|S| - \operatorname{rank}(A[S])}.$$

Theorem 13.1 (Arratia, Bollobás, and Sorkin [5, Theorem 12], Aigner and van der Holst [3, Corollary 1]). *The interlace polynomial is the unique map q satisfying the following two conditions.*

(i) *If a graph G has an edge uv, then*

$$q(G,x) = q(G-u,x) + q(G \wedge uv - u, x).$$

(ii) *If a graph G has n vertices and no edges, then $q(G,x) = x^n$.*

Aigner and van der Holst [3] observed the following corollary of Theorem 13.1.

Corollary 13.2. *If G and G' are pivot-equivalent, then $q(G,x) = q(G',x)$.*

A theorem of Martin [84] implies that the interlace polynomial of a fundamental graph of a planar graph H is a restriction of the Tutte polynomial of H. The following generalization shows that the interlace polynomial of a bipartite graph is identical to a restriction of the Tutte polynomial of a corresponding binary matroid.

Theorem 13.3 (Aigner and van der Holst [3, Theorem 3], Bouchet [21]). *Let M be a binary matroid and G be its fundamental graph. Then $q(G,x) = T_M(x,x)$, where $T_M(x,y)$ is the Tutte polynomial of M.*

Aigner and van der Holst [3] defined another polynomial $Q(G,x)$ satisfying a 3-term recurrence relation. Let A be the adjacency matrix of a graph G over the binary field and for a set X of vertices, let I_X be the $V(G) \times V(G)$ diagonal matrix over the binary field such that the (v,v)-entry is 1 if and only if $v \in X$. We define the *global interlace polynomial* as

$$Q(G,x) = \sum_{T \subseteq S \subseteq V(G)} (x-2)^{|S| - \operatorname{rank}((A+I_T)[S])}.$$

Theorem 13.4 (Aigner and van der Holst [3, Section 4]). *The global interlace polynomial is the unique map Q satisfying the following two conditions.*

(i) *If a graph G has an edge uv, then*

$$Q(G,x) = Q(G-u,x) + Q(G * u - u, x) + Q(G \wedge uv - u, x).$$

(ii) *If a graph G has n vertices and no edges, then $Q(G, x) = x^n$.*

This theorem implies that two locally equivalent graphs yield the same global interlace polynomial.

Corollary 13.5 (Aigner and van der Holst [3, Corollary 4])**.** *If G and G' are locally equivalent, then $Q(G, x) = Q(G', x)$.*

14 Conclusions

We conclude this paper by presenting several conjectures motivated by the Graph Minors Project of Robertson and Seymour. The following conjecture is one of the central open problems on vertex-minors.

Conjecture 9.7. *Graphs are well-quasi-ordered by the vertex-minor relation.*

If true, then for every class $\mathcal{C}$ of graphs closed under taking vertex-minor, there is a finite list of graphs $G_1, \ldots, G_k$ such that a graph H is in $\mathcal{C}$ if and only if H has no vertex-minor isomorphic to any G_i.

Here is a conjecture analogous to the algorithm of Robertson and Seymour [100] on testing minors.

Conjecture 14.1. *For every fixed graph H, there is a polynomial-time algorithm to decide whether an input graph G contains a vertex-minor isomorphic to H.*

If Conjectures 9.7 and 14.1 hold, then for every class $\mathcal{C}$ of graphs closed under taking vertex-minors, there is a polynomial-time algorithm to decide whether an input graph is in $\mathcal{C}$.

For the application to graph states in the quantum information theory, it is interesting to find a vertex-minor on a fixed set of vertices.

Conjecture 14.2. *For every fixed graph H, there is a polynomial-time algorithm to decide whether an input graph G such that $V(H) \subseteq V(G)$ contains H as a vertex-minor.*

Motivated by the graph structure theorem [101] and the Matroid Minors Project [57], Geelen proposed a structural conjecture for vertex-minors. For an integer $k \geqslant 1$, we say a graph G is *k-rank-connected* if $|V(G)| \geqslant 2k$ and $\rho_G(X) \geqslant \min(|X|, |V(G) - X|, k)$ for each $X \subseteq V(G)$. A graph G is a *rank-k perturbation* of a graph H if the adjacency matrix of G can be obtained from the adjacency matrix of H by adding a diagonal matrix and a matrix of rank at most k over the binary field.

Conjecture 14.3 (Weak structural conjecture; see [85])**.** *For every fixed graph H, there are positive integers k and p such that every k-rank-connected graph having no vertex-minor isomorphic to H is a rank-p perturbation of a circle graph.*

Geelen proposed a stronger version of this conjecture, see [85, page 23].

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