Vertex-minors of graphs

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Abstract

For a graph with a vertex v, the local complementation at v is an operation that replaces the neighborhood of v by its complement graph. Two graphs are locally equivalent if one is obtained from the other by a sequence of local complementations. A graph H is a vertex-minor of a graph G if H is an induced subgraph of a graph locally equivalent to G. Although this concept was introduced in the 1980s, it was not widely known and except for the survey paper of Bouchet published in 1990, there is no comprehensive survey listing all the new developments. We survey classic and recent theorems and conjectures on vertex-minors and related concepts such as circle graphs, cut-rank functions, rank-width, interlace polynomials, and isotropic systems.

1 Introduction

We aim to survey known results and conjectures for vertex-minors of graphs. In this paper, all graphs are simple, meaning that neither loops nor parallel edges are allowed. For a vertex v of a graph G, we write $G - v$ to denote the graph obtained by deleting the vertex v and all edges incident with v.

Let us start with their definitions. Vertex-minors are defined in terms of two graph operations, local complementations and vertex deletions. The *local complementation* of a graph G at a vertex v is an operation to obtain a new graph denoted by $G * v$ from G by 'toggling' the adjacencies between all pairs of neighbors of v, see Figure 1. In other words, two distinct vertices x and y are adjacent in $G * v$ if and only if exactly one of the following holds.

(a) x and y are adjacent in G.

(b) Both x and y are neighbors of v in G.

Two graphs are *locally equivalent* if one is obtained from the other by a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph locally equivalent to G. The name 'vertex-minor' first appeared in Oum [Oum05] but it appeared previously under the various names such as l-reduction [Bou94] and i-minor [Bou87d]. According to Bouchet [Bou90], local complementations were introduced by Kotzig [Kot68, Kot77].

For two adjacent vertices x and y, we write $G \wedge xy$ to denote $G*x*y*x$. It is easy to check that $G*x*y*x = G*y*x*y$ and so this operation is well defined and is called the *pivot* operation. We note that $G \wedge xy$ could be obtained from G by toggling the adjacency between

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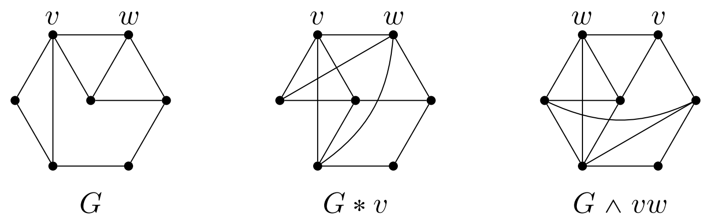

Figure 1: Examples of local complementations and pivotings.

every pair of vertices in two different sets among $N_G(x) - (N_G(y) \cup \{y\})$, $N_G(x) \cap N_G(y)$, and $N_G(y) - (N_G(x) \cup \{x\})$ and then switching labels of x and y, see Figure 1. Another graphical description was given in Oum [Oum05] along with the proof of the following well-known fact:

$$G \wedge xy \wedge yz = G \wedge xz.$$

Because of this, it is often convenient to write G/v to denote $(G \wedge vw) - v$ for some neighbor w of v, because the choice of w does not change the outcome up to pivot-equivalence, as $(G \wedge vw) - v = ((G \wedge vw') - v) \wedge ww'$. Exceptionally, if v has no neighbor, we define $G/v := G - v$. Two graphs are *pivot-equivalent* if one is obtained from the other by a sequence of pivotings. A graph H is a *pivot-minor* of a graph G if H is an induced subgraph of a graph pivot-equivalent to G.

In Sections 2 and 3 we review some basic properties of local equivalences and vertex-minors. In Sections 4–7 we survey the cut-rank function of a graph, which is a connectivity function for vertex-minors. In Sections 8 and 9 we give structural theorems for the vertex-minor, and in Section 10 we present recent progress on χ-boundedness for vertex-minor-closed classes of graphs. Section 11 provides some algorithmic results on vertex-minors, and in Sections 12 and 13 we review interlace polynomials and isotropic systems, respectively.

2 Basic theorems

It may be seen that there are many inequivalent ways to reduce a graph to a vertex-minor with one fewer vertex. However, the following theorem, proved by Bouchet and Fon-Der-Flaass independently, ensures that essentially there are only three ways to remove one vertex to have a vertex-minor.

Theorem 2.1 (Bouchet [Bou88a, (9.2)] and Fon-Der-Flaass [FDF88, Corollary 4.3])**.** *Let H be a vertex-minor of a graph G and v be a vertex in $V(G) - V(H)$. Then H is a vertex-minor of $G - v$, $G * v - v$, or G/v.*

Bouchet [Bou88a] used isotropic systems to prove the previous theorem. Fon-Der-Flaass [FDF88] proved the theorem in graph-theoretic notions. Sometimes it is convenient to use the following form presented in Geelen and Oum [GO09, Lemma 3.2], which immediately implies Theorem 2.1.

Lemma 2.2 (Geelen and Oum [GO09, Lemma 3.1])**.** *Let G be a graph and v and w be vertices.*

(i) If $v \neq w$ and v is non-adjacent to w, then $(G * w) - v$, $(G * w * v) - v$, and $(G * w)/v$ are locally equivalent to $G - v$, $(G * v) - v$, and G/v, respectively.

(ii) If $v \neq w$ and v is adjacent to w, then then $(G * w) - v$, $(G * w * v) - v$, and $(G * w)/v$ are locally equivalent to $G - v$, G/v, and $(G * v) - v$, respectively.

(iii) If $v = w$, then $(G * w) - v$, $(G * w * v) - v$, and $(G * w)/v$ are locally equivalent to $(G * v) - v$, $G - v$, and G/v, respectively.

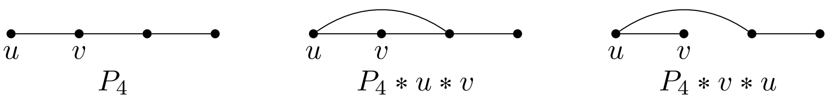

Figure 2: Local complementations are not commutable.

We remark that bipartite graphs with the pivot-minor relation are essentially equivalent to binary matroids with the matroid minor relation. Many theorems on vertex-minors and pivot-minors are motivated in part by theorems on minors of graphs or matroids. See [Oum05, Section 3] for more discussion on relating pivot-minors of bipartite graphs and minors of binary matroids.

3 Various properties on locally equivalent graphs

Local complementations do not commute; see Figure 2. Hence when we enumerate a sequence of local complementations, there might have many vertices appearing more than once. Fon-Der-Flaass [FDF88] showed that for every pair of local equivalent graphs, there is a good sequence of local complementations and pivotings avoiding redundancy.

Theorem 3.1 (Fon-Der-Flaass [FDF88, Corollary 4.2])**.** *Let G and H be locally equivalent graphs. Then there is a sequence of local complementations and pivotings such that all vertices used for local complementations and all ends of edges used for pivotings appear only once.*

Pivot-equivalent graphs can be explained by a matrix operation originated from Tucker [Tuc60]. For a $V \times V$ matrix A and a subset $X \subseteq V$, we write $A[X]$ to denote its $X \times X$ principal submatrix. Let

$$A = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}.$$

If $A[X] = \alpha$ is nonsingular, then let

$$A * X = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

This operation is called the *pivoting* or the *principal pivoting.* Tucker [Tuc60] proved that $(A*X)[Y]$ is nonsingular if and only if $A[X \triangle Y]$ is nonsingular. Here, $X \triangle Y = (X-Y) \cup (Y-X)$.

Let us write $A(G)$ to denote the adjacency matrix of a graph G over the binary field. It turns out that

$$A(G \wedge uv) = A(G) * \{u, v\}.$$

Thus we observe the following.

Lemma 3.2. *Two graphs G and H on the vertex set V are pivot-equivalent if and only if $A(G) = A(H) * X$ for some $X \subseteq V$ with nonsingular $A(H)[X]$.*

This means that if $H = G \wedge e_1 \wedge e_2 \cdots \wedge e_k$, then $V(e_1) \triangle V(e_2) \triangle \cdots \triangle V(e_k)$ determines H and conversely if H and G are pivot-equivalent, then we can choose edges e_1, e_2, …, e_k such that $H = G \wedge e_1 \wedge e_2 \cdots \wedge e_k$ and $V(e_i) \cap V(e_j) = \varnothing$ for all $i \neq j$, because for any nonsignular non-trivial skew-symmetric matrix X, one can find a 2×2 nonsingular principal submatrix to apply pivoting.

Also, note that if $A(G) = A(H) * X$, then $A(G)[Y]$ is nonsingular if and only if $A(H)[X \triangle Y]$ is nonsingular by Tucker [Tuc60]. So we deduce the following.

Lemma 3.3. *Let G and H be graphs on the vertex set V. Then G and H are pivot-equivalent if and only if there exists $X \subseteq V$ such that*

$$\{Y : Y \subseteq V \text{ and } \det A(G)[Y] \neq 0\} = \{Y \triangle X : Y \subseteq V \text{ and } \det A(H)[Y] \neq 0\}.$$

The above lemma is the essence of *even binary delta-matroids.* For a graph G on V, the pair $(V, \mathcal{F})$ with $\mathcal{F} = \{X \subseteq V : \det A(G)[X] \neq 0\}$ defines even binary delta-matroids introduced by Bouchet [Bou88b]. Bouchet introduced delta-matroids more generally [Bou87a] and pivot-minors of graphs can be explained as minors of even binary delta-matroids. This relation is useful when studying pivot-minors of graphs.

Here is another theorem on two locally equivalent graphs.

Theorem 3.4 (Fon-Der-Flaass [FDF89, FDF96])**.** *Let G and H be locally equivalent graphs. Then there are vertices $v_1, \ldots, v_k$ and edges $e_1, \ldots, e_\ell, e'_1, \ldots, e'_m$ such that*

(i) $e_i \in E(G \wedge e_1 \wedge \cdots \wedge e_{i-1})$ *for* $1 \leqslant i \leqslant \ell$,

(ii) $\{v_1, \ldots, v_k\}$ *is an independent set in* $G \wedge e_1 \wedge \cdots \wedge e_\ell$,

(iii) $e'_j \in E(G \wedge e_1 \wedge \cdots \wedge e_\ell * v_1 * \cdots * v_k \wedge e'_1 \wedge \cdots \wedge e'_{j-1})$ *for* $1 \leqslant j \leqslant m$, *and*

(iv) $H = G \wedge e_1 \wedge \cdots \wedge e_\ell * v_1 * \cdots * v_k \wedge e'_1 \wedge \cdots \wedge e'_m$.

Fon-Der-Flaass proved the preceding theorem in Russian. A paper by the authors [KO22] includes a proof based on isotropic systems. As an easy consequence of Theorem 3.4, Fon-Der-Flaass [FDF96] proved that two locally equivalent bipartite graphs are pivot-equivalent. This was a conjecture of Bouchet [Bou90, Conjecture 3.5] and is also implied by Theorem 4.3.

Theorem 3.5 (Fon-Der-Flaass [FDF88, Theorem 2.2])**.** *If a graph G is locally equivalent to a tree T, then G has a subgraph isomorphic to T.*

Theorem 3.5 implies the following theorem, which was initially conjectured by Mulder at the Oberwolfach meeting in 1986. Bouchet also solved this conjecture independently.

Theorem 3.6 (Bouchet [Bou88c] and Fon-Der-Flaass [FDF88])**.** *If two trees are locally equivalent, then they are isomorphic.*

Jeong, Kwon, and Oum [JKO14] used the idea of the above theorem to prove the following theorem on block graphs. A *block graph* is a graph in which every block is a complete graph. A vertex is *simplicial* if its neighbors are pairwise adjacent.

Theorem 3.7 (Jeong, Kwon, and Oum [JKO14, Theorem 1.4])**.** *If two block graphs without simplicial vertices of degree 2 are locally equivalent, then they are isomorphic.*

Theorem 3.8 (Fon-Der-Flaass [FDF88, Theorem 5.1])**.** *Let G be a graph and $n \geqslant 5$. If G is locally equivalent to C_n, then G has a subgraph isomorphic to C_n.*

Theorem 3.8 implies that if an n-vertex bipartite graph is locally equivalent to a cycle, then n is even, proved by Allys [All94, Lemma 5.2] later by using isotropic systems. Another proof of this fact is presented in the appendix of [KO22].

4 Connectity: Cut-rank functions

For a graph G, the cut-rank function ρ_G is defined over the subsets X of $V(G)$ such that $\rho_G(X)$ is the rank of the $X \times (V(G) - X)$ matrix over the binary field whose entry is 1 if and only if two vertices representing the column and the row are adjacent in G. Cut-rank functions play an important role in the study of vertex-minors because they are invariant under taking local complementations. As elementary row operations do not change the rank and $1 + 1 = 0$ in the binary field, it is easy to observe the following theorem.

Theorem 4.1 (Bouchet [Bou89, Corollary 2]; see Oum [Oum05]). *If G and H are locally equivalent, then they have an identical cut-rank function.*

Since deleting a row or a column does not increase the rank of a matrix, we deduce the following corollary.

Corollary 4.2. *If H is a vertex-minor of a graph G, then $\rho_H(X \cap V(H)) \leqslant \rho_G(X)$ for all $X \subseteq V(G)$.*

Bouchet [Bou90] conjectured that the converse of Theorem 4.1 is true but Fon-Der-Flaass disproved the conjecture. A counterexample with 10 vertices is presented in [FDF96]. According to a computer search, the smallest example is a pair of two graphs on 9 vertices that are not locally equivalent but have an identical cut-rank function.

The converse of Theorem 4.1 is true for bipartite graphs in the stronger sense that we only need pivoting. This follows from the result of Seymour [Sey88].

Theorem 4.3. *Let G_1, G_2 be bipartite graphs on the same set V of vertices. If*

$$\rho_{G_1}(X) = \rho_{G_2}(X) \text{ for all subsets } X \subseteq V,$$

then G_1 is pivot-equivalent to G_2.

Proof. We may assume that both G_1 and G_2 are connected because the cut-rank function of a graph determines components. For $i = 1, 2$, let (S_i, T_i) be a bipartition of $V(G_i)$, let A_i be an $S_i \times V$ matrix over the binary field such that for $v \in S_i$ and $w \in V$,

$$A_i(v, w) = \begin{cases} 1 & \text{if } v = w, \\ 1 & \text{if } w \in T_i \text{ and } v \text{ and } w \text{ are adjacent in } G_i, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 1, 2$, let M_i be the binary matroid represented by A_i and let λ_i be the matroid connectivity function of M_i, that is $\lambda_i(X) = r_i(X) + r_i(V - X) - r_i(V)$ for the rank function r_i of M_i. This construction is to make G_i the fundamental graph of M_i with respect to S_i. Then, it is well known that the matroid connectivity function coincides with the cut-rank function of its fundamental graph, that is $\rho_{G_i}(X) = \lambda_i(X)$, see [Oum05].

Seymour [Sey88] showed that if two connected binary matroids M_1 and M_2 have the same matroid connectivity function, then $M_1 = M_2$ or $M_1 = M_2^*$. We may assume that $M_1 = M_2$ because we may swap S_2 and T_2 to replace M_2 with M_2^*. Any two fundamental graphs of a binary matroid are pivot-equivalent, see Oum [Oum05, Corollary 3.5] and therefore G_1 and G_2 are pivot-equivalent. □

5 Vertex-minors certifying high connectivity

By Corollary 4.2, if G has a vertex-minor H having a cut of large cut-rank, then all cuts of G inducing the same cut on H will have large cut-rank. The following theorem shows that the converse holds. This is an analog of Tutte's linking theorem [Tut65] on matroids.

Theorem 5.1 (Oum [Oum05, Theorem 6.1]). *Let G be a graph and X and Y be disjoint subsets of $V(G)$. The following are equivalent.*

(i) *$\rho_G(Z) \geqslant k$ for all Z with $X \subseteq Z \subseteq V(G) - Y$.*

(ii) *G has a vertex-minor H on $X \cup Y$ such that $\rho_H(X) \geqslant k$.*

(iii) *G has a pivot-minor H on $X \cup Y$ such that $\rho_H(X) \geqslant k$.*

Theorem 5.1 allows us to find a small vertex-minor certifying that any cuts separating a pair of disjoint sets X and Y have high connectivity. However, often it is convenient to keep the adjacency between X and Y and it may be enough to have a rough structure. The following theorem provides such a structure.

Theorem 5.2 (Geelen, Kwon, McCarty, and Wollan [GKMW23, Lemma 4.3])**.** *There is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $k > 0$, if G is a graph and X and Y are disjoint subsets of $V(G)$ such that*

$$\rho_G(Z) \geqslant f(k) \text{ for all } Z \text{ with } X \subseteq Z \subseteq V(G) - Y$$

and $\rho_{G[X \cup Y]}(X) < k$, then there exists a graph H locally equivalent to G such that $H[X \cup Y] = G[X \cup Y]$ and either

(i) *there is a set $L \subseteq V(G) - (X \cup Y)$ of size k such that $\rho_{G[X \cup L]}(L) = \rho_{G[Y \cup L]}(L) = k$, or*

(ii) *there are disjoint subsets $L_1, L_2 \subseteq V(G) - (X \cup Y)$ of size k such that $\rho_{G[X \cup L_1]}(L_1) = \rho_{G[L_1 \cup L_2]}(L_1) = \rho_{G[Y \cup L_2]}(L_2) = k$, all vertices in L_1 have the same set of neighbors in Y, and all vertices in L_2 have the same set of neighbors in X.*

The following theorem of Lee and Oum [LO23] implies that it is possible to find a small vertex-minor certifying that any cuts separating any of two pairs have large cut-rank.

Theorem 5.3 (Lee and Oum [LO23])**.** *Let k, ℓ be integers. Let G be a graph and Q, R, S, T be subsets of $V(G)$ such that $Q \cap R = S \cap T = \varnothing$,*

$$\rho_G(Z_1) \geqslant k \text{ for all } Z_1 \text{ with } Q \subseteq Z_1 \subseteq V(G) - R,$$

and

$$\rho_G(Z_2) \geqslant \ell \text{ for all } Z_2 \text{ with } S \subseteq Z_2 \subseteq V(G) - T.$$

If $|V(G) - (Q \cup R \cup S \cup T)| \geqslant (2\ell + 1)2^{2k}$, then there exists a vertex $v \in V(G) - (Q \cup R \cup S \cup T)$ satisfying at least two of the following.

(i) *$\rho_{G-v}(Z_1) \geqslant k$ for all Z_1 with $Q \subseteq Z_1 \subseteq V(G) - R - \{v\}$, and $\rho_{G-v}(Z_2) \geqslant \ell$ for all Z_2 with $S \subseteq Z_2 \subseteq V(G) - T - \{v\}$.*

(ii) *$\rho_{G*v-v}(Z_1) \geqslant k$ for all Z_1 with $Q \subseteq Z_1 \subseteq V(G) - R - \{v\}$, and $\rho_{G*v-v}(Z_2) \geqslant \ell$ for all Z_2 with $S \subseteq Z_2 \subseteq V(G) - T - \{v\}$.*

(iii) *$\rho_{G/v}(Z_1) \geqslant k$ for all Z_1 with $Q \subseteq Z_1 \subseteq V(G) - R - \{v\}$, and $\rho_{G/v}(Z_2) \geqslant \ell$ for all Z_2 with $S \subseteq Z_2 \subseteq V(G) - T - \{v\}$.*

6 Split decompositions and prime graphs

A *split* of a graph G is a partition (X, Y) of the vertex set of G such that $|X|, |Y| \geqslant 2$ and $\rho_G(X) \leqslant 1$. Note that $\rho_G(X) \leqslant 1$ if there are subsets $X' \subseteq X$ and $Y' \subseteq V(G) - X$ for which the set of adjacent pairs of a vertex $a \in X$ and a vertex $b \in V(G) - X$ is precisely $X' \times Y'$. We say a graph is *prime* if it has no splits. By Theorem 4.1, locally equivalent graphs have the same splits and therefore if a graph is locally equivalent to a prime graph then it is prime.

If a graph admits a split, it can be built by the 1-join of two graphs. The 1-*join* of two graphs G_1 and G_2 with marker vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ is defined to be a graph obtained from the disjoint union of $G - v_1$ and $G - v_2$ by adding an edge between every neighbor of v_1 in G_1 and every neighbor of v_2 in G_2. Whenever a graph G admits a split (X, Y), we choose $v_1 \in Y$ and $v_2 \in X$ so that v_1 have neighbors in X and v_2 have neighbors in Y if there is at least one edge between X and Y. Then G is the 1-join of $G[X \cup \{v_1\}]$ and $G[Y \cup \{v_2\}]$

with v_1, v_2. Thus if G is not prime, it can be decomposed into two smaller graphs with at least three vertices by the 1-join operation.

Now starting from a set $\{G\}$ consisting of a single graph, we recursively pick a graph H in the set having a split and replace it with two smaller graphs so that their 1-join is H as long as H is not a complete graph or a star, until no further replacement is possible. We can associate a tree by having one node for each graph in the resulting set and adding an edge between two nodes if the corresponding pairs have marker vertices that are used when making the 1-join at some point. This decomposition is called the *split decomposition* of a graph. Cunningham [Cun82] proved that a connected graph has a unique split decomposition and therefore this is sometimes called a *canonincal decomposition.* Note that complete graphs and stars have exponentially many splits and for the sake of having the unique split decomposition, the recursive process stops at those graphs. If we do not care much about the uniqueness, then we could decompose a connected graph into prime graphs and build any graph from prime graphs by repeatedly taking 1-join.

There are linear-time algorithms to find a split decomposition [Dah00, CdMR12].

Prime graphs are important building blocks of graphs, analogous to the fact that every connected graph can be decomposed into blocks. As planar graphs can be decided by investigating the planarity of blocks, there are properties of graphs that can be determined by the properties of prime graphs.

Circle graphs are one of the major examples of classes of graphs closed under taking vertex-minors. A *circle graph* is the intersection graph of chords in a circle. In other words, a circle graph is represented by a chord diagram where vertices are chords and two vertices are adjacent if and only if they are intersecting. Here is a theorem on the unique representation of circle graphs up to local equivalence.

Theorem 6.1 (Bouchet [Bou87c])**.** *Prime circle graphs on at least* 5 *vertices have a unique representation up to cyclic equivalence.*

Furthermore, if G_1 and G_2 are circle graphs, then so is their 1-join [Bou87c, (4.2)]. We can rewrite as follows because any prime induced subgraph is an induced subgraph of some graph in the canonical decomposition.

Theorem 6.2 (Bouchet [Bou87c])**.** *A graph is a circle graph if and only if all of its prime induced subgraphs are circle graphs.*

So in order to decide whether a graph is a circle graph, it is enough to check whether all graphs in its canonical decomposition are circle graphs.

A similar property holds for graphs of rank-width at most k. First let us define the rank-width, introduced by Oum and Seymour [OS06]. A *rank-decomposition* of a graph G is a pair (T, L) of a tree T of maximum degree at most 3 and a bijection from $V(G)$ to the set of leaves of T. For every edge e, $T - e$ gives a partition (A_e, B_e) of the leaves of T and the *width* of an edge e in T is defined as $\rho_G(L^{-1}(A_e))$. The *width* of (T, L) is defined as the maximum width of all edges of T. The *rank-width* of a graph G is the minimum width of all rank-decompositions of G. If $|V(G)| < 2$, then it admits no rank-decompositions and we define the rank-width to be 0.

As the rank-width of the 1-join of two graphs is equal to the maximum rank-width of G or H, Hliněný et al. observed the following.

Theorem 6.3 (Hliněný, Oum, Seese, and Gottlob [HOSG08, Theorem 4.3])**.** *The rank-width of a graph is the maximum rank-width of all prime induced subgraphs.*

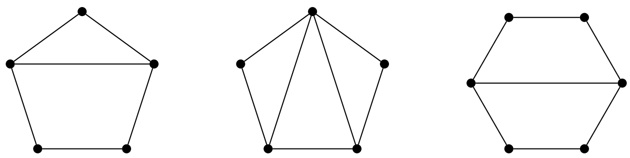

Figure 3: Unavoidable induced subgraphs in prime graphs that are not cycles.

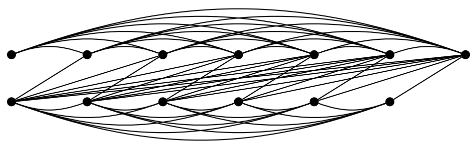

Figure 4: The graph H_{13}. No vertex v other than the unique vertex adjacent to all other vertices has the property that $H_{13} - v$ or $H_{13} * v - v$ is prime.

7 Chain theorems

Graph theory employs chain theorems as a means to reduce the size of a graph while maintaining its connectivity. A detailed overview of these theorems can be found in Sections 7.2 and 7.3 of Chapter 2 in [GGL95], which offer a comprehensive survey on chain theorems of graph minors. This survey describes variants of Tutte's chain theorem and examines their applications.

A significant chain theorem for vertex-minors was proven by Bouchet [Bou87d]. This theorem serves as a critical tool used by Bouchet for recognizing circle graphs in polynomial time.

Theorem 7.1 (Bouchet [Bou87d]). *Let G be a prime graph with at least 6 vertices. Then G has a prime vertex-minor H with $|V(H)| = |V(G)| - 1$.*

Theorem 7.2 (Bouchet [Bou87c, Lemma 3.1]). *Every prime graph on 4 or 5 vertices is locally equivalent to C_5.*

Gabor, Hsu, and Supowit [GSH89] provide an $O(|V(G)| \times |E(G)|)$-time algorithm recognizing a circle graph for an input graph G, using the next result.

Theorem 7.3 (Gabor, Hsu, and Supowit [GSH89]). *Every prime graph with at least 4 vertices contains an induced subgraph isomorphic to a cycle of length at least 5 or a graph in Figure 3.*

Allys proved a strengthening of Theorem 7.1 by using isotropic systems.

Theorem 7.4 (Allys [All94]). *Let G be a prime graph with at least 6 vertices. Then G has a vertex v such that $G - v$ or $G * v - v$ is prime.*

Kim and Oum extended the theorem of Allys (Theorem 7.4) as follows. Let H_n be a graph on $\{v_1, v_2, \ldots, v_n\}$ such that two vertices v_i and v_j with $i < j$ are adjacent if and only if i is even or j is odd. See Figure 4 for an illustration of H_{13}.

Theorem 7.5 (Kim and Oum [KO22]). *Let G be a prime graph with at least 6 vertices and x be a vertex of G. Then G has a vertex $v \neq x$ such that $G - v$ or $G * v - v$ is prime, unless G is isomorphic to $H_{|V(G)|}$, x is adjacent to all other vertices of G, and $|V(G)|$ is odd.*

Theorem 7.5 is proved by using the following theorem on the number of vertices that admit at least two ways to be removed while keeping the primeness.

Theorem 7.6 (Kim and Oum [KO22]). *Let G be a prime graph with at least 4 vertices.*

(i) *G has at least two vertices v such that at least two of $G - v$, $G * v - v$, and G/v are prime, unless G is locally equivalent to C_5.*

(ii) *G has at least three vertices v such that at least two of $G-v$, $G*v-v$, and G/v are prime, unless G is locally equivalent to a cycle or a graph consisting of at least three internally-disjoint paths between two fixed vertices, none of which has length 2.*

Here is an easy corollary of the previous theorem.

Corollary 7.7 (Kim and Oum [KO22]). *Let G be a prime graph with at least 6 vertices. For every pair of vertices $x, y \in V(G)$, G has a prime vertex-minor H containing both x and y such that $|V(H)| = |V(G)| - 1$.*

The following theorem concerns the number of ways to obtain a fixed graph H as a vertex-minor.

Theorem 7.8 (Geelen and Oum [GO09, Theorem 1.10]). *Let H be a vertex-minor of G. If $|V(G)| \geqslant 2^{|V(H)|}$, then there is a vertex v such that at least two of $G-v$, $G*v-v$, and G/v have H as a vertex-minor.*

For integers k and ℓ, a graph G is called *$k^{+\ell}$-rank-connected* if $\rho_G(X) < k$ implies $\min(|X|, |V(G)-X|) < k+\ell$ for all subsets X of $V(G)$. Note that a graph is prime if and only if it is 2^{+0}-rank-connected.

Theorem 7.9 (Oum [Oum23, Proposition 2.7]). *If G is a $k^{+\ell}$-rank-connected and v is a vertex, then $G-v$ or G/v is $k^{+(2\ell+k-1)}$-rank-connected.*

Theorem 7.10 (Oum [Oum23]). *If G is a prime 3^{+2}-rank-connected graph with at least 10 vertices, then there is a vertex v such that $G-v$ or G/v is prime 3^{+3}-rank-connected.*

For a graph G, a set $X \subseteq V(G)$ is *sequential* if its elements admit an ordering $v_1, \ldots, v_k$ such that such that $\rho_G(\{a_1, \ldots, a_i\}) \leqslant 2$ for each $1 \leqslant i \leqslant k$. A graph G is *sequentially 3-rank-connected* if it is prime and for every $X \subseteq V(G)$, $\rho_G(X) > 2$ unless X or $V(G) - X$ is sequential.

Theorem 7.11 (Lee and Oum [LO21]). *Every sequentially 3-rank-connected graph G has a sequentially 3-rank-connected vertex-minor with one fewer vertex, unless $|V(G)| \leqslant 12$.*

Geelen proved a stronger theorem in his Ph.D. thesis, analogous to the splitter theorem for 3-connected graphs by Seymour [Sey88]. The proof is based on purely graph-theoretic methods.

Theorem 7.12 (Geelen [Gee96, Lemma 5.9]). *Let G be a prime graph and H be a prime induced subgraph of G with at least 4 vertices. If $|V(H)| < |V(G)|$, then at least one of the following holds.*

(i) *G has a vertex v such that $G-v$ is prime and has a vertex-minor isomorphic to H.*

(ii) *G has a vertex w of degree 2 such that $G*w-w$ is prime and has a vertex-minor isomorphic to H.*

From Theorems 7.3 and 7.12, Geelen derived the following corollary, stronger than Theorem 7.4.

Corollary 7.13 (Geelen [Gee96, Corollary 5.10]). *Let G be a prime graph with at least 6 vertices. Then at least one of the following holds.*

(i) *G has a vertex v such that $G-v$ is prime.*

(ii) *G has a vertex w of degree 2 such that $G*w-w$ is prime.*

Here is an alternative, slightly weaker, formulation of Theorem 7.12.

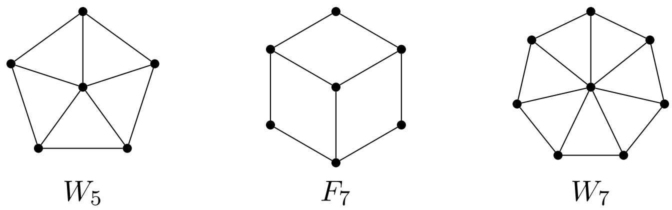

Figure 5: Vertex-minor obstructions for the class of circle graphs.

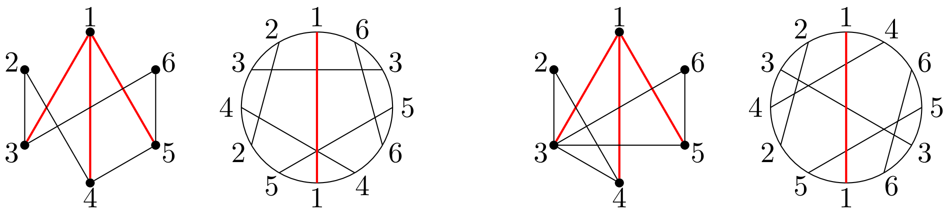

Figure 6: Two circle graphs G and $G * 1$ with their chord diagrams. The chord diagram for $G * 1$ is obtained by flipping one side of the circle divided by the chord representing 1.

Corollary 7.14 (Geelen [Gee96, Corollary 5.11]). *Let G and H be prime graphs such that $4 \leqslant |V(H)| < |V(G)|$. If H is a vertex-minor of G, then there is a sequence of prime graphs $G_1 := G, G_2, \ldots, G_m$ such that*

- *for each $i < m$, G_{i+1} is a vertex-minor of G_i with $|V(G_{i+1})| = |V(G_i)| - 1$ and*
- *G_m is isomorphic to H.*

8 Forbidden vertex-minor characterizations

8.1 Circle graphs

It is straightforward to see that if G is a circle graph, then so is $G*v$ for every vertex v of G; this can be achieved by taking the chord represented by v in the chord diagram representing G and reversing one side of the circle to obtain the chord diagram of $G*v$, see Figure 6. Bouchet [Bou94] proved the following analog of Kuratowski's theorem for circle graphs. Lee [Lee17] presented an alternative proof.

Theorem 8.1 (Bouchet [Bou94]). *A graph is a circle graph if and only if it has no vertex-minor isomorphic to W_5, F_7, or W_7 in Figure 5.*

Geelen [Gee96] described the structure of graphs with no vertex-minors isomorphic to W_5. Let Q_3 be the 3-dimensional cube graph. Note that for any vertex v in Q_3, $Q_3 - v$ is isomorphic to F_7.

Theorem 8.2 (Geelen [Gee96, Theorem 5.14]). *A graph G has no vertex-minor isomorphic to W_5 if and only if one of the following holds.*

(i) *G is a prime circle graph.*

(ii) *G is locally equivalent to a graph isomorphic to W_7, F_7, or Q_3.*

(iii) *G is the 1-join of two smaller graphs G_1 and G_2, both of which are graphs with no vertex-minors isomorphic to W_5.*

Theorem 7.8 immediately implies the following.

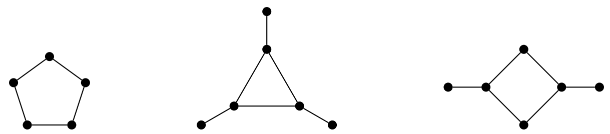

Figure 7: Vertex-minor obstructions for the class of graphs of linear rank-width 1.

Theorem 8.3 (Geelen and Oum [GO09, Theorem 1.3])**.** *Let $\mathcal{G}$ be a class of graphs closed under taking vertex-minors. If every vertex-minor-minimal graph not in $\mathcal{G}$ has at most k vertices, then every pivot-minor-minimal graph not in $\mathcal{G}$ has at most $2^k - 1$ vertices.*

As a corollary, we deduce that every pivot-minor-minimal graph which is not a circle graph has at most $2^8 - 1$ vertices. In [GO09], Geelen and Oum found all such graphs by computer search, which are total 15 graphs up to pivot-equivalence and isomorphism. This implies that there is a list of 15 graphs such that a graph is a circle graph if and only if it has no pivot-minor isomorphic to a graph in the list. Geelen and Oum [GO09] explain how this implies Kuratowski's theorem regarding planar graphs.

Bouchet [Bou99] proved a theorem on the complement of a circle graph. An alternative proof is given by Esperet and Stehlík [ES20].

Theorem 8.4 (Bouchet [Bou99])**.** *Let G be a bipartite graph. If the complement of G is a circle graph, then so is G.*

8.2 Distance-hereditary graphs

A graph is *distance-hereditary* [BM86] if for every connected induced subgraph H and two vertices x and y of H, the distance between x and y in H is equal to the distance between x and y in G. Two vertices x and y are *twins* of a graph G if they have the same set of neighbors in $V(G) - \{x, y\}$. Bandelt and Mulder [BM86] showed that all distance-hereditary graphs can be built from K_1 by creating twins and adding an isolated vertex or a pendant vertex to a distance-hereditary graph. Oum [Oum05] observed that a graph is distance-hereditary if and only if its rank-width is at most 1. Kwon and Oum [KO14b] showed the following obtained by combining this observation with results of Bouchet [Bou88c, Bou87c].

Theorem 8.5 (Bouchet [Bou88c, Bou87c], Oum [Oum05], and Kwon and Oum [KO14b, Theorem 4.1])**.** *Let G be a graph. The following are equivalent.*

(i) *G is distance-hereditary.*

(ii) *G has rank-width at most* 1*.*

(iii) *G has no vertex-minor isomorphic to C_5.*

(iv) *G is a vertex-minor of a tree.*

8.3 Linear rank-width at most 1

The linear rank-width of a graph G is the minimum k such that there is an ordering $v_1, v_2, \ldots, v_n$ of all vertices of G with the property that

$$\rho_G(\{v_1, \ldots, v_i\}) \leqslant k$$

for all $i = 1, 2, \ldots, n := |V(G)|$. It is easy to see that the rank-width of a graph is less than or equal to the linear rank-width of a graph. So, the class of graphs of linear rank-width at most 1 is a subclass of the class of distance-hereditary graphs. To describe the structure of graphs of linear rank-width at most 1, Ganian [Gan11] defined thread graphs and proved that a graph is a thread graph if and only if it has linear rank-width at most 1.

Here is a theorem characterizing the class of graphs of linear rank-width at most 1.

Theorem 8.6 (Adler, Farley, and Proskurowski [AFP14] and Kwon and Oum [KO14a, Theorem 4.3])**.** *Let G be a graph. The following are equivalent.*

(i) *G has linear rank-width at most* 1.

(ii) *G has no vertex-minor isomorphic to any graphs in Figure 7.*

(iii) *G is a vertex-minor of a path.*

Here is an interesting theorem on the linear rank-width of a tree.

Theorem 8.7 (Adler and Kanté [AK15])**.** *For every forest T, the linear rank-width of T is equal to the path-width of T.*

8.4 Rank-width and linear rank-width

Theorem 8.8 (Oum [Oum05])**.** *For each k, there exists a finite list of graphs each having at most* $(6^{k+1} - 1)/5$ *vertices such that a graph has rank-width at most k if and only if it has no pivot-minor isomorphic to a graph in the list.*

Theorem 8.9 (Kanté, Kim, Kwon, and Oum [KKKO23])**.** *For each k, there exists a finite list of graphs each having at most* $2^{2^{O(k^2)}}$ *vertices such that a graph has linear rank-width at most k if and only if it has no pivot-minor isomorphic to a graph in the list.*

Note that if a class $\mathcal{C}$ of graphs is closed under taking vertex-minors and has a list L of graphs such that

$\mathcal{C}$ is the class of all graphs having no pivot-minor isomorphic to any graph in L,

then the same list can be used to state that

$\mathcal{C}$ is the class of all graphs having no vertex-minor isomorphic to any graph in L.

Thus, we can replace pivot-minors with vertex-minors in both Theorem 8.8 and Theorem 8.9. The converse of the above observation is not true in general, as witnessed by circle graphs. However, a weak converse is true, as we discussed in Theorem 8.3.

8.5 Well-quasi-ordering

So far we have witnessed several instances of graph classes closed under taking vertex-minors that admit characterizations in terms of finitely many forbidden vertex-minors or pivot-minors. The celebrated graph minors theorem of Robertson and Seymour [RS04] states that every proper minor-closed class of graphs is characterized by a finite set of forbidden minors. This property can be seen easily equivalent to the following statement; Every infinite sequence G_1, G_2, ... of graphs has a pair $i < j$ such that G_i is isomorphic to a minor of G_j.

We can extend this to quasi-orders. A binary relation $\preceq$ on X is a *quasi-order* if $x \preceq x$ for all $x \in X$ and $x \preceq y$ and $y \preceq z$ implies $x \preceq z$. A quasi-order $\preceq$ on X is a *well-quasi-ordering* if every infinite sequence of $x_1, x_2, x_3, \ldots \in X$ admits a pair $i < j$ such that $x_i \preceq x_j$. If so, we call the set X *well-quasi-ordered* by the relation $\preceq$. The graph minors theorem can be equivalently stated that graphs are well-quasi-ordered by the minor relation.

Motivated by the graph minors theorem, it is very natural to propose the following conjectures.

Conjecture 8.10. *Graphs are well-quasi-ordered by the vertex-minor relation.*

Conjecture 8.11 (Oum [Oum12, Oum17])**.** *Graphs are well-quasi-ordered by the pivot-minor relation.*

Bouchet [Bou94] conjectured that circle graphs are well-quasi-ordered by the vertex-minor relation. This is implied by the well-quasi-ordering of 4-regular graphs by the immersion relation, proved by Robertson and Seymour [RS10]. The relation between the immersions on 4-regular graphs with their eulerian circuits and vertex-minors of circle graphs was observed by Kotzig [Kot68] and is described in the survey of Bouchet [Bou90].

Theorem 8.12 (Implied by Robertson and Seymour [RS10]; see McCarty [McC21])**.** *Circle graphs are well-quasi-ordered by the vertex-minor relation.*

Pivot-minors of bipartite graphs are associated with minors of binary matroids, see Oum [Oum05]. Geelen, Gerards, and Whittle [GGW14] proved that for every fixed finite field $\mathbb{F}$, $\mathbb{F}$-representable matroids are well-quasi-ordered by the minor relation. This in particular means that binary matroids are well-quasi-ordered by the minor relation and so we deduce the following. As far as the authors know, its proof is still being written.

Theorem 8.13 (Implied by Geelen, Gerards, and Whittle [GGW14])**.** *Bipartite graphs are well-quasi-ordered by the pivot-minor relation.*

Oum showed that graphs of bounded rank-width are well-quasi-ordered by the pivot-minor relation.

Theorem 8.14 (Oum [Oum08a])**.** *Graphs of bounded rank-width are well-quasi-ordered by the pivot-minor relation.*

Oum [Oum09] studied pivot-minors of line graphs and found a formulation so that pivot-minors of line graphs correspond to graft minors where a graft is a pair (G, T) of a graph G and a vertex set T. In the graft minors, contracting an edge e will make the new vertex belong to T if and only if exactly one end of e belongs to T. This can be seen as minors of group-labelled graphs where vertices have labels from the binary field. Geelen, Gerards, and Whittle announced that while proving their theorem for matroids [GGW14], they proved that group-labelled graphs with labels from the finite field are well-quasi-ordered by the minor relation and this implies that grafts are well-quasi-ordered by the graft minor relations. So this would imply that

pivot-minors of line graphs are well-quasi-ordered by the pivot-minor relation,

based on [Oum09].

For two graphs H_1 and H_2, let $\mathcal{C}$ be the class of graphs having no vertex-minor isomorphic to H_1 or having no vertex-minor isomorphic to H_2. Clearly, $\mathcal{C}$ is closed under taking vertex-minors. If Conjecture 8.10 holds, then there will be a list L of finitely many graphs such that a graph belongs to $\mathcal{C}$ if and only if it has no vertex-minor isomorphic to a graph in L. Each graph in L is required to have both a vertex-minor isomorphic to H_1 and a vertex-minor isomorphic to H_2. So the conjecture 8.10 implies that there are only finitely many vertex-minor-minimal graphs containing both a vertex-minor isomorphic to H_1 and a vertex-minor isomorphic to H_2. This is verified by Geelen and Oum [GO09] with an explicit bound by using Theorem 7.8.

Theorem 8.15 (Geelen and Oum [GO09, Theorem 1.11])**.** *For graphs H_1 and H_2, if G is a vertex-minor-minimal graphs containing both a vertex-minor isomorphic to H_1 and a vertex-minor isomorphic to H_2, then*

$$|V(G)| \leqslant 2^{|V(H_1)|} + 2^{|V(H_2)|} - 2.$$

It is open whether the analog holds for pivot-minors, which would be a consequence of Conjecture 8.11.

Conjecture 8.16 (Lee and Oum [LO23])**.** *For graphs H_1 and H_2, up to isomorphisms, there are only finitely many pivot-minor-minimal graphs containing both a pivot-minor isomorphic to H_1 and a pivot-minor isomorphic to H_2.*

9 Approximate characterizations in terms of vertex-minors

9.1 Large graphs

Here are easy observations derived from Ramsey's theorem, observed in Kwon and Oum [KO14b, Theorem 3.1]. Here, nK_2 means the disjoint union of n copies of K_2.

Theorem 9.1 (Kwon and Oum [KO14b, Theorem 3.1])**.** *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors.*

(i) *Graphs in $\mathcal{C}$ have bounded number of vertices if and only if $\{\overline{K_n} : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

(ii) *Graphs in $\mathcal{C}$ have bounded number of edges if and only if $\{K_n : n \geqslant 1\} \nsubseteq \mathcal{C}$ and $\{nK_2 : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

(iii) *Connected graphs in $\mathcal{C}$ have bounded number of vertices if and only if $\{K_n : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

Although we stated the above theorem as an exact theorem on classes of graphs closed under taking vertex-minors, we could interpret it as a rough structure theorem for graphs excluding certain vertex-minors. For instance, Theorem 9.1(i) can be restated as follows:

> For every integer n, there exists N such that graphs without a vertex-minor isomorphic to $\overline{K_n}$ have at most N vertices.

In fact, this is how Kwon and Oum [KO14b] stated their theorem. All theorems in this section can be restated in such a form.

For a positive integer n, $K_n \boxminus K_n$ is the graph obtained from the disjoint union of two complete graphs on n vertices by adding a perfect matching joining those two complete graphs.

Theorem 9.2 (Kwon and Oum [KO14b, Theorem 7.1])**.** *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Prime graphs in $\mathcal{C}$ have bounded number of vertices if and only if $\{C_n : n \geqslant 3\} \nsubseteq \mathcal{C}$ and $\{K_n \boxminus K_n : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

Conjecture 9.3 (McCarty [McC21, Conjecture 3.5.4])**.** *For every positive integer k, there exists an integer N such that if a graph G has a vertex set A with the property that more than $c|A|^2$ vertices in $V(G) - A$ have pairwise distinct neighborhoods in A, then every k-vertex graph is isomorphic to a vertex-minor of G.*

9.2 Large rank-depth or rank-brittleness

A *decomposition* of a graph G is a pair (T, σ) of a tree T having at least one internal node and a bijection σ from $V(G)$ to the set of leaves of T. The *radius* of a decomposition (T, σ) is the radius of the tree T. Each internal node of T induces a partition $\mathcal{P}_v$ of $V(G)$ by taking vertices of G mapped to the same component of $T - v$ by σ as one part. The ρ_G-width of a partition $\mathcal{P}_v$ is defined as $\max\{\rho_G(\bigcup_{X \in \mathcal{Q}} X) : \varnothing \neq \mathcal{Q} \subseteq \mathcal{P}_v\}$. The *width* of a decomposition (T, σ) is the maximum width of $\mathcal{P}_v$ among all internal nodes v of T. The *depth-d rank-brittleness* of a graph G [KO21] is the minimum width of a decomposition of G having radius at most d. (If $|V(G)| < 2$, then we define the depth-d rank-brittleness to be 0.) The *rank-depth* of a graph G [DKO20] is the minimum integer k such that G has a decomposition of radius at most

k and width at most k. DeVos, Kwon, and Oum [DKO20] showed that a class of graphs has bounded rank-depth if and only if it has bounded shrub-depth and so we omit the definition of shrub-depth [GHN$^+$19] in this survey. We denote by $T_{2,n}$ the 1-subdivision of $K_{1,n}$.

Theorem 9.4 (Kwon and Oum [KO21]). *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors.*

(i) *Graphs in $\mathcal{C}$ have bounded depth-1 rank-brittleness if and only if $\{nK_2 : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

(ii) *Graphs in $\mathcal{C}$ have bounded depth-2 rank-brittleness if and only if $\{P_n : n \geqslant 1\} \nsubseteq \mathcal{C}$ and $\{nT_{2,n} : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

A class of graphs is *rank k-scattered* if there is an integer N such that every graph G admits a partition of its vertex set into sets of size at most k such that $\rho_G(X) \leqslant N$ whenever X is the union of a collection of those sets.

Theorem 9.5 (Kwon and Oum [KO20]). *Let k be a positive integer. Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Then $\mathcal{C}$ is rank k-scattered if and only if*

$$\{nH : n \geqslant 1\} \nsubseteq \mathcal{C}$$

for every connected graph H on $k+1$ vertices.

Theorem 9.6 (Kwon, McCarty, Oum, and Wollan [KMOW21]). *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Graphs in $\mathcal{C}$ have bounded rank-depth if and only if $\{P_n : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

In prime graphs, having a long cycle as a vertex-minor is equivalent to having a long path as a vertex-minor, as shown by Kwon and Oum.

Lemma 9.7 (Kwon and Oum [KO14b, Theorem 6.1]). *Every prime graph with an induced path of length $\lceil 6.75n^7 \rceil$ has a vertex-minor isomorphic to C_n.*

By Theorem 9.6 and the previous lemma, we deduce the following.

Corollary 9.8. *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Prime graphs in $\mathcal{C}$ have bounded rank-depth if and only if $\{C_n : n \geqslant 1\} \nsubseteq \mathcal{C}$.*

The previous corollary can be restated equivalently as follows. For every integer $n \geqslant 3$, there exists an integer N such that if G has no vertex-minor isomorphic to C_n, then G can be built from graphs of rank-depth at most N by repeatedly applying 1-join.

Theorem 9.9 (Hliněný, Kwon, Obdržálek, and Ordyniak [HKOO16]). *A class $\mathcal{C}$ of graphs has bounded rank-depth if and only if there is a class $\mathcal{D}$ of graphs of bounded tree-depth such that every graph in $\mathcal{C}$ is a vertex-minor of a graph in $\mathcal{D}$.*

9.3 Large linear rank-width

Conjecture 9.10 (Kanté and Kwon [KK18]). *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Graphs in $\mathcal{C}$ have bounded linear rank-width if and only if there exists a tree not in $\mathcal{C}$.*

A natural analog of Conjecture 9.10 for pivot-minors is false, shown by Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [DDJ$^+$21]. A *caterpillar* is a tree having a path such that every vertex is within distance 1 from the path.

Theorem 9.11 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [DDJ$^+$21, Corollary 1.6]). *Let T be a tree. The class of distance-hereditary graphs having no pivot-minors isomorphic to T has bounded linear rank-width if and only if T is a caterpillar.*

Conjecture 9.12 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [DDJ+21, Conjecture 3])**.** *For every positive integer n, the class of graphs having no pivot-minors isomorphic to P_n has bounded linear rank-width.*

Theorem 9.13 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [DDJ+21, Theorem 1.5])**.** *For every integer $n > 2$, distance-hereditary graphs having no pivot-minors isomorphic to P_n have linear rank-width at most $2n - 5$.*

Theorem 9.14 (Kanté and Kwon [KK18])**.** *Let T be a tree. Every distance-hereditary graph of sufficiently large linear rank-width has a vertex-minor isomorphic to T.*

Theorem 9.15 (Kwon and Oum [KO14a])**.** *The following are equivalent for a class $\mathcal{C}$ of graphs.*

(i) *Graphs in $\mathcal{C}$ have bounded linear rank-width.*

(ii) *There is a class $\mathcal{D}$ of graphs of bounded path-width such that every graph in $\mathcal{C}$ is a pivot-minor of a graph in $\mathcal{D}$.*

(iii) *There is a class $\mathcal{D}$ of graphs of bounded path-width such that every graph in $\mathcal{C}$ is a vertex-minor of a graph in $\mathcal{D}$.*

Note that Kwon and Oum [KO14a] shows the implication from (i) to (ii). The implication from (iii) to (i) follows from the following theorem of Adler and Kanté [AK15].

Theorem 9.16 (Adler and Kanté [AK15, Lemma 5])**.** *The linear rank-width of a graph is less than or equal to its path-width.*

9.4 Large rank-width

The following theorem, initially asked by Oum [Oum05, Section 8], proves that circle graphs are precisely the vertex-minor obstruction for having large rank-width.

Theorem 9.17 (Geelen, Kwon, McCarty, and Wollan [GKMW23])**.** *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Graphs in $\mathcal{C}$ have bounded rank-width if and only if there exists a circle graph not in $\mathcal{C}$.*

Oum [Oum09] conjectured that bipartite circle graphs are precisely the pivot-minor obstruction for having large rank-width.

Conjecture 9.18 (Oum [Oum09])**.** *Let $\mathcal{C}$ be a class of graphs closed under taking pivot-minors. Graphs in $\mathcal{C}$ have bounded rank-width if and only if there exists a bipartite circle graph not in $\mathcal{C}$.*

This conjecture has been verified for a small number of classes.

(i) True if $\mathcal{C}$ is a class of bipartite graphs. This is implied by the theorem on binary matroids due to Geelen, Gerards, and Whittle [GGW07] and is explained in [Oum05, Corollary 3.9].

(ii) True if $\mathcal{C}$ is a class of pivot-minors of line graphs. This was shown by Oum [Oum09].

(iii) True if $\mathcal{C}$ is a class of circle graphs, if we assume the result in the Ph.D. thesis of Johnson [Joh02, Theorem 2.5], as explained in Oum [Oum09].

We remark that Conjecture 9.18 implies both the grid minor theorem of Robertson and Seymour [RS86] and its analog for binary matroids by Geelen, Gerards, and Whittle [GGW07]. This is explained in [Oum08a, GGW07].

Theorem 9.19 (Kwon and Oum [KO14a]). *The following are equivalent for a class $\mathcal{C}$ of graphs.*

(i) *Graphs in $\mathcal{C}$ have bounded rank-width.*

(ii) *There is a class $\mathcal{D}$ of graphs of bounded tree-width such that every graph in $\mathcal{C}$ is a pivot-minor of a graph in $\mathcal{D}$.*

(iii) *There is a class $\mathcal{D}$ of graphs of bounded tree-width such that every graph in $\mathcal{C}$ is a vertex-minor of a graph in $\mathcal{D}$.*

Note that the implication from (iii) to (i) is implied by the following theorem.

Theorem 9.20 (Oum [Oum08b]). *Let G be a graph with tree-width t and rank-width r. Then $r \leqslant t + 1$.*

A class $\mathcal{G}$ of graphs closed under taking induced subgraph is called *induced*-tw-*bounded* if there is a function f such that every graph $G \in \mathcal{G}$ of tree-width at least $f(t)$ contains K_t, $K_{t,t}$, a subdivision of the $t \times t$-wall, or the line graph of a subdivision of the $t \times t$-wall as an induced subgraph.

Theorem 9.21 (Hickingbotham, Illingworth, Mohar, and Wood [HIMW22, Theorem 23]). *Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. Then $\mathcal{C}$ has bounded rank-width if and only if it is induced*-tw-*bounded.*

10 χ-boundedness

For a graph G, we write $\chi(G)$ to denote its chromatic number and $\omega(G)$ to denote the maximum size of a clique. A class of graphs is χ*-bounded* if there is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $\chi(G) \leqslant f(\omega(G))$ for every induced subgraph G of a graph in the class. Such a function f is called the χ*-bounding* function.

Geelen (see [DK12]) conjectured in 2009 at the DIMACS workshop held at Princeton University that every proper vertex-minor closed class of graphs is χ-bounded. This had been verified for circle graphs by Gyárfás [Gyá85, Gyá86], graphs of bounded rank-width and graphs without a vertex-minor isomorphic to W_5 by Dvořák and Král' [DK12], and graphs with no vertex-minor isomorphic to a fixed wheel graph by Choi, Kwon, Oum, and Wollan [CKOW19].

James Davies resolved the conjecture as follows.

Theorem 10.1 (Davies [Dav22b]). *For every graph H, there is a function f such that for every graph G with no vertex-minor isomorphic to H, we have $\chi(G) \leqslant f(\omega(G))$.*

If the χ-bounding function can be taken as a polynomial, then the class is called *polynomially χ-bounded*. Kim, Kwon, Oum, and Sivaraman [CKOW19] proposed the following conjecture and proved it when H is a cycle.

Conjecture 10.2 (Kim, Kwon, Oum, and Sivaraman [CKOW19]). *Let H be a graph. The class of graphs with no vertex-minor isomorphic to H is polynomially χ-bounded.*

10.1 Circle graphs

Gyárfás [Gyá85, Gyá86] showed that circle graphs are χ-bounded but until recently it was an open problem to decide whether or not circle graphs are polynomially χ-bounded. This is now proven by Davies and McCarty.

Theorem 10.3 (Davies and McCarty [DM21]). *If G is a circle graph, then $\chi(G) \leqslant 7\omega(G)^2$.*

Davies further improves the χ-bounding function.

Theorem 10.4 (Davies [Dav22a]). *If G is a circle graph, then*

$$\chi(G) \leqslant 2\omega(G)\log_2(\omega(G)) + 2\omega(G)\log_2\log_2\omega(G) + 10\omega(G).$$

He also improves an existing lower bound for the χ-bounding function for circle graphs.

Theorem 10.5 (Davies [Dav22a]). *For every positive integer k, there is a circle graph G such that $\omega(G) \leqslant k$ and $\chi(G) \geqslant k\ln k - 2k$.*

For the class $\mathcal{G}$ of graphs closed under taking induced subgraphs, let $\mathcal{G}^{\&}$ be its closure under disjoint union and 1-join.

Theorem 10.6 (Dvořák and Král' [DK12], Kim [Kim11]). *Let $\mathcal{G}$ be a class of graphs closed under taking induced subgraphs. If $\mathcal{G}$ is χ-bounded, then so is $\mathcal{G}^{\&}$.*

Theorem 10.7 (Kim, Kwon, Oum, and Sivaraman [KKOS20]). *Let $\mathcal{G}$ be a class of graphs closed under taking induced subgraphs. If $\mathcal{G}$ is polynomially χ-bounded, then so is $\mathcal{G}^{\&}$.*

As a corollary of Theorems 8.2 and 10.3, we deduce the following.

Theorem 10.8. *The class of graphs having no vertex-minors isomorphic to W_5 is polynomially χ-bounded.*

10.2 Rank-width and polynomially χ-boundedness

Theorem 10.9 (Bonamy and Pilipczuk [BP20]). *Every class of graphs with bounded rank-width is polynomially χ-bounded.*

Now, Theorem 10.9 is now implied by results on twin-width, a relatively new width parameter introduced by Bonnet, Kim, Thomassé, and Watrigant [BKTW22]. They showed that if a class of graphs has bounded rank-width, then it has bounded twin-width. Bonnet, Geniet, Kim, Thomassé, and Watrigant [BGK$^+$20, BGK$^+$21] showed that every class of graphs of bounded twin-width is χ-bounded. Pilipczuk and Sokołowski [PS22] announced that the χ-bounding function can be taken as a quasi-polynomial. Recently, Bourneuf and Thomassé [BT23] announced that a class of graphs of bounded twin-width is polynomially χ-bounded. This would imply Theorem 10.9.

Here is a corollary of Theorems 10.9 and 9.17 which verifies Conjecture 10.2 when H is a circle graph.

Corollary 10.10. *Let H be a circle graph. The class of graphs with no vertex-minor isomorphic to H is polynomially χ-bounded.*

Corollary 10.10 implies the result of Kim, Kwon, Oum, and Sivaraman [CKOW19] for graphs having no vertex-minor isomorphic to C_n.

The degree of a χ-bounding function for the class of graphs of rank-width at most k cannot be independent of k.

Theorem 10.11 (Bonamy and Pilipczuk [BP20, Lemma 5.2]). *For every k, if f is a χ-bounding polynomial for the class of graphs of rank-width at most k, then the degree of f is at least $\Omega(\log k)$.*

10.3 Linear rank-width and linearly χ-boundedness

If the χ-bounded function can be taken as a linear, then the class is called *linearly χ-bounded.*

Theorem 10.12 (Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz [NOdMRS21]). *Every class of graphs with bounded linear rank-width is linearly χ-bounded.*

By Theorem 9.6, graphs without a vertex-minor isomorphic to P_n have bounded rank-depth. Since graphs of bounded rank-depth have bounded linear rank-width, we deduce the following result from Theorem 10.12. This improves the previous result by Kim, Kwon, Oum, and Sivaraman [CKOW19], who showed that a class of graphs with no vertex-minor isomorphic to P_n is polynomially χ-bounded.

Corollary 10.13. *Let n be a positive integer. Every class of graphs with no vertex-minors isomorphic to P_n is linearly χ-bounded.*

10.4 Erdős-Hajnal property

A class $\mathcal{G}$ of graphs closed under taking induced subgraphs has the *Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that every graph G in $\mathcal{G}$ has an independent set or a clique of size at least $|V(G)|^\varepsilon$, where a *clique* in a graph is a set of pairwise adjacent vertices, and an *independent set* in a graph is a set of pairwise non-adjacent vertices. Erdős and Hajnal [EH89] conjectured that for every graph H, the class of graphs with no induced subgraph isomorphic to H has the Erdős-Hajnal property. Chudnovsky and Oum [CO18] proved the affirmative result for a class of graphs forbidding any graph as a vertex-minor.

Theorem 10.14 (Chudnovsky and Oum [CO18])**.** *For every graph H, the class of graphs with no vertex-minor isomorphic to H has the Erdős-Hajnal property.*

They indeed proved the following stronger theorem. In a graph, two disjoint sets A and B of vertices is *complete* if every vertex in A is adjacent to all vertices in B, and *anticomplete* if there is no edge between A and B.

Theorem 10.15 (Chudnovsky and Oum [CO18])**.** *For every graph H, there exists $\varepsilon > 0$ such that every n-vertex graph with no vertex-minor isomorphic to H has a pair of disjoint vertex sets A and B such that $|A|, |B| \geqslant \varepsilon n$ and A is either complete or anticomplete to B.*

We remark that Conjecture 10.2 implies Theorem 10.14. Davies [Dav23] announced that one can replace the vertex-minor with the pivot-minor in Theorem 10.14.

11 Testing vertex-minors

Vertex-minors are related to graph states studied in the field of quantum information theory and quantum computing. Van den Nest, Dehaene, and De Moor [VdNDM04] discovered that vertex-minor operations on graphs represent local Clifford operations on graph states. They are interested in finding a particular vertex-minor from a given graph, as it has applications on graph states and so some of the papers in this section are actually published in a physics journal.

Bouchet [Bou91a] presented a polynomial-time algorithm deciding whether two graphs are locally equivalent, by observing this problem is equivalent to finding solutions of a system of equations over the binary field.

Theorem 11.1 (Bouchet [Bou91a])**.** *There is an $O(n^4)$-time algorithm to decide whether two n-vertex graphs on the same vertex set are locally equivalent.*

Dahlberg, Helsen, and Wehner [DHW20a] proved that counting locally equivalent graphs is #P-complete by reducing this problem on circle graphs to the problem of counting Eulerian circuits in a 4-regular graph.

Theorem 11.2 (Dahlberg, Helsen, and Wehner [DHW20a, Theorem V.1])**.** *Computing the number of graphs locally equivalent to an input graph is #P-complete, even if the input graph is a circle graph.*

In contrast to the local equivalence, it is NP-complete to decide whether one graph is a vertex-minor of another graph.

Theorem 11.3 (Dahlberg, Helsen, and Wehner [DHW20b, Theorem 3.1])**.** *Deciding whether H is a vertex-minor of G for two input graphs G and H with $V(H) \subseteq V(G)$ is NP-complete, even if H is a complete graph and G is a circle graph.*

Theorem 11.4 (Dahlberg, Helsen, and Wehner [DHW22])**.** *Deciding whether H is isomorphic to a vertex-minor of G for two input graphs G and H is NP-complete, even if both G and H are circle graphs.*

Theorem 11.5 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma [DDJ+18])**.** *Deciding whether H is isomorphic to a pivot-minor of G for two input graphs G and H is NP-complete, even if H is a star graph and G is bipartite.*

Courcelle and Oum [CO07] constructed a modulo-2 counting monadic second-order transduction that maps a graph into the set of all vertex-minors and this allows them to prove the following theorem.

Theorem 11.6 (Courcelle and Oum [CO07])**.** *Let H be a fixed graph and t be an integer. There is an $O(n^3)$-time algorithm to decide whether an input n-vertex graph of rank-width at most t contains a vertex-minor isomorphic to H.*

By Theorems 9.17 and 11.6, we deduce the following.

Theorem 11.7. *For each fixed circle graph H, there is an $O(n^3)$-time algorithm to decide whether an input n-vertex graph G contains a vertex-minor isomorphic to H.*

It is not known whether we can eliminate the condition that H is a circle graph in the previous theorem, in contrast that by [RS95], there is an $O(n^3)$-time algorithm deciding whether one fixed graph is isomorphic to a graph minor of an input n-vertex graph. By using the method of Courcelle and Oum [CO07], it is straightforward to deduce the following algorithm to check vertex-minors without isomorphisms.

Theorem 11.8 (Courcelle and Oum [CO07])**.** *Let H be a fixed graph and t be an integer. There is an $O(n^3)$-time algorithm to decide whether an input n-vertex graph G of rank-width at most t with $V(H) \subseteq V(G)$ contains H as a vertex-minor.*

Theorem 11.9 (Dahlberg, Helsen, and Wehner [DHW20b, Theorem 4.7])**.** *There is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each positive integer t, we have an $f(t)n^{O(1)}$-time algorithm to decide whether an input n-vertex circle graph contains a complete vertex-minor on an input set of at most t vertices.*

12 Interlace polynomials

This section will introduce interlace polynomials and global interlace polynomials. For more details, there is a well-written survey by Brijder and Jan Hoogeboom [BJH22].

Arratia, Bollobás, and Sorkin [ABS04] defined (single-variable) interlace polynomials of graphs recursively but later Aigner and van der Holst [AvdH04] presented the following equivalent definition. Let A be the adjacency matrix of a graph G over the binary field. For a subset S of $V(G)$, we write $A[S]$ to denote the $S \times S$ principal submatrix of A. The (single-variable) *interlace polynomial* of G is defined as

$$q(G, x) = \sum_{S \subseteq V(G)} (x-1)^{|S| - \operatorname{rank}(A[S])}.$$

Theorem 12.1 (Arratia, Bollobás, and Sorkin [ABS04, Theorem 12], Aigner and van der Holst [AvdH04, Corollary 1]). *The interlace polynomial is the unique map q satisfying the following two conditions.*

(i) *If a graph G has an edge uv, then*

$$q(G, x) = q(G - u, x) + q(G \wedge uv - u, x).$$

(ii) *If a graph G is a graph with n vertices and no edges, then* $q(G, x) = x^n$.

Aigner and van der Holst [AvdH04] observed the following.

Corollary 12.2. *If G and G′ are pivot-equivalent, then* $q(G, x) = q(G', x)$.

A special case of the following theorem was proved by Balister, Bollobás, Cutler, and Pebody [BBCP02] for $x = -1$.

Theorem 12.3 (Aigner and van der Holst [AvdH04, Corollary 1]). *Let A be the adjacency matrix of a graph G over the binary field. For a subset S of V, we write A[S] to denote the S × S submatrix of A. Then,*

$$q(G, x) = \sum_{S \subseteq V(G)} (x - 1)^{|S| - \mathrm{rank}(A[S])}.$$

Martin [Mar77] showed the relation between the Tutte polynomial of a planar graph H and the interlace polynomial of a fundamental graph G of H, where G is a circle graph by de Fraysseix [dF81]. This result is generalized as follows.

Theorem 12.4 (Aigner and van der Holst [AvdH04, Theorem 3], Bouchet [Bou91b]). *Let M be a binary matroid and G be its fundamental graph. Then* $q(G, x) = T_M(x, x)$*, where* $T_M(x, y)$ *is the Tutte polynomial of M.*

Theorem 12.5 (Aigner and van der Holst [AvdH04]). *For a graph G,* $q(G, 1)$ *equals the number of induced subgraphs of G with an odd number of perfect matchings. In particular, if G is a forest, then* $q(G, 1)$ *equals the number of matchings of G.*

Aigner and van der Holst [AvdH04] defined another polynomial $Q(G, x)$ satisfying a 3-term recursion. Let A be the adjacency matrix of a graph G over the binary field and for a set X of vertices, let I_X be the $V(G) \times V(G)$ diagonal matrix over the binary field such that the (v, v)-entry is 1 if and only if $v \in X$. We use the notation $(A + I_T)[S]$ to denote the $S \times S$ submatrix of $A + I_T$. We define the *global interlace polynomial* as

$$Q(G, x) = \sum_{T \subseteq S \subseteq V(G)} (x - 2)^{|S| - \mathrm{rank}((A + I_T)[S])}.$$

Theorem 12.6 (Aigner and van der Holst [AvdH04, Section 4], Bouchet [Bou91b]). *The global interlace polynomial Q satisfies the following.*

(i) *If a graph G has an edge uv, then*

$$Q(G, x) = Q(G - u, x) + Q(G * u - u, x) + Q(G \wedge uv - u, x).$$

(ii) *If a graph G is a graph with n vertices and no edges, then* $Q(G, x) = x^n$.

Corollary 12.7 (Aigner and van der Holst [AvdH04, Corollary 4]). *If G and G′ are locally equivalent, then* $Q(G, x) = Q(G', x)$.

A graph is *Eulerian* if every vertex has even degree.

Theorem 12.8 (Aigner and van der Holst [AvdH04, Theorem 5])**.** *For an n-vertex graph G, $Q(G,4)/2^n$ equals the number of induced Eulerian subgraphs.*

By Theorem 12.8 and Corollary 12.7, we deduce that locally equivalent graphs have the same number of induced Eulerian subgraphs.

We remark that Bouchet [Bou91b] defined the Tutte-Martin polynomial and the restricted Tutte-Martin polynomial of an isotropic system, which are essentially equivalent to the interlace polynomial and the global interlace polynomial of a graph, respectively, by [AvdH04, Section 6].

13 Isotropic systems

Isotropic systems, defined by Bouchet [Bou87b], are maximally isotropic subspace of even-dimensional vector space over the binary field equipped with a non-degenerate symmetric bilinear form. An isotropic system is particularly useful for graphs with the vertex-minor relation since it corresponds to a local equivalence class of graphs [Bou88a], and its connectivity function is equal to the cut-rank function of its fundamental graphs [Bou89].

Let K be a 2-dimensional vector space over the binary field, and we denote by α, β, γ the non-zero elements of K. Let $\langle \binom{a}{b}, \binom{c}{d} \rangle_K := ad + bc$ with elements a, b, c, d in the binary field. For a finite set V, a subspace L of a vector space K^V is *isotropic* if for every $\mathbf{x}, \mathbf{y} \in L$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ where $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i \in V} \langle \mathbf{x}_i, \mathbf{y}_i \rangle_K$. Note that $\langle \cdot, \cdot \rangle$ is a nondegenerate bilinear form, and so the dimension of every maximal isotropic space is at most $\frac{1}{2} \dim K^V = |V|$. An *isotropic system* is a pair $S = (V, L)$ of a finite set V and an isotropic subspace L of K^V with $\dim L = |V|$. A vector $\mathbf{a} \in K^n$ is an *Eulerian* vector of an isotropic system W if $\mathbf{a}[X] \notin L$ for every nonempty subset X of V, where $\mathbf{a}[X]$ is a vector in K^V such that $\mathbf{a}[X](v) = \mathbf{a}(v)$ if $v \in X$ and $\mathbf{a}[X](v) = 0$ otherwise.

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *supplementary* if $\mathbf{a}(v)$ and $\mathbf{b}(v)$ are nonzero and distinct for every $v \in V$.

Theorem 13.1 (Bouchet [Bou88a, (3.1)])**.** *Let G be a graph on the vertex set V and let $\mathbf{a}$ and $\mathbf{b}$ be supplementary vectors in K^V. Let L be the vector space spanned by $\mathbf{a}[N_G(v)] + \mathbf{b}[\{v\}]$ with $v \in V$. Then $S = (V, L)$ is an isotropic system.*

In the previous theorem, we call a triple $(G, \mathbf{a}, \mathbf{b})$ a *graphic presentation* of the resulting isotropic system, and call G a *fundamental graph.* Conversely, each isotropic system induces a graph as follows.

Theorem 13.2 (Bouchet [Bou88a, (4.6)])**.** *Every isotropic system admits a graphic presentation.*

Theorem 13.3 (Bouchet [Bou88a, (7.1) and (7.6))] and Oum [Oum04, Proposition 10.1])**.** *Let $(G_1, \mathbf{a}_1, \mathbf{b}_1)$ and $(G_2, \mathbf{a}_2, \mathbf{b}_2)$ be graphic presentations of the same isotropic system. Then they are locally equivalent. Furthermore, if $\{\mathbf{a}_1(v), \mathbf{b}_1(v)\} = \{\mathbf{a}_2(v), \mathbf{b}_2(v)\}$ for every $v \in V(G_1)$, then G_1 and G_2 are pivot-equivalent.*

An isotropic system might have distinct graphic presentations inducing the same fundamental graph. However, Bouchet [Bou91a, (2.5)] showed that regardless of a fundamental graph G of an isotropic system S, the number of graphic presentations $(H, \mathbf{a}, \mathbf{b})$ of S such that $H = G$ is the same.

Theorem 13.4 (Bouchet [Bou88a, (7.6) and (8.3)])**.** *Let $S = (V, L)$ be an isotropic system and $(G, \mathbf{a}, \mathbf{b})$ be its graphic presentation. For a vertex u and an edge vw of G, both $(G * u, \mathbf{a} + \mathbf{b}[\{u\}], \mathbf{b} + \mathbf{a}[N_G(u)])$ and $(G \wedge vw, \mathbf{a}[V - \{v, w\}] + [\{v, w\}], \mathbf{b}[V - \{v, w\}] + \mathbf{a}[\{v, w\}])$ are graphic presentations of S.*

Minors of an isotropic system are defined well by the next theorem. For $L \subseteq K^V$, $x \in K - \{0\}$, and $v \in V$, let $L|_x^v := \{p(\mathbf{x}) : \langle \mathbf{x}(v), x \rangle_K = 0\}$ where $p : K^V \to K^{V-\{v\}}$ be the canonical projection.

Theorem 13.5 (Bouchet [Bou87b, (8.1)] and [Bou88a, (9.1)])**.** *Let $S = (V, L)$ be an isotropic system with the graphic presentation $(G, \mathbf{a}, \mathbf{b})$. For each $x \in K$ and $v \in V$, $S|_x^v := (V - \{v\}, L|_x^v)$ is an isotropic system and one of the following is its graphic presentation.*

(i) $(G - v, p(\mathbf{a}), p(\mathbf{b}))$ *if $x = \mathbf{a}(v)$ or v is an isolated vertex in G,*

(ii) $(G \wedge vw - v, p(\mathbf{a} + \mathbf{b}[\{w\}]), p(\mathbf{b_a}[\{w\}]))$ *if $x = \mathbf{b}(v)$ and w is a neighbor of v in G, and*

(iii) $(G * v - v, p(\mathbf{a}), p(\mathbf{b} + \mathbf{a}[N_G(v)]))$ *otherwise.*

Theorem 2.1 is an immediate corollary of Theorem 13.5.

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