석 사 학 위 논 문
Master's Thesis

3-leaf power 꼭짓점 제거 문제의 다항식 커널

A polynomial kernel for 3-leaf power deletion

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한 국 과 학 기 술 원

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A polynomial kernel for 3-leaf power deletion

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The study was conducted in accordance with Code of Research Ethics[1].

[1] Declaration of Ethical Conduct in Research: I, as a graduate student of Korea Advanced Institute of Science and Technology, hereby declare that I have not committed any act that may damage the credibility of my research. This includes, but is not limited to, falsification, thesis written by someone else, distortion of research findings, and plagiarism. I confirm that my thesis contains honest conclusions based on my own careful research under the guidance of my advisor.

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초 록

주어진 트리 T의 ℓ-leaf power란, T의 모든 리프들을 꼭짓점으로 가지고, T에서 거리가 ℓ 이하인 두 리프를 변으로 연결한 그래프를 말한다. 주어진 그래프 G에서 최대 k개의 꼭짓점을 제거해 어떤 트리의 3-leaf power가 되도록 하는 게 가능한지를 판별하는 문제를 3-Leaf Power Deletion이라고 한다. 그래프 G에서 그러한 최대 k개의 꼭짓점이 존재하면 (G, k)를 yes-instance라고 하자. 그리고 (G, k)가 yes-instance라는 조건과 (G', k')가 yes-instance라는 조건이 동치일 때, (G, k)와 (G', k')를 동치라고 하자. 우리는 주어진 (G, k)와 동치이며 최대 $O(k^{14} \log^{12} k)$개의 꼭짓점을 가지는 그래프 G'와 $k' \leqslant k$로 구성된 (G', k')을 다항식 시간 안에 생성하는 알고리즘을 제시했다.

핵 심 낱 말 ℓ-leaf power, 꼭짓점 제거 문제, fixed-parameter 알고리즘, 커널화

Abstract

A graph G is an *ℓ-leaf power of a tree T* if $V(G)$ is equal to the set of leaves of T, and distinct vertices v and w of G are adjacent if and only if the distance between v and w in T is at most ℓ. Given a graph G, 3-Leaf Power Deletion asks whether there is a set $S \subseteq V(G)$ of size at most k such that $G \backslash S$ is a 3-leaf power of some tree T. We provide a polynomial kernel for this problem. More specifically, we present a polynomial-time algorithm for an input instance (G, k) to output an equivalent instance (G', k') such that $k' \leqslant k$ and G' has at most $O(k^{14} \log^{12} k)$ vertices.

Keywords ℓ-leaf power, vertex deletion problem, fixed-parameter algorithm, kernelization

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Chapter 1. Introduction

Nishimura, Ragde, and Thilikos [37] introduced an ℓ-leaf power of a tree to understand the structure of phylogenetic trees in computational biology. A graph G is an *ℓ-leaf power of a tree T* if $V(G)$ is equal to the set of leaves of T, and distinct vertices v and w of G are adjacent if and only if the distance between v and w in T is at most ℓ. We say that G is an *ℓ-leaf power* if G is an ℓ-leaf power of some tree. Note that an ℓ-leaf power could have more than one component. For instance, an ℓ-leaf power of a path of length at least $\ell + 1$ has two distinct trivial components. We are interested in the following *vertex deletion* problem.

3-Leaf Power Deletion
Input : A graph G and a non-negative integer k
Parameter : k
Question : Is there a set $S \subseteq V(G)$ with $\lvert S\rvert \leqslant k$ such that $G \backslash S$ is a 3-leaf power?

The following theorem is our main result.

Theorem 1.1. *For a graph G and a non-negative integer k, we can find a graph G' having at most $k^{O(1)}$ vertices and a non-negative integer $k' \leqslant k$ satisfying the following in time bounded above by a polynomial on $\lvert V(G)\rvert$:*

> *G admits a vertex set S of size at most k such that $G \backslash S$ is a 3-leaf power if and only if G' admits a vertex set S' of size at most k' such that $G' \backslash S'$ is a 3-leaf power.*

Vertex deletion problems include some of the best studied NP-hard problems in theoretical computer science, including Vertex Cover and Feedback Vertex Set. In general, the problem asks whether it is possible to delete at most k vertices from an input graph so that the resulting graph belongs to a specified graph class. Lewis and Yannakakis [34] showed that every vertex deletion problem to a non-trivial hereditary (i.e., closed under taking induced subgraphs) graph class is NP-hard. Since the class of 3-leaf powers is non-trivial and hereditary, it follows that 3-Leaf Power Deletion is NP-hard.

Chordal graphs are graphs with no induced cycles of length at least 4. The class of chordal graphs contains various interesting subclasses, including distance-hereditary graphs and block graphs; a graph G is *distance-hereditary* if for every connected induced subgraph H and vertices v and w of H, the distance between v and w in H is equal to that in G, and G is a *block graph* if every 2-connected induced subgraph of G is a complete graph. It is known that ℓ-leaf powers are chordal, for every positive integer ℓ [8, 26].

Vertex deletion problems have been investigated on various subclasses of chordal graphs through the parameterized complexity paradigm [17, 20], which measures the performance of algorithms not only with respect to the input size but also with respect to an additional numerical parameter. The notion of vertex deletion allows a highly natural choice of the parameter, specifically the size of the deletion set k. Chordal Deletion is a problem of deciding whether a graph has a vertex set of size at most k whose deletion makes it chordal. Marx [36] showed that Chordal Deletion is fixed-parameter tractable by presenting an algorithm with running time $f(k) \cdot n^{O(1)}$ where n is the number of vertices of an input graph and f is a computable function on k. Cao and Marx [11] improved this result by presenting an algorithm with running time $2^{O(k \log k)} \cdot n^{O(1)}$. Distance-Hereditary Deletion is a problem of deciding whether a graph has a vertex set of size at most k whose deletion makes it distance-hereditary.

Eiben, Ganian, and Kwon [22] presented a single-exponential fixed-parameter algorithm for Distance-Hereditary Deletion, that is an algorithm with running time $c^k \cdot n^{O(1)}$ for input size n and some constant c. Block Deletion is a problem of deciding whether a graph has a vertex set of size at most k whose deletion makes it a block graph. Kim and Kwon [31] showed that Block Deletion is fixed-parameter tractable by presenting an algorithm with running time $10^k \cdot n^{O(1)}$. Agrawal, Kolay, Lokshtanov, and Saurabh [1] improved this result by presenting an algorithm with running time $4^k \cdot n^{O(1)}$. Dom, Guo, Hüffner, and Niedermeier [18] showed that 3-Leaf Power Deletion is fixed-parameter tractable, based on a reduction to another problem, called Weighted Feedback Vertex Set.

Beyond the fixed-parameter tractability of parameterized problems, one of the most natural follow-up questions in parameterized complexity is whether the problem admits a polynomial kernel. A *kernel* is basically a polynomial-time preprocessing algorithm that transforms the given instance of the problem into an equivalent instance whose size is bounded above by some function $f(k)$ of the parameter. The function $f(k)$ is usually referred to as the *size* of the kernel. A *polynomial kernel* is then a kernel with size bounded above by some polynomial on k. The existence of polynomial kernels for vertex deletion problems has been widely investigated [6, 19, 28, 35]. Jansen and Pilipczuk [29] presented a kernel with $O(k^{161} \log^{58} k)$ vertices for Chordal Deletion, and Agrawal, Lokshtanov, Misra, Saurabh, and Zehavi [3] improved this result by presenting a kernel with $O(k^{12} \log^{10} k)$ vertices. Kim and Kwon [32] presented a kernel with $O(k^{30} \log^5 k)$ vertices for Distance-Hereditary Deletion. Kim and Kwon [31] presented a kernel with $O(k^6)$ vertices for Block Deletion. We restate our main result, Theorem 1.1, in these terminologies as follows.

Theorem 1.2. 3-Leaf Power Deletion *admits a kernel with* $O(k^{14} \log^{12} k)$ *vertices.*

The first step of our kernel is to find a *"good"* approximate solution, called a *good modulator* of an input graph G, that is a set $S \subseteq V(G)$ of size $O(k^2 \log^2 k)$ such that $G \backslash (S \setminus \{v\})$ is a 3-leaf power for each vertex v in S. Afterward, we design a series of reduction rules that allows us to bound the number and the size of components of $G \backslash S$. We remark that Bessy, Paul, and Perez [7] presented a kernel with $O(k^3)$ vertices for 3-leaf power edge modification problems including editing, completion, and edge-deletion.

We organize this paper as follows. In Chapter 2, we summarize some terminologies in graph theory and parameterized complexity, and introduce 3-leaf powers, distance-hereditary graphs, and a relation between them. In Chapter 3, we present two ways to give a fixed-parameter algorithm to 3-Leaf Power Deletion, which are different from the way of Dom, Guo, Hüffner, and Niedermeier [18]. In Chapter 4, we introduce a good modulator of a graph, and present an algorithm that either confirms that an input instance (G, k) is a no-instance, or constructs a small good modulator of G. In Chapters 5 and 6, we design a series of reduction rules that allows us to bound the number of vertices outside of a good modulator of a graph. In Chapter 7, we combine the above steps to present a kernel with $O(k^{14} \log^{12} k)$ vertices. In Chapter 8, we conclude this paper with some open problems.

This is a joint work with Eduard Eiben, O-joung Kwon, and Sang-il Oum [4].

Chapter 2. Preliminaries

In this paper, all graphs are finite and simple. For a vertex v and a set X of vertices of a graph G, let $N_G(v)$ be the set of neighbors of v in $V(G)$, $N_G(X)$ be the set of vertices not in X that are adjacent to some vertices in X, and $N_G[X] := N_G(X) \cup X$. We may omit the subscripts of these notations if it is clear from the context. For disjoint sets X and Y of vertices of G, we say that X *is complete to* Y if each vertex in X is adjacent to all vertices in Y, and X *is anti-complete to* Y if each vertex in X is non-adjacent to all vertices in Y. Let $G\backslash X$ be a graph obtained from G by removing all vertices in X and all edges incident with some vertices in X, and $G[X] := G\backslash(V(G)\backslash X)$. We may write $G\backslash v$ instead of $G\backslash\{v\}$ for each vertex v of G. For a set T of edges of G, let $G\backslash T$ be a graph obtained from G by removing all edges in T.

A graph G is *trivial* if $|V(G)| = 1$, and *non-trivial*, otherwise. A *clique* is a set of pairwise adjacent vertices. A graph is *complete* if every pair of two distinct vertices is adjacent, and *incomplete*, otherwise. An *independent set* is a set of pairwise non-adjacent vertices. Distinct vertices v and w of G are *twins* in G if $N_G(v)\backslash\{w\} = N_G(w)\backslash\{v\}$. Twins v and w in G are *true* if v and w are adjacent, and *false* if v and w are non-adjacent. A *twin-set* in G is a set of pairwise twins in G. A twin-set is *true* if it is a clique, and *false* if it is an independent set.

A vertex of a graph G is a *cut-vertex* if $G\backslash v$ has more components than G. A set X of vertices of G is a *vertex cut* if $G\backslash X$ has more components than G. A *clique cut-set* is a vertex cut which is a clique.

A vertex is *isolated* if it has no neighbors. A node of a tree is a *leaf* if it has exactly one neighbor, and is *branching* if it has at least three neighbors. For graphs $G_1, \ldots, G_m$, a graph G is $(G_1, \ldots, G_m)$*-free* if G has no induced subgraphs isomorphic to one of $G_1, \ldots, G_m$.

We say that a reduction rule is *safe* if each input instance is equivalent to the resulting instance obtained from the input instance by applying the rule.

2.1 Parameterized problems and kernels

For a fixed finite set Σ of alphabets, an *instance* is an element in $\Sigma^* \times \mathbb{N}$. For an instance (I, k), k is called a *parameter*. A *parameterized problem* is a set $L \subseteq \Sigma^* \times \mathbb{N}$. A parameterized problem Π is *fixed-parameter tractable* if there is an algorithm, called a *fixed-parameter algorithm* for Π, that correctly decides whether an input instance $(I, k) \in \Pi$ in time $O(f(k) \cdot n^c)$ for a computable function $f : \mathbb{N} \to \mathbb{N}$ and a constant c where n is the size of I. A fixed-parameter algorithm for a parameterized problem is *single-exponential* if it takes $O(\alpha^k \cdot n^c)$ time for some constants $\alpha > 1$ and c.

An instance (I, k) is a *yes-instance* for a parameterized problem Π if $(I, k) \in \Pi$, and a *no-instance*, otherwise. Instances (I, k) and (I', k') are *equivalent* with respect to Π if (I, k) is a yes-instance for Π if and only if (I', k') is a yes-instance for Π. A *kernel* for Π is a polynomial-time algorithm that given an instance (I, k), outputs an instance (I', k') equivalent to (I, k) with respect to Π such that $|I'| + k' \leqslant g(k)$ for some computable function $g : \mathbb{N} \to \mathbb{N}$. Such a function $g(k)$ is the *size* of the kernel. A *polynomial kernel* for Π is a kernel for Π with the size as some polynomial in k. We may omit the term "for Π" and "with respect to Π" of all these definitions if it is clear from the context. There is a relationship between the fixed-parameter tractability and the existence of a kernel for parameterized problems.

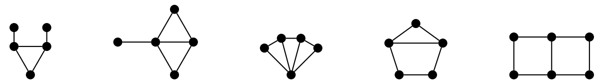

Figure 2.1: A bull, a dart, a gem, a house, and a domino.

Theorem 2.1 (See Downey and Fellows [20]). *A parameterized problem* Π *is fixed-parameter tractable if and only if* Π *admits a kernel.*

2.2 Characterizations of 3-leaf powers

Brandstädt and Le [9] presented a linear-time algorithm to recognize 3-leaf powers.

Theorem 2.2 (Brandstädt and Le [9, Theorem 15]). *Given a graph* G*, we can either confirm that* G *is not a* 3*-leaf power, or find a tree of which* G *is a* 3*-leaf power in linear time.*

Figure 2.1 shows three graphs called a *bull*, a *dart*, and a *gem*. A *hole* is an induced cycle of length at least 4. A graph is *chordal* if it has no holes. Dom, Guo, Hüffner, and Niedermeier [18] presented the following characterization of 3-leaf powers.

Theorem 2.3 (Dom, Guo, Hüffner, and Niedermeier [18, Theorem 1]). *A graph* G *is a* 3*-leaf power if and only if* G *is (bull, dart, gem)-free and chordal.*

We say that a graph H is an *obstruction* if H either is a hole, or is isomorphic to one of the bull, the dart, and the gem. An obstruction H is *small* if $|V(H)| \leqslant 5$. We see the following six observations about obstructions.

Observation 1 (O1). *No obstructions have true twins.*

Observation 2 (O2). *No obstructions have independent sets of size at least* 4.

Observation 3 (O3). *No obstructions have* K_4 *or* $K_{2,3}$ *as a subgraph.*

Observation 4 (O4). *No obstruction* H *has a cut-vertex* v *such that* $H\backslash v$ *has exactly two components* H_1 *and* H_2 *with* $|V(H_1)| = |V(H_2)|$.

Observation 5 (O5). *If an obstruction* H *has false twins* v *and* w*, then both* v *and* w *have degree* 2 *in* H.

Observation 6 (O6). *If a vertex* v *of an obstruction* H *has exactly one neighbor* w *in* $V(H)$*, then* w *has degree at least* 3 *in* H.

Observation 7 (O7). *A graph* H *is an obstruction having three distinct vertices of degree* 2 *in* H *if and only if* H *is a hole.*

Brandstädt and Le [9] showed that a graph G is a 3-leaf power if and only if G is obtained from some forest F by replacing each node u of F with a non-empty clique B_u of arbitrary size, and each edge vw of F with the edges whose one end is in B_v, and the other end is in B_w. We rephrase this characterization by using the following definition.

A *tree-clique decomposition* of a graph G is a pair $(F, \{B_u : u \in V(F)\})$ of a forest F and a family $\{B_u : u \in V(F)\}$ of non-empty subsets of $V(G)$ satisfying the following two conditions.

(1) The family $\{B_u : u \in V(F)\}$ is a partition of $V(G)$.

(2) Distinct vertices x and y of G are adjacent if and only if F has either a node u such that $\{x, y\} \subseteq B_u$, or an edge vw such that $x \in B_v$ and $y \in B_w$.

We call B_u a *bag of* u for each node u of F. We say that B is a *bag of* G if B is a bag of some node of F. We remark that each bag is a clique by (2).

Theorem 2.4 (Brandstädt and Le [9, Theorem 9]). *A graph G is a 3-leaf power if and only if G has a tree-clique decomposition. Moreover, if G is connected and incomplete, then G has a unique tree-clique decomposition.*

We remark that every connected incomplete 3-leaf power has at least three bags. Brandstädt and Le [9] showed that for a connected incomplete 3-leaf power G, distinct vertices v and w of G are in the same bag of G if and only if v and w are true twins in G. Thus, for such a graph G, B is a bag of G if and only if B is a maximal true twin-set in G.

2.3 Characterizations of distance-hereditary graphs

A graph G is *distance-hereditary* if for every connected induced subgraph H of G and vertices v and w of H, the distance between v and w in H is equal to the distance between v and w in G. Figure 2.1 shows two graphs called a *house* and a *domino*. Bandelt and Mulder [5] presented the following two characterizations of distance-hereditary graphs.

Theorem 2.5 (Bandelt and Mulder [5, Theorem 1]). *A graph G is distance-hereditary if and only if G is constructed from by a sequence of the following four operations on graphs.*

(1) Create a new vertex which is isolated in the resulting graph.

(2) Create a new vertex and make it adjacent to exactly one other vertex.

(3) Create a new vertex and make it a twin of some other vertex.

Theorem 2.6 (Bandelt and Mulder [5, Theorem 2]). *A graph G is distance-hereditary if and only if G is (house, domino, gem)-free, and has no holes of length at least 5.*

Since both the house and the domino have a hole of length 4, every 3-leaf power is distance-hereditary by Theorems 2.3 and 2.6.

The following lemma shows when graphs are not distance-hereditary.

Lemma 2.7. *Let P be an induced path of length at least 3 in a graph G. If G has a vertex v adjacent to both ends of P, then $G[V(P) \cup \{v\}]$ is not distance-hereditary.*

Proof. Let v_1 and v_2 be the ends of P, and $G' := G[V(P) \cup \{v\}]$. Since P has length at least 3, the distance between v_1 and v_2 in P is at least 3. Since v is adjacent to both v_1 and v_2, the distance between v_1 and v_2 in G' is exactly 2. Therefore, G' is not distance-hereditary. □

Chapter 3. Fixed-parameter algorithms for 3-Leaf Power Deletion

It is already known that 3-Leaf Power Deletion is fixed-parameter tractable due to Dom, Guo, Hüffner, and Niedermeier [18]. However, we present two alternative fixed-parameter algorithms solving 3-Leaf Power Deletion. In Section 3.1, we review width parameters on graphs. In Section 3.2, we present a fixed-parameter algorithm solving 3-Leaf Power Deletion based on algorithms of Hliněný and Oum [27], Jeong, Kim, and Oum [30], and Courcelle, Makowsky, and Rotics [14]. In Section 3.3, we present a single-exponential fixed-parameter algorithms solving 3-Leaf Power Deletion, based on a single-exponential fixed-parameter algorithm of Eiben, Ganian, and Kwon [22] solving Distance-Hereditary Deletion.

3.1 Width parameters

Oum and Seymour [40] introduced the rank-width and a rank-decomposition of a graph. Let M be an $m \times n$ matrix with entries a_{ij} for $i \in [m]$ and $j \in [n]$, and $M[I, J] := (a_{ij} : i \in I, j \in J)$ for sets $I \subseteq [m]$ and $J \subseteq [n]$. Let $\mathrm{rk}(M)$ be the rank of a matrix M, and $A(G)$ be the adjacency matrix of a graph G over the binary field. The *cut-rank function* cutrk_G of G is a function on $2^{V(G)}$ to $\mathbb{N} \cup \{0\}$ such that for each set X of vertices of G, $\mathrm{cutrk}_G(X) = \mathrm{rk}(A(G)[X, V(G)\backslash X])$. Note that cutrk is *symmetric*, that is, $\mathrm{cutrk}(X) = \mathrm{cutrk}(V(G)\backslash X)$, because $A(G)$ is symmetric. A tree T is *subcubic* if $|V(T)| \geqslant 2$ and the maximum degree of T is at most 3. For a graph G with at least two vertices, a *rank-decomposition* of G is a pair (T, ℓ) consisting of a subcubic tree T and a bijection ℓ between $V(G)$ and the set L of leaves of T. We say that G has no rank-decomposition if $|V(G)| \leqslant 1$.

For each edge e of T, $T\backslash e$ has exactly two components, thus, $T\backslash e$ induces a partition of L. Since ℓ is a bijection between $V(G)$ and L, the partition of L and ℓ induce a partition (X, Y) of $V(G)$. The *width of e in (T, ℓ)* is $\mathrm{cutrk}_G(X)$, and the *width of (T, ℓ)* is the maximum width of an edge in T. The *rank-width* of G, denoted by $\mathrm{rwd}(G)$, is the minimum width of a rank-decomposition of G.

Oum [38] showed that distance-hereditary graphs are precisely graphs of rank-width at most 1.

Theorem 3.1 (Oum [38, Proposition 7.3])**.** *A graph G is distance-hereditary if and only if G has rank-width at most* 1.

For the completeness of the paper, we include a proof of Theorem 3.1. To prove Theorem 3.1, we will use the following two lemmas.

Lemma 3.2. *If a graph G has twins v and w, and $G\backslash v$ has at least one edge, then G and $G\backslash v$ have the same rank-width.*

Proof. For every induced subgraph H of a graph G, $\mathrm{rwd}(H) \leqslant \mathrm{rwd}(G)$. Thus, it suffices to show that $\mathrm{rwd}(G) \leqslant \mathrm{rwd}(G\backslash v)$.

Let (T, ℓ) be a rank-decomposition of $G\backslash v$ of width $\mathrm{rwd}(G\backslash v)$. Let $x := \ell(w)$ and y be the neighbor of x in T, and T' be a tree obtained by subdividing xy with a new vertex z and adding a new pendant vertex x' adjacent to z. Let ℓ' be a bijection between $V(G)$ and the set of leaves of T' such that $\ell'(v) = x'$ and $\ell'(u) = \ell(u)$ for every vertex $u \neq v$. Thus, (T', ℓ') is a rank-decomposition of G. Note that if e is neither xz nor $x'z$, then the width of e in (T', ℓ') is equal to the width of e in (T, ℓ), and if e is one of xz and $x'z$, then the width of e in (T', ℓ') is at most 1. Therefore, the width of (T', ℓ') is at most $\mathrm{rwd}(G\backslash v)$. In other words, $\mathrm{rwd}(G) \leqslant \mathrm{rwd}(G\backslash v)$. $\square$

Lemma 3.3. *Let G is a graph of rank-width at most 1 with at least two vertices. Then G has distinct vertices v and w such that either v and w are twins, or exactly one of v and w has neighbors in $V(G)\backslash\{v, w\}$.*

Proof. We may assume that $|V(G)| \geqslant 3$, because otherwise the vertices of G are twins. Let (T, ℓ) be a rank-decomposition of G of width at most 1. Since T has at least three leaves, it has an internal node u adjacent to at least two leaves. Let $\ell(v)$ and $\ell(w)$ be distinct leaves adjacent u, and u' be a neighbor of u distinct to both $\ell(v)$ and $\ell(w)$. Then the width of uu' is equal to $\mathrm{cutrk}_G(\{v, w\})$ which is at most 1. If the width of uu' is equal to 0, then both v and w have no neighbors in $V(G)\backslash\{v, w\}$, and therefore v and w are twins. Otherwise, either v and w are twins, or exactly one of v and w has neighbors in $V(G)\backslash\{v, w\}$. $\square$

Proof of Theorem 3.1. Suppose that G is distance-hereditary. By Theorem 2.5 and Lemma 3.2, we may assume that G is a forest. It is easy to check that every forest has rank-width at most 1.

Conversely, suppose that G has rank-width at most 1. We may assume that G has at least two vertices. By Lemma 3.3, G has a vertex w such that G is obtained from $G\backslash w$ by applying one of four operations in Theorem 2.5. Thus, we may assume that $|V(G)| = 2$. Since every graph with exactly two vertices can be constructed from nothing by applying the operation (1) twice or the operations (1) and (2) in order, G is distance-hereditary. $\square$

Courcelle, Engelfriet, and Rozenberg [13] introduced a p-expression, and Courcelle and Olariu [15] introduced the clique-width of a graph. A *p-graph* is a labeled graph G where each vertex is labeled by an integer in $\{1, \ldots, p\}$. For a positive integer i, we denote by $i(v)$ the p-graph with exactly one vertex v labeled by i. Let us define the following three operations p-graphs.

(1) $\oplus$ denotes a disjoint union of two labeled graphs.

(2) For integers $i < j$ in $\{1, \ldots, p\}$, $\eta_{i,j}$ denotes a unary operation making every vertex labeled by i adjacent to each vertex labeled by j.

(3) For distinct integers i and j in $\{1, \ldots, p\}$, $\rho_{i\to j}$ denotes a unary operation relabeling every vertex labeled by i into j.

A *p-expression* is an algebraic operation using $i(v)$ for integers $i \in \{1, \ldots, p\}$ and these three operations in order. A *p-expression of a graph G* is a p-expression constructing a labeled graph whose underlying graph is isomorphic to G. For instance, here is a 3-expression of the dart.

$$\eta_{2,3}(2(v_5) \oplus \eta_{1,3}(3(v_4) \oplus \eta_{1,2}(1(v_1) \oplus 1(v_2) \oplus 2(v_3)))).$$

The *clique-width* of a graph G, denoted by $\mathrm{cwd}(G)$, is the minimum integer p such that there is p-expression of G. The clique-width of a graph is well defined, because for a graph G with $V(G) = \{v_1, \ldots, v_n\}$, there is an n-expression of G by applying (2) iteratively on $1(v_1) \oplus \cdots \oplus n(v_n)$.

Oum and Seymour [40] showed that the rank-width and the clique-width of a graph are tied in the sense of boundedness.

Theorem 3.4 (Oum and Seymour [40, Proposition 6.3]). *For a graph G, $\mathrm{rwd}(G) \leqslant \mathrm{cwd}(G) \leqslant 2^{\mathrm{rwd}(G)+1} - 1$. In addition, one can convert a k-expression of G into a rank-decomposition of G of width at most k in $O(|V(G)|)$ time, and a rank-decomposition of G of width k into a $(2^{k+1} - 1)$-expression of G in $O(|V(G)|^2)$ time.*

Golumbic and Rotics [24] showed that every distance-hereditary graph has clique-width at most 3, and Theorems 3.1 and 3.4 give us a simpler proof for this result.

Oum and Seymour presented a fixed-parameter algorithm which either confirms that an input graph G has rank-width larger than k, or finds a rank-decomposition of G of width at most $3k + 1$.

Theorem 3.5 (Oum and Seymour [40, Theorem 1.1]). *For each non-negative integer k, there is an algorithm which either confirms that an input graph G has rank-width at least $k + 1$, or finds a rank-decomposition of G with rank-width at most $3k+1$ in $O(f(k)\cdot|V(G)|^9 \log|V(G)|)$ time for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

Oum [39] improved this result by presenting a cubic-time algorithm.

Theorem 3.6 (Oum [39, Theorem 5.8]). *For a graph G and a fixed positive integer k, there is a $O(|V(G)|^3)$-time algorithm that either confirms that G has rank-width larger than k, or finds a rank-decomposition of G of width at most $3k - 1$.*

Hliněný and Oum [27], and Jeong, Kim, and Oum [30] improved this result by presenting a cubic-time algorithm that confirming a stronger condition.

Theorem 3.7 (Hliněný and Oum [27, Theorem 7.3], and Jeong, Kim, and Oum [30, Theorem 2.3]). *For a graph G and a fixed positive integer k, there is a $O(|V(G)|^3)$-time algorithm that either confirms that G has rank-width larger than k, or finds a rank-decomposition of G of width at most k.*

3.2 A fixed-parameter algorithm based on the meta theorem

In this section, we show that 3-Leaf Power Deletion is fixed-parameter tractable by applying Theorem 3.7 and a theorem in Courcelle, Makowsky, and Rotics [14], called the *meta theorem*.

For a finite set D and a positive integer m, a *relation symbol on D with arity m* is a mapping $R : D^m \to \{\textsc{True}, \textsc{False}\}$. A *set predicate on D with arity m* is a mapping $R : (2^D)^m \to \{\textsc{True}, \textsc{False}\}$. We denote by $\rho(R)$ the arity of a relation symbol or a set predicate R. We say that a relation symbol or a set predicate R on D is *unary* if $\rho(R) = 1$, and *binary* if $\rho(R) = 2$. A *relational structure S* is a pair $(D, \mathcal{R})$ of a finite set D and a collection $\mathcal{R}$ of relation symbols and set predicates. For a relational structure $S := (D, \mathcal{R})$, *first-order variables* represent elements in D, and *set variables* represent subsets of D. For instance, a graph G with the vertex set V is a relational structure $(V, \{\textsc{Edg}\})$ where $\textsc{Edg}$ is a binary relation symbol on V such that $\textsc{Edg}(v_1, v_2) = \textsc{True}$ if and only if v_1 is adjacent to v_2.

Let $\mathcal{R}$ be a set of relation symbols and set predicates on D. The *atomic formulas* are $x = y$, $R(x_1, \ldots, x_{\rho(R)})$ for variables $x, y, x_1, \ldots, x_{\rho(R)}$ and $R \in \mathcal{R}$. As usual, the *propositional connectives* are $\neg$, $\wedge$, and $\vee$, and *quantifications* are $\exists x$ and $\forall x$ for a variable x. The *monadic second-order formulas* over $\mathcal{R}$ are formulas consisting of atomic formulas, propositional connectives, and quantifications such that every set predicate in $\mathcal{R}$ is unary. A monadic second-order formula over $\mathcal{R}$ is a *first-order formula* if $\mathcal{R}$ consists of only relation symbols.

For the sake of convenience, we use the following terminologies to express properties in monadic second-order formula. Let ϕ and ψ be monadic second-order formulas.

(1) $x_1 \neq x_2 := \neg(x_1 = x_2)$, $x \notin X := \neg(x \in X)$.

(2) $\nexists X[\phi] := \neg(\exists X[\phi])$.

(3) $\phi \Rightarrow \psi := (\neg\phi) \vee \psi$.

(4) $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$.

(5) $\forall x \in X[\phi] := \forall x[(x \in X) \Rightarrow \phi]$.

(6) $Y \subseteq X := \forall x[(x \in Y) \Rightarrow (x \in X)]$.

We are going to formulate 3-Leaf Power Deletion in monadic second-order formula. To do this, we formulate the following three properties in monadic second-order formula.

Lemma 3.8. *For vertices x_1 and x_2 of a graph G and a set X of vertices of G, one can formulate the following three properties in monadic second-order formulas.*

(1) There is a path P from x_1 to x_2 such that $V(P) \subseteq X$, denoted by $\textsc{Path}(x_1, x_2, X)$.

(2) A graph $G[X]$ is chordal, denoted by $\textsc{Chor}(X)$.

(3) A graph $G[X]$ has a graph H with the vertex set $\{u_1, \ldots, u_m\}$ as an induced subgraph, denoted by $\textsc{Ind}_H(X)$.

Proof. We prove this lemma by presenting exact monadic second-order formulas.

$$\begin{aligned}
\textsc{Path}(x_1, x_2, X) :=& (x_1 \in X) \wedge (x_2 \in X) \wedge ((x_1 = x_2) \vee (\forall Y \subseteq X[((\forall y \in Y \forall z \in X[\textsc{Edg}(y, z) \Rightarrow (z \in Y)]) \\
&\wedge (x_1 \in Y)) \Rightarrow (x_2 \in Y)]])). \\
\textsc{Chor}(X) :=& \nexists Y \subseteq X[(\forall x_1 \in Y \forall x_2 \in Y[\textsc{Path}(x_1, x_2, Y)]) \wedge (\forall x \in Y[\exists x_1 \in Y \exists x_2 \in Y \exists x_3 \in Y \\
&[(x \neq x_1) \wedge (x \neq x_2) \wedge (x \neq x_3) \wedge (x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge (x_2 \neq x_3) \wedge \textsc{Edg}(x, x_1) \\
&\wedge \textsc{Edg}(x, x_2) \wedge (\forall y \in Y[((y \neq x_1) \wedge (y \neq x_2)) \Rightarrow \neg\textsc{Edg}(x, y)]]])]. \\
\textsc{Ind}_H(X) :=& \exists v_1 \in X, \ldots, \exists v_m \in X \left[\bigwedge_{i=1}^{m} \bigwedge_{j=1}^{m} [(i \neq j) \Rightarrow ((v_i \neq v_j) \wedge (\textsc{Edg}(v_i, v_j) \Leftrightarrow \textsc{Edg}(u_i, u_j)))] \right].
\end{aligned}$$

Therefore, the three properties are expressible in monadic second-order formula. □

Now, we are ready to show that 3-Leaf Power Deletion is fixed-parameter tractable. A set S of vertices of a graph G is a *modulator of* G if $G \backslash S$ is a 3-leaf power. An instance (G, k) is a yes-instance if and only if G has a modulator of size at most k. By the monadic second-order formulas in Lemma 3.8, one can formulate a property saying that an input graph G has a modulator of size at most k in monadic second-order formula as follows.

$$\exists x_1 \cdots \exists x_k[\forall X[((x_1 \notin X) \wedge \cdots \wedge (x_k \notin X)) \Rightarrow (\textsc{Chor}(X) \wedge \neg\textsc{Ind}_{bull}(X) \wedge \neg\textsc{Ind}_{dart}(X) \wedge \neg\textsc{Ind}_{gem}(X))]].$$

Therefore, 3-Leaf Power Deletion can be expressed in monadic second-order formula.

We remark that the deletion of one vertex of G decreases the rank-width of G by at most 1. Therefore, if G has a modulator of size at most k, then it has rank-width at most $k + 1$, because every 3-leaf power is distance-hereditary, and so has rank-width at most 1. By applying the algorithm in Theorem 3.7, we may assume that for every input instance (G, k) for 3-Leaf Power Deletion, G has rank-width at most $k + 1$.

Theorem 3.9 (Courcelle, Makowsky, and Rotics [14, Theorem 4])**.** *Let $\mathcal{C}$ be a class of p-graphs of clique-width at most k with a given $O(f(|V(G)|, |E(G)|))$-time algorithm constructing a k-expression of G for each p-graph G in $\mathcal{C}$. Then every problem expressible in monadic second-order formula on $\mathcal{C}$ can be solved in time $O(f(|V(G)|, |E(G)|))$.*

Since we assume that every input graph has bounded rank-width, and since 3-Leaf Power Deletion is expressible in monadic second-order formula, 3-Leaf Power Deletion is fixed-parameter tractable by Corollary 3.10, which is deduced from Theorems 3.4, 3.5, and 3.9.

Corollary 3.10. *Given an instance (G, k), one can correctly solve* 3-Leaf Power Deletion *in $O(f(k)|V(G)|^3)$ time for a computable function $f : \mathbb{N} \to \mathbb{N}$.*

3.3 A single-exponential fixed-parameter algorithm based on an algorithm for Distance-Hereditary Deletion

In this section, we show that 3-Leaf Power Deletion admits a single-exponential fixed-parameter algorithm, based on a single-exponential fixed-parameter algorithm of Eiben, Ganian, and Kwon [22].

Theorem 3.11 (Eiben, Ganian, and Kwon [22, Theorem 1.1])**.** *For a graph G and a non-negative integer k, we can decide whether there is a set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G \backslash S$ is distance-hereditary in time $O(37^k|V(G)|^7(|V(G)| + |E(G)|))$.*

Theorem 3.12. *Given an instance (G, k), one can correctly solve* 3-Leaf Power Deletion *in time $O(37^k|V(G)|^7(|V(G)| + |E(G)|))$.*

Proof. We proceed by induction on k. If $k = 0$, then 3-Leaf Power Deletion is solved in linear time by Theorem 2.2. Thus, we may assume that $k > 0$. We investigate every 5-element subset of $V(G)$ to find a small obstruction in time $O(|V(G)|^5)$. If we find a small obstruction H in G, then we branch on each vertex of H to be included in the solution, and solve each of the at most five instances in time $O(37^{k-1}|V(G)|^7(|V(G)| + |E(G)|))$ by the induction hypothesis. Otherwise, we apply Theorem 3.11 for (G, k). Then since G has no small obstructions, an induced subgraph of G is a 3-leaf power if and only if it is distance-hereditary, and therefore the answer for 3-Leaf Power Deletion is equal to the answer obtained from G and k by Theorem 3.11. This can be done in time $O(37^k|V(G)|^7(|V(G)|+|E(G)|))$. □

Chapter 4. Good modulators

A modulator S of a graph G is *good* if $G\backslash(S\backslash\{v\})$ is a 3-leaf power for each vertex v in S. Note that if G has a modulator S, then for every induced subgraph G' of G, $S \cap V(G')$ is a modulator of G'. This means that if (G,k) is a yes-instance, then so is (G',k). We remark that if G has an obstruction H and a good modulator S, then H has at least two vertices in S. To find a small good modulator, we first find a modulator by combining a maximal packing of small obstructions with outcomes of the following approximation algorithms for Chordal Deletion by Kim and Kwon [33], and by Agrawal et al. [2]:

Theorem 4.1 (Kim and Kwon [33, Theorem 1.1])**.** *Given a graph G and a positive integer k, we can either find $k+1$ vertex-disjoint holes in G, or find a set $S \subseteq V(G)$ of size $O(k^2 \log k)$ such that $G\backslash S$ is chordal in time bounded above by a polynomial in $|V(G)|$.*

Theorem 4.2 (Agrawal, Lokshtanov, Misra, Saurabh, and Zehavi [2, Theorem 2])**.** *For a graph G and a non-negative integer k, we can either confirm that there is no set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G\backslash S$ is chordal, or find a set $S \subseteq V(G)$ of size $O(k \log^2 |V(G)|)$ such that $G\backslash S$ is chordal in time bounded above by a polynomial in $|V(G)|$.*

Corollary 4.3. *Given an instance (G,k) with $k>0$, we can either confirm that G has no modulators of size at most k, or find a modulator of G having size $O(\min(k^2 \log k, k \log^2 |V(G)|))$ in time bounded above by a polynomial in $|V(G)|$.*

Proof. We can find a maximal packing $H_1, \ldots, H_m$ of vertex-disjoint small obstructions in G in time $O(|V(G)|^6)$. If $m \geqslant k+1$, then we confirm that G has no modulators of size at most k. Thus, we may assume that $m \leqslant k$. Let $X := \bigcup_{i \in \{1,\ldots,m\}} V(H_i)$. We apply Theorems 4.1 and 4.2 for $G\backslash X$ and k. Note that $|X| \leqslant 5k$, and $G\backslash X$ has no small obstructions.

If any of the algorithms in Theorems 4.1 and 4.2 confirms that $G\backslash X$ has no set S of size at most k such that $G\backslash(X \cup S)$ is chordal, then we confirm that G has no modulators of size at most k. Thus, let S_1 be the output of the algorithm in Theorem 4.1 having size $O(k^2 \log k)$, and S_2 be the output of the algorithm in Theorem 4.2 having size $O(k \log^2 |V(G)|)$. Then we choose S as one of S_1 and S_2 so that $|S| = \min(|S_1|, |S_2|)$. Then $X \cup S$ is a modulator of G, and $|X \cup S| = |X| + |S| \leqslant 5k + O(\min(k^2 \log k, k \log^2 |V(G)|)) = O(\min(k^2 \log k, k \log^2 |V(G)|))$. ☐

With a modulator of size $O(\min(k^2 \log k, k \log^2 |V(G)|))$ at hand, we are ready to find a small good modulator. We note that, in principle, a *small* good modulator might not exists, but if that is the case, we are able to identify a vertex that has to be in every modulator of size at most k. Then we can remove it from the input graph, and decrease the parameter k by 1.

Reduction Rule 1 (R1)**.** *Given an instance (G,k) with $k>0$, if G has $k+1$ obstructions $H_1, \ldots, H_{k+1}$ and a vertex v such that $V(H_i) \cap V(H_j) = \{v\}$ for every distinct i and j in $\{1, \ldots, k+1\}$, then replace (G,k) with $(G\backslash v, k-1)$.*

Proof of Safeness. It suffices to show that if G has a modulator S of size at most k, then S contains v. Suppose not. Then S contains at least one vertex of $H_i\backslash v$ for each $i \in \{1, \ldots, k+1\}$. Therefore, $|S| \geqslant k+1$, a contradiction. ☐

To find the obstructions $H_1, \dots, H_{k+1}$, we make use of the following lemma, which we slightly rephrase to better fit our application.

Lemma 4.4 (Jansen and Pilipczuk [29, Lemma 1.3])**.** *Given a graph G, a non-negative integer k, and a vertex v, if $G \backslash v$ is chordal, then we can either find holes $H_1, \dots, H_{k+1}$ in G such that $V(H_i) \cap V(H_j) = \{v\}$ for every distinct i and j in $\{1, \dots, k+1\}$, or find a set $S \subseteq V(G) \backslash \{v\}$ of size at most $12k$ such that $G \backslash S$ is chordal in time bounded above by a polynomial in $|V(G)|$.*

Lemma 4.5. *Given an instance (G, k) with $k > 0$, we can find an equivalent instance (G', k') such that $|V(G')| \leqslant |V(G)|$ and $k' \leqslant k$, and a good modulator of G' having size $O(\min(k^3 \log k, k^2 \log^2 |V(G)|))$ in time bounded above by a polynomial in $|V(G)|$.*

Proof. We first try to find a modulator S of G having size $O(\min(k^2 \log k, k \log^2 |V(G)|))$ by using Corollary 4.3. If it fails, then (G, k) is a no-instance, and therefore we take $(K_{2,2}, 0)$ as (G', k') and $V(K_{2,2})$ as a good modulator of G'. Otherwise, for each vertex v in S, let $G_v := G \backslash (S \backslash \{v\})$, and $F_1^v, \dots, F_{m(v)}^v$ be a maximal packing of small obstructions in G_v such that $V(F_i^v) \cap V(F_j^v) = \{v\}$ for every distinct i and j in $\{1, \dots, m(v)\}$. Finally, let $G'_v := G_v \backslash ((V(F_1^v) \cup \cdots \cup V(F_{m(v)}^v)) \backslash \{v\})$. If $m(v) \geqslant k+1$ for some vertex $v \in S$, then we apply our algorithm recursively for $(G \backslash v, k-1)$. This is safe, because (R1) is safe. Therefore, we may assume that $m(v) \leqslant k$ for every vertex $v \in S$.

By Lemma 4.4 for G'_v, $k - m(v)$, and v, we can either

(1) find $k - m(v) + 1$ holes $H_1^v, \dots, H_{k-m(v)+1}^v$ in G'_v such that $V(H_i^v) \cap V(H_j^v) = \{v\}$ for every distinct i and j in $\{1, \dots, k - m(v) + 1\}$, or

(2) find a set $S'_v \subseteq V(G'_v) \backslash \{v\}$ of size at most $12(k - m(v))$ such that $G'_v \backslash S'_v$ is chordal.

If (1) holds, then we apply our algorithm recursively for $(G \backslash v, k-1)$. This is safe, because (R1) is safe. Therefore, we may assume that (2) holds for every vertex v in S. Then let $S_v := (V(F_1^v) \cup \cdots \cup V(F_{m(v)}^v) \cup S'_v) \backslash \{v\}$. Note that $|S_v| \leqslant 4m(v) + 12(k - m(v)) \leqslant 12k$ and $G_v \backslash S_v$ is a 3-leaf power.

We take (G, k) as (G', k') and $X := S \cup \bigcup_{v \in S} S_v$ as a good modulator of G. Clearly, $|X| \leqslant |S| + 12k|S| = O(\min(k^3 \log k, k^2 \log^2 |V(G)|))$. It remains to argue that X is a good modulator of G. Suppose that H is an obstruction in G. Since S is a modulator of G, H has a vertex $v \in S$. If $|V(H) \cap S| = 1$, then H is an induced subgraph of G_v, and therefore H has at least one vertex in S_v. Since S_v and S are disjoint, H has at least two vertices in X. Therefore, X is a good modulator of G. □

Chapter 5. Bounding the number of components outside of a good modulator

Let S be a good modulator of a graph G. In this chapter, we bound the number of non-trivial components of $G\backslash S$. In Section 5.1, we introduce a complete split of a graph, and present two lemmas observing obstructions with a complete split of a graph. Then we define a blocking pair for a set of vertices, and present a characterization of a complete split of a graph and a lemma observing obstructions with a common blocking pair for two sets of vertices. All lemmas introduced in this subsection will be used in the next subsection to bound the number of non-trivial components of $G\backslash S$. In Section 5.2, we partition S into S^+ and S^-, and bound the number of components of $G\backslash S$ having neighbors of S^-. Afterward, we design a reduction rule to bound the number of non-trivial components of $G\backslash S$ having no neighbors of S^-, and in Section 5.3, we bound the number of isolated vertices of $G\backslash S$.

5.1 Complete splits and blocking pairs

Cunningham [16] introduced a split of a graph. A *split* of a graph G is a partition (A, B) of $V(G)$ such that $|A| \geqslant 2$, $|B| \geqslant 2$, and $N(A)$ is complete to $N(B)$. We say that a split (A, B) of G is *complete* if $N(A) \cup N(B)$ is a clique. The following two lemmas observe obstructions from the view of a complete split of a graph.

Lemma 5.1. *Let (A, B) be a complete split of a graph G. If G has a hole H, then $V(H) \cap A = \varnothing$ or $V(H) \cap B = \varnothing$.*

Proof. Suppose not. Since $N(A) \cup N(B)$ is a clique, H has at most two vertices in $N(A) \cup N(B)$, because otherwise H has K_3 as a subgraph. Since both $V(H) \cap A$ and $V(H) \cap B$ are non-empty, and H is connected, H has at least two vertices in $N(A) \cup N(B)$. Therefore, H has exactly two vertices x_1 and x_2 in $N(A) \cup N(B)$, a contradiction, because $H\backslash x_1x_2$ is disconnected. □

Lemma 5.2. *Let (A, B) be a complete split of a graph G. If G has an obstruction H having exactly two vertices a_1 and a_2 in A, then a_1 is adjacent to a_2, one of a_1 and a_2 has degree 1 in H, and the other vertex has degree 3 in H.*

Proof. Suppose that both a_1 and a_2 have neighbors in B. Since $N(A) \cup N(B)$ is a clique, a_1 and a_2 are adjacent, and have the same set of neighbors in B. Then a_1 and a_2 are true twins in H, a contradiction by (O1). Therefore, either a_1 or a_2, say a_1, has no neighbors in B. Since H is connected, a_1 is adjacent to a_2. Thus, a_1 has degree 1 in H. By (O3), a_2 has at most three neighbors in H. By (O6), a_2 has at least three neighbors in H. Therefore, a_2 has degree 3 in H. □

We remark that for a complete split (A, B) of a graph G, if G has an obstruction H having exactly two vertices in A, then H is isomorphic to the bull.

Now, we define a blocking pair for a set $X \subseteq V(G)$. A *blocking pair for X* is an unordered pair $\{v, w\}$ of distinct vertices in $N(X)$ such that if v and w are adjacent, and $N(v) \cap X = N(w) \cap X$, then $N(v) \cap X$ is not a clique. Note that if $v, w \in N(X)$ are not adjacent, or $N(v) \cap X \neq N(w) \cap X$, then $\{v, w\}$ is a blocking pair for X. We say that X is *blocked by* $\{v, w\}$ if $\{v, w\}$ is a blocking pair for X. We remark that if $N(X)$ has a blocking pair $\{v, w\}$ for some subset of X, then X is blocked by $\{v, w\}$ as well. This definition is motivated by the following lemma.

Lemma 5.3. *Let (A, B) be a partition of the vertex set of a graph G such that $|A| \geqslant 2$ and $|B| \geqslant 2$. Then (A, B) is a complete split of G if and only if $N(B)$ is a clique, and B has no blocking pairs for A.*

Proof. It is clear that if (A, B) is a complete split of G, then $N(B)$ is a clique, and B has no blocking pairs for A.

Conversely, suppose that $N(B)$ is a clique, and B has no blocking pairs for A. We may assume that $|N(A)| \geqslant 2$, because otherwise $N(A) \cup N(B)$ is a clique, and (A, B) is a complete split of G. Since B has no blocking pairs for A, $N(A)$ is a clique, because if $N(A)$ has a non-edge vw, then $\{v, w\}$ is a blocking pair for A. Moreover, $N(v) \cap A = N(w) \cap A$ for all vertices v and w in $N(A)$, because otherwise $\{v, w\}$ is a blocking pair for A. This means that $N(A)$ is complete to $N(B)$. Therefore, $N(A) \cup N(B)$ is a clique, and (A, B) is a complete split of G. □

The following lemma shows that a blocking pair $\{v, w\}$ for a set $X \subseteq V(G)$ tells us not only that $(X, V(G)\backslash X)$ is not a complete split of G, but also that G is not a 3-leaf power if $G[X]$ has two distinct components whose vertex sets are blocked by $\{v, w\}$.

Lemma 5.4. *Let (A, B) be a partition of the vertex set of a graph G such that $|A| \geqslant 2$ and $|B| \geqslant 2$. If $G[A]$ has distinct components C_1 and C_2 such that both $V(C_1)$ and $V(C_2)$ are blocked by $\{v, w\}$ of vertices in B, then $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power.*

Proof. If $N_G(v) \cap V(C_1) \neq N_G(w) \cap V(C_1)$, then we may assume that C_1 has a vertex u_1 adjacent to v and non-adjacent to w, because otherwise we can swap v and w. Let u_2 be a neighbor of v in $V(C_2)$, and P be an induced path in $G[V(C_1) \cup V(C_2) \cup \{w\}]$ from u_1 to u_2. Note that the length of P is at least 3, because P must intersect w that is non-adjacent to u_1. Since v is adjacent to both ends of P, $G[V(P) \cup \{v\}]$ is not distance-hereditary by Lemma 2.7. Therefore, $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power.

Therefore, $N_G(v) \cap V(C_i) = N_G(w) \cap V(C_i)$ for $i = 1, 2$. If v and w are non-adjacent, then for a neighbor u_1 of v in $V(C_1)$ and a neighbor u_2 of v in $V(C_2)$, $G[\{v, w, u_1, u_2\}]$ is a hole. Therefore, we may assume that v and w are adjacent. Then since $\{v, w\}$ is a blocking pair for $V(C_1)$, $N_G(v) \cap V(C_1)$ has a non-edge u_1u_2. Let P be an induced path in C_1 from u_1 to u_2. Since v is adjacent to both ends of P, we may assume that the length of P is exactly 2 by Lemma 2.7. Let u_3 be a common neighbor of u_1 and u_2 in $V(P)$, and u_4 be a neighbor of v in $V(C_2)$. Then $G[\{v, u_1, u_2, u_3, u_4\}]$ is isomorphic to the dart if u_3 is adjacent to v, and has a hole of length 4 if u_3 is non-adjacent to v. Therefore, $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power. □

5.2 The number of non-trivial components

Let S^+ be the set of vertices v in S such that for each component C of $G\backslash S$, $N_G(v) \cap V(C)$ is a true twin-set in C, and $S^- := S\backslash S^+$. The following proposition shows that $G\backslash S$ has at most $|S^-|$ components having neighbors of S^-.

Proposition 5.5. *Let S be a good modulator of a graph G, v be a vertex in S, and C be a component of $G\backslash S$. If $N_G(v) \cap V(C)$ contains distinct vertices w_1 and w_2 that are not true twins in C, then no components of $G\backslash S$ different from C have neighbors of v.*

Proof. Suppose that there is a component of $G\backslash S$ different from C having a neighbor w of v. If w_1 and w_2 are adjacent, then we may assume that C has a vertex w_3 adjacent to exactly one of w_1 and w_2, because w_1 and w_2 are not true twins in C. Then $G[\{v, w, w_1, w_2, w_3\}]$ is isomorphic to the dart if w_3 is adjacent to v, and the bull if w_3 is non-adjacent to v, a contradiction, because it has exactly one vertex v in S, which is a good modulator of G.

Therefore, w_1 and w_2 are non-adjacent. Let P be an induced path in C from w_1 to w_2. Since v is adjacent to both ends of P, and S is a good modulator of G, we may assume that the length of P is exactly 2 by Lemma 2.7. Let w_3 be a common neighbor of w_1 and w_2 in $V(P)$. Then $G[\{v, w, w_1, w_2, w_3\}]$ is isomorphic to the dart if w_3 is adjacent to v, and has a hole of length 4 if w_3 is non-adjacent to v, a contradiction, because it has exactly one vertex v in S. □

We present a reduction rule to bound the number of non-trivial components of $G\backslash S$ having no neighbors of S^-. We will use the following definition to design such a reduction rule.

Let X be a set of vertices of a graph Q. For a non-negative integer ℓ, a set $M \subseteq E(Q)$ is an (X, ℓ)-*matching* of Q if every vertex in X is incident with at most ℓ edges in M, and every vertex in $V(Q)\backslash X$ is incident with at most one edge in M.

Reduction Rule 2 (R2). *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, let S^+ be the set of vertices u in S such that for each component C of $G\backslash S$, $N_G(u) \cap V(C)$ is a true twin-set in C, X be the set of 2-element subsets of S^+, and Y be the set of non-trivial components of $G\backslash S$ having no neighbors of $S\backslash S^+$. Let Q be a bipartite graph on $(X \times \{1, 2, 3\}, Y)$ such that the following three statements are true.*

(1) Elements $(\{v, w\}, 1) \in X \times \{1\}$ and $C \in Y$ are adjacent in Q if and only if $V(C)$ is blocked by $\{v, w\}$.

(2) Elements $(\{v, w\}, 2) \in X \times \{2\}$ and $C \in Y$ are adjacent in Q if and only if C has a vertex adjacent to both v and w.

(3) Elements $(\{v, w\}, 3) \in X \times \{3\}$ and $C \in Y$ are adjacent in Q if and only if C has an edge xy such that x is adjacent to both v and w, and y is non-adjacent to both v and w.

If Q has a maximal $(X \times \{1, 2, 3\}, k+2)$-matching M avoiding some element U in Y, then replace (G, k) with $(G\backslash E(U), k)$.

Proof of Safeness. Let $G' := G\backslash E(U)$. We need to show that (G, k) is a yes-instance if and only if (G', k) is a yes-instance.

Suppose that G has a modulator S' of size at most k, and $G'\backslash S'$ has an obstruction H. Since $G\backslash S'$ is a 3-leaf power, H has vertices b_1 and b_2 such that $b_1b_2 \in E(U\backslash S')$. Thus, $|V(U)\backslash S'| \geqslant 2$.

Claim 1. We claim that $(V(U)\backslash S', V(G)\backslash(V(U) \cup S'))$ is a split of $G'\backslash S'$.

We first show that $|V(G)\backslash(V(U) \cup S')| \geqslant 2$. If H is a hole of length 4, then H has at most two vertices of $U\backslash S'$, because no holes of length 4 have independent sets of size at least 3, and $V(U)\backslash S'$ is an independent set of $G'\backslash S'$. Therefore, H has at least two vertices of $G\backslash(V(U) \cup S')$.

If H is not a hole of length 4, then $|V(H)| \geqslant 5$. By (O2), H has at most three vertices of $U\backslash S'$, because $V(U)\backslash S'$ is an independent set of $G'\backslash S'$. Therefore, H has at least two vertices of $G\backslash(V(U) \cup S')$, and $|V(G)\backslash(V(U) \cup S')| \geqslant 2$.

Now, suppose that $(V(U)\backslash S', V(G)\backslash(V(U) \cup S'))$ is not a split of $G'\backslash S'$. Then $G\backslash(V(U) \cup S')$ has vertices v and w such that both v and w have neighbors in $V(U)\backslash S'$, and $N_G(v) \cap (V(U)\backslash S') \neq$

$N_G(w) \cap (V(U)\backslash S')$. Thus, $\{v, w\}$ is a blocking pair for $V(U)\backslash S'$, so for $V(U)$. Then U is adjacent to $(\{v, w\}, 1)$ in Q. Since M is maximal, Y has distinct elements $C_1, \ldots, C_{k+2}$ different from U such that $V(C_i)$ is blocked by $\{v, w\}$ for each $i \in \{1, \ldots, k+2\}$. Since $|S'| \leqslant k$, two of them, say C_1 and C_2, have no vertices in S'. Then $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power by Lemma 5.4, a contradiction, because it is an induced subgraph of $G\backslash S'$, and this proves the claim.

Since $V(U)\backslash S'$ is an independent set of $G'\backslash S'$, and H is connected, both b_1 and b_2 have neighbors in $V(G)\backslash(V(U) \cup S')$. Then by Claim 1, b_1 and b_2 are false twins in $G'\backslash S'$. By (O5), both b_1 and b_2 have degree 2 in H. Let z_1 and z_2 be the neighbors of b_1 in $V(H) \cap S$. Then U is adjacent to $(\{z_1, z_2\}, 2)$ in Q. Since M is maximal, Y has distinct elements $C'_1, \ldots, C'_{k+2}$ different from U such that C'_i has a vertex adjacent to both z_1 and z_2 for each $i \in \{1, \ldots, k+2\}$. Since $|S'| \leqslant k$, two of them, say C'_1 and C'_2, have no vertices in S'. Note that S' has no vertices of H, because H is an induced subgraph of $G'\backslash S'$.

If z_1 and z_2 are non-adjacent, then $G[V(C'_1) \cup V(C'_2) \cup \{z_1, z_2\}]$ has a hole of length 4, a contradiction, because it is an induced subgraph of $G\backslash S'$. Therefore, z_1 and z_2 are adjacent. Since $G[\{b_1, z_1, z_2\}]$ is isomorphic to K_3, H is not a hole, and therefore $|V(H)| = 5$. Let a be a vertex of H different from b_1, b_2, z_1, and z_2. We may assume that a is not in $V(C'_1)$, because otherwise we may swap C'_1 and C'_2. Let c be a vertex of C'_1 adjacent to both z_1 and z_2. Note that $G[\{b_1, b_2, z_1, z_2\}]$ is isomorphic to $K_4\backslash b_1b_2$.

Since the dart and a hole of length 4 are the only obstructions having false twins, H is isomorphic to the dart. Thus, we may assume that $N_H(a) = \{z_1\}$. Then $G[\{a, b_1, c, z_1, z_2\}]$ is isomorphic to the gem if c is adjacent to a, and the dart if c is non-adjacent to a, a contradiction, because it is an induced subgraph of $G\backslash S'$. Therefore, if (G, k) is a yes-instance, then so is (G', k).

Conversely, suppose that G' has a modulator S' of size at most k, and $G\backslash S'$ has an obstruction H. Since $G'\backslash S'$ is a 3-leaf power, H has an edge of $U\backslash S'$. Thus, $|V(U)\backslash S'| \geqslant 2$. Since S is a good modulator of G, H has at least two vertices in $S\backslash S'$. Then $|V(G)\backslash(V(U) \cup S')| \geqslant 2$, because $S\backslash S' \subseteq V(G)\backslash(V(U) \cup S')$.

Claim 2. We claim that $(V(U)\backslash S', V(G)\backslash(V(U) \cup S'))$ is a complete split of $G\backslash S'$.

Suppose not. We first show that $V(G)\backslash(V(U) \cup S')$ has a blocking pair for $V(U)\backslash S'$. Since U is a component of $G\backslash S$, and has no neighbors of $S\backslash S^+$, it suffices to show that $S^+\backslash S'$ has a blocking pair for $V(U)\backslash S'$. We may assume that for all vertices v and w in $S^+\backslash S'$ having neighbors in $V(U)\backslash S'$, v and w are adjacent, and have the same set of neighbors in $V(U)\backslash S'$, because otherwise $\{v, w\}$ is a blocking pair for $V(U)\backslash S'$. For each vertex v in $S^+\backslash S'$ having neighbors in $V(U)\backslash S'$, the set of neighbors of v in $V(U)\backslash S'$ is a true twin-set in $U\backslash S'$, that is, a clique. Therefore, $N_G(S^+\backslash S') \cap (V(U)\backslash S')$ is a clique of $U\backslash S'$. Thus, by Lemma 5.3, $S^+\backslash S'$ has a blocking pair $\{v, w\}$ for $V(U)\backslash S'$, so for $V(U)$.

Since $V(U)$ is blocked by $\{v, w\}$, U is adjacent to $(\{v, w\}, 1)$ in Q. Since M is maximal, Y has distinct elements $C_1, \ldots, C_{k+2}$ different from U such that $V(C_i)$ is blocked by $\{v, w\}$ for each $i \in \{1, \ldots, k+2\}$. Since $|S'| \leqslant k$, two of them, say C_1 and C_2, have no vertices in S'. Then $G[V(C_1) \cup V(C_2) \cup \{v, w\}]$ is not a 3-leaf power by Lemma 5.4, a contradiction, because it is an induced subgraph of $G'\backslash S'$, and this proves the claim.

Since both $U\backslash S'$ and $G\backslash(V(U) \cup S')$ have vertices of H, H is not a hole by Lemma 5.1 and Claim 2, and therefore $|V(H)| = 5$. Let $t_1, \ldots, t_p$ be the vertices of H in $V(U)\backslash S'$, and $s_1, \ldots, s_q$ be the vertices of H in $V(G)\backslash(V(U) \cup S')$. Note that both p and q are at least 2. Since $|V(H)| = 5$, $(p, q) = (3, 2)$ or $(p, q) = (2, 3)$.

If $(p, q) = (3, 2)$, then we may assume that $N_H(s_1) = \{s_2\}$ and $N_H(s_2) = \{s_1, t_1, t_2\}$ by Lemma 5.2

and Claim 2. Since U has no neighbors of $S\backslash S^+$, s_2 is in S^+. Thus, t_1 and t_2 are true twins in $U\backslash S'$, a contradiction by (O1).

Therefore, $(p,q) = (2,3)$. By Lemma 5.2 and Claim 2, we may assume that $N_H(t_1) = \{t_2\}$ and $N_H(t_2) = \{t_1, s_1, s_2\}$. Note that s_1 and s_2 are in $S\backslash S'$. Then U is adjacent to $(\{s_1, s_2\}, 3)$ in Q. Since M is maximal, Y has distinct elements $C''_1, \ldots, C''_{k+2}$ different from U such that C''_i has an edge x_iy_i such that x_i is adjacent to both s_1 and s_2, and y_i is non-adjacent to both s_1 and s_2 for each $i \in \{1, \ldots, k+2\}$. Since $|S'| \leqslant k$, two of them, say C''_1 and C''_2, have no vertices in S'. We may assume that s_3 is not in $V(C''_1)$, because otherwise we may swap C''_1 and C''_2. We remark that the bull is the only possible graph to which H is isomorphic. Thus, s_1 and s_2 are adjacent, and s_3 is adjacent to exactly one of s_1 and s_2 in H. Then $G[\{x_1, y_1, s_1, s_2, s_3\}]$ is isomorphic to the gem if both x_1 and y_1 are adjacent to s_3, the bull if both x_1 and y_1 are non-adjacent to s_3, and the dart if x_1 is adjacent to s_3 and y_1 is non-adjacent to s_3, and has a hole of length 4 if x_1 is non-adjacent to s_3 and y_1 is adjacent to s_3, a contradiction, because it is an induced subgraph of $G'\backslash S'$. Therefore, if (G', k) is a yes-instance, then so is (G, k). □

Proposition 5.6. *Given an instance (G,k) with $k > 0$ and a non-empty good modulator S of G, if (R2) is not applicable to (G,k), then $G\backslash S$ has at most $2(k+2)|S|^2$ non-trivial components.*

Proof. Let S^+ be the set of vertices u in S such that for each component C of $G\backslash S$, $N_G(u) \cap V(C)$ is a true twin-set in C, and $S^- := S\backslash S^+$. By Proposition 5.5, each vertex in S^- is adjacent to at most one component of $G\backslash S$. Therefore, $G\backslash S$ has at most $|S^-|$ non-trivial components having neighbors of S^-.

Let Q and M be defined as in (R2). Since (R2) is not applicable to (G,k), each non-trivial component of $G\backslash S$ having no neighbors of S^- is incident with exactly one edge in M. Since each edge in M is incident with some element in $X \times \{1,2,3\}$, and each element in $X \times \{1,2,3\}$ is incident with at most $k+2$ edges, $|M| \leqslant (k+2) \cdot |X \times \{1,2,3\}| \leqslant (k+2) \cdot 3\binom{|S^+|}{2} \leqslant 3(k+2)|S|^2/2$. Then $|S^-| + |M| \leqslant |S| + 3(k+2)|S|^2/2 \leqslant (k+2)|S|^2/2 + 3(k+2)|S|^2/2 = 2(k+2)|S|^2$, and therefore $G\backslash S$ has at most $2(k+2)|S|^2$ non-trivial components. □

5.3 The number of isolated vertices

We present a reduction rule to bound the number of isolated vertices of $G\backslash S$.

Reduction Rule 3 (R3)**.** *Given an instance (G,k) with $k > 0$ and a non-empty good modulator S of G, let $\mathcal{A}$ be the set of ordered pairs (A_1, A_2) of disjoint subsets of S such that $2 \leqslant |A_1| + |A_2| \leqslant 4$, and X be the set of isolated vertices of $G\backslash S$. For each element (A_1, A_2) in $\mathcal{A}$, let X_{A_1,A_2} be a maximal set of vertices v in X such that $N_G(v) \cap (A_1 \cup A_2) = A_1$ and $|X_{A_1,A_2}| \leqslant k+3$. If X has a vertex u not in $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$, then replace (G,k) with $(G\backslash u, k)$.*

Proof of Safeness. We need to show that if $(G\backslash u, k)$ is a yes-instance, then so is (G,k). Suppose that $G\backslash u$ has a modulator S' of size at most k, and $G\backslash S'$ has an obstruction H. Then $u \in V(H)$, because $G\backslash(S' \cup \{u\})$ is a 3-leaf power.

If H is a hole, then u has exactly two neighbors v_1 and v_2 in $V(H) \cap S$ such that v_1 is non-adjacent to v_2. By the construction of $X_{\{v_1,v_2\},\varnothing}$, $X_{\{v_1,v_2\},\varnothing}$ contains distinct vertices $u_1, \ldots, u_{k+3}$ different from u. Note that H has at most one of $u_1, \ldots, u_{k+3}$, because v_1 and v_2 have at most two common neighbors in $V(H)$ including u. Then since $|S'| \leqslant k$, two of them, say u_1 and u_2, are not in $S' \cup V(H)$. Thus, $G[\{v_1, v_2, u_1, u_2\}]$ is a hole, a contradiction, because it is an induced subgraph of $G\backslash(S' \cup \{u\})$.

Suppose that H is isomorphic to the bull, the dart, or the gem. Then $2 \leqslant |S \cap V(H)| \leqslant 4$, because H has exactly five vertices including u, and S is a good modulator of G. Let $B_1 := (S \cap V(H)) \cap N_G(u)$, and $B_2 := (S \cap V(H)) \backslash N_G(u)$. By the construction of X_{B_1,B_2}, X_{B_1,B_2} contains distinct vertices $u_1, \ldots, u_{k+3}$ different from u. Since $|V(H)| = 5$ and $2 \leqslant |S \cap V(H)| \leqslant 4$, H has at most three vertices in X including u. Thus, H has at most two of $u_1, \ldots, u_{k+3}$. Then since $|S'| \leqslant k$, one of them, say u_1, is not in $S' \cup V(H)$. Thus, $G[(V(H)\backslash\{u\}) \cup \{u_1\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(S' \cup \{u\})$. □

Proposition 5.7. *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, if (R3) is not applicable to (G, k), then $G\backslash S$ has at most $2(k + 3)|S|^4/3$ isolated vertices.*

Proof. Let $\mathcal{A}$, X, and X_{A_1,A_2} be defined as in (R3). If $|S| \leqslant 1$, then $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$ is empty, and therefore X is empty. Thus, we may assume that $|S| \geqslant 2$. Let $s := |S|$. For each m-element subset T of S with $2 \leqslant m \leqslant 4$, $\mathcal{A}$ contains exactly 2^m elements (A_1, A_2) such that $T = A_1 \cup A_2$. Therefore, $|\mathcal{A}|$ is at most

$$\begin{aligned}
2^4 \cdot \binom{s}{4} + 2^3 \cdot \binom{s}{3} + 2^2 \cdot \binom{s}{2} &\leqslant \frac{2}{3}(s-1)^4 + \frac{4}{3}(s-1)^3 + 2s(s-1) \\
&= \frac{2}{3}(s-1)^2((s-1)^2 + 2(s-1)) + 2s(s-1) \\
&\leqslant \frac{2}{3}(s-1)^2 s^2 + 2s(s-1) \\
&= 2s(s-1)(s^2 - s + 3)/3 \\
&\leqslant 2s(s-1)(s^2 + s)/3 = 2s^2(s^2-1)/3 \leqslant 2s^4/3.
\end{aligned}$$

For each element (A_1, A_2) in $\mathcal{A}$, $|X_{A_1,A_2}| \leqslant k + 3$. Therefore, $|\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}| \leqslant 2(k + 3)|S|^4/3$. Since (R3) is not applicable to (G, k), every isolated vertex of $G\backslash S$ is in $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$. Therefore, $G\backslash S$ has at most $2(k + 3)|S|^4/3$ isolated vertices. □

Chapter 6. Bounding the size of components outside of a good modulator

Let S be a good modulator of a graph G. In this chapter, we bound the size of each component of $G\backslash S$. Section 6.1 is about complete components of $G\backslash S$, and Section 6.2 is about incomplete components of $G\backslash S$.

6.1 The size of each complete component

We present a reduction rule to bound the size of each complete component of $G\backslash S$.

Reduction Rule 4 (R4)**.** *Given an instance (G,k) with $k>0$ and a non-empty good modulator S of G, let $\mathcal{A}$ be the set of ordered pairs (A_1,A_2) of disjoint subsets of S such that $2 \leqslant |A_1|+|A_2| \leqslant 4$, and C be a complete component of $G\backslash S$. For each element (A_1,A_2) in $\mathcal{A}$, let X_{A_1,A_2} be a maximal set of vertices v of C such that $N_G(v)\cap(A_1\cup A_2)=A_1$ and $|X_{A_1,A_2}| \leqslant k+3$. If C has a vertex u not in $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$, then replace (G,k) with $(G\backslash u,k)$.*

Proof of Safeness. We need to show that if $(G\backslash u,k)$ is a yes-instance, then so is (G,k). Suppose that $G\backslash u$ has a modulator S' of size at most k, and $G\backslash S'$ has an obstruction H. Then $u \in V(H)$, because $G\backslash(S'\cup\{u\})$ is a 3-leaf power.

Suppose that H is a small obstruction. Since H has at most five vertices including u, and S is a good modulator of G, $2 \leqslant |S\cap V(H)| \leqslant 4$. Let $B_1 := (S\cap V(H))\cap N_G(u)$, and $B_2 := (S\cap V(H))\backslash N_G(u)$. By the construction of X_{B_1,B_2}, X_{B_1,B_2} contains distinct vertices $u_1,\ldots,u_{k+3}$ different from u. Since $|V(H)| \leqslant 5$ and $2 \leqslant |S\cap V(H)| \leqslant 4$, H has at most three vertices of C including u. Thus, H has at most two of $u_1,\ldots,u_{k+3}$. Then since $|S'| \leqslant k$, one of them, say u_1, is not in $S'\cup V(H)$. Thus, $G[(V(H)\backslash\{u\})\cup\{u_1\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(S'\cup\{u\})$.

Therefore, H is a hole of length at least 6. Note that H has at most two vertices of C, because C is complete. Suppose that H has exactly one vertex u of C. In this case, u is adjacent to distinct vertices v_1 and v_2 in $V(H)\cap S$. Then $H\backslash u$ is an induced path of length at least 4 from v_1 to v_2. By the construction of $X_{\{v_1,v_2\},\varnothing}$, $X_{\{v_1,v_2\},\varnothing}$ contains distinct vertices $u_1,\ldots,u_{k+3}$ different from u. Since $|S'| \leqslant k$, one of them, say u_1, is not in S'. Then $G[(V(H)\backslash\{u\})\cup\{u_1\}]$ is not distance-hereditary by Lemma 2.7, a contradiction, because it is an induced subgraph of $G\backslash(S'\cup\{u\})$.

Therefore, H has exactly two vertices u and u' of C. In this case, u is adjacent to a vertex v_1 in $V(H)\cap S$, and u' is adjacent to another vertex v_2 in $V(H)\cap S$. Note that u' is non-adjacent to v_1. Then $H\backslash u$ is an induced path of length at least 4 from v_1 to u'. By the construction of $X_{\{v_1\},\{v_2\}}$, $X_{\{v_1\},\{v_2\}}$ contains distinct vertices $u_1,\ldots,u_{k+3}$ different from u. Since $|S'| \leqslant k$, one of them, say u_1, is not in S'. Then $G[(V(H)\backslash\{u\})\cup\{u_1\}]$ is not distance-hereditary by Lemma 2.7, a contradiction, because it is an induced subgraph of $G\backslash(S'\cup\{u\})$. Therefore, if $(G\backslash u,k)$ is a yes-instance, then so is (G,k). □

Proposition 6.1. *Given an instance (G,k) with $k>0$ and a non-empty good modulator S of G, if (R4) is not applicable to (G,k), then every complete component of $G\backslash S$ has at most $2(k+3)|S|^4/3$ vertices.*

Proof. Let $\mathcal{A}$, C, and X_{A_1,A_2} be defined as in (R4). Since (R4) is not applicable to (G,k), every vertex of C is in $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$. If $|S| \leqslant 1$, then $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$ is empty, contradicting the assumption that every vertex of C is in $\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}$. Thus, $|S| \geqslant 2$. For each m-element subset T of S with $2 \leqslant m \leqslant 4$, $\mathcal{A}$ contains exactly 2^m elements (A_1, A_2) such that $T = A_1 \cup A_2$. Therefore, $|\mathcal{A}| \leqslant 2^4 \cdot \binom{|S|}{4} + 2^3 \cdot \binom{|S|}{3} + 2^2 \cdot \binom{|S|}{2} \leqslant 2|S|^4/3$, as in the proof of Proposition 5.7. For each element (A_1, A_2) in $\mathcal{A}$, $|X_{A_1,A_2}| \leqslant k + 3$. Therefore, $|\bigcup_{(A_1,A_2)\in\mathcal{A}} X_{A_1,A_2}| \leqslant 2(k+3)|S|^4/3$, and C has at most $2(k+3)|S|^4/3$ vertices. □

6.2 The size of each incomplete component

We present a reduction rule to bound the size of each true twin-set in G.

Reduction Rule 5 (R5)**.** *Given an instance (G,k) with $k > 0$, if G has a true twin-set X such that $v \in X$ and $|X| \geqslant k + 2$, then replace (G,k) with $(G\backslash v, k)$.*

Proof of Safeness. We need to show that if $(G\backslash v, k)$ is a yes-instance, then so is (G,k). Suppose that $G\backslash v$ has a modulator S of size at most k, and $G\backslash S$ has an obstruction H. Then $v \in V(H)$, because $G\backslash(S \cup \{v\})$ is a 3-leaf power. By (O1), $V(H) \cap X = \{v\}$. Since $|S| \leqslant k$, X contains a vertex w not in $S \cup \{v\}$. Then $G[(V(H)\backslash\{v\}) \cup \{w\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(S \cup \{v\})$. □

We present a reduction rule to remove some bags of $G\backslash S$ anti-complete to S.

Reduction Rule 6 (R6)**.** *Given an instance (G,k) with $k > 0$ and a non-empty good modulator S of G, let B be a non-empty true twin-set in $G\backslash S$. If $G\backslash(S \cup B)$ has a component D having no neighbors of S such that $V(D)\backslash N_G(B)$ is non-empty, then replace (G,k) with $(G\backslash(V(D)\backslash N_G(B)), k)$.*

Proof of Safeness. Let $G' := G\backslash(V(D)\backslash N_G(B))$. We need to show that if (G', k) is a yes-instance, then so is (G,k). Suppose that G' has a modulator S' of size at most k, and $G\backslash S'$ has an obstruction H. Since $G'\backslash S'$ is a 3-leaf power, H has at least one vertex of $D\backslash N_G(B)$. Since H is connected, and D has no neighbors of S, H has at least one vertex in $V(D) \cap N_G(B)$. Thus, H has at least two vertices of D. Since $V(H) \cap S \neq \varnothing$, $V(H) \cap B$ is a clique cut-set of H and therefore H is not a hole. Thus, $|V(H)| = 5$. Since S is a good modulator of G, $|V(H) \cap S| = 2$, $|V(H) \cap B| = 1$, and $|V(H) \cap V(D)| = 2$, contradicting by (O4). □

We present two reduction rules to reduce the number of bags of $G\backslash S$.

Reduction Rule 7 (R7)**.** *Given an instance (G,k) with $k > 0$ and a non-empty good modulator S of G, let B be a non-empty true twin-set in $G\backslash S$. If $G\backslash(S \cup B)$ has distinct components $D_1, \ldots, D_{k+4}$ such that $N_G(V(D_1)) = \cdots = N_G(V(D_{k+4}))$, and either $V(D_1) \cup \cdots \cup V(D_{k+4}) \subseteq N_G(B)$, or $\varnothing \neq V(D_i) \cap N_G(B) \neq V(D_i)$ for each $i \in \{1, \ldots, k+4\}$, then replace (G,k) with $(G\backslash V(D_1), k)$.*

To show that (R7) is safe, we will use the following two lemmas. Lemma 6.2 will be useful because it implies that for a good modulator S of G, a subset B of $V(G)\backslash S$ is a true twin-set in $G\backslash S$ if and only if it is a true twin-set in G.

Lemma 6.2. *Let G be a 3-leaf power having a vertex v such that $G\backslash v$ is connected and incomplete. Then vertices t_1 and t_2 in $V(G)\backslash\{v\}$ are true twins in G if and only if t_1 and t_2 are true twins in $G\backslash v$.*

Proof. It is clear that if t_1 and t_2 are true twins in G, then so are in $G\backslash v$.

Conversely, suppose that t_1 and t_2 are true twins in $G\backslash v$, and v is adjacent to t_1, and non-adjacent to t_2. Note that $|N_G(t_2)| \geqslant 2$, because otherwise $G\backslash v$ is isomorphic to K_2.

If $N_G(t_2)$ is a clique, then $G\backslash v$ has at least one vertex not in $N_G(t_2)$, because otherwise $G\backslash v$ is complete. Thus, G has an edge xy such that x is adjacent to both t_1 and t_2, and y is non-adjacent to both t_1 and t_2, because $G\backslash v$ is connected. Then $G[\{v, x, y, t_1, t_2\}]$ is isomorphic to the gem if both x and y are adjacent to v, the bull if both x and y are non-adjacent to v, and the dart if x is adjacent to v and y is non-adjacent to v, and has a hole of length 4 if x is non-adjacent to v and y is adjacent to v, a contradiction, because it is an induced subgraph of G.

Therefore, t_2 has distinct neighbors x and y such that x is non-adjacent to y. Then $G[\{v, x, y, t_1, t_2\}]$ has a hole of length 4 if both x and y are adjacent to v, and is isomorphic to the gem if exactly one of x and y is adjacent to v, and the dart if both x and y are non-adjacent to v, a contradiction, because it is an induced subgraph of G. □

Lemma 6.3. *Let (A, B) be a complete split of a graph G, and S be a non-empty good modulator of G. If G has an obstruction H, and $S \subseteq B\backslash N(A)$, then H has at most one vertex in A.*

Proof. Suppose not. Since S is a good modulator of G, H has at least two vertices in S. Thus, H has vertices in both A and B. Since (A, B) is a complete split of G, H is not a hole by Lemma 5.1, and therefore $|V(H)| = 5$. Then $|V(H) \cap N(A)| \leqslant 5 - |V(H) \cap A| - |V(H) \cap S| \leqslant 5 - 2 - 2$, a contradiction, because no obstruction has a cut vertex partitioning its vertex set into two sets both having size 2. □

Proof of Safeness for (R7). We need to show that if $(G\backslash V(D_1), k)$ is a yes-instance, then so is (G, k). Suppose that $G\backslash V(D_1)$ has a modulator S' of size at most k, and $G\backslash S'$ has an obstruction H. Since $G\backslash(V(D_1) \cup S')$ is a 3-leaf power, H has at least one vertex of D_1. Since S is a good modulator of G, $G\backslash(S\backslash\{v\})$ is a 3-leaf power for each vertex v in S. Thus, if v has a neighbor in a true twin-set X in $G\backslash S$, then $\{v\}$ is complete to X by Lemma 6.2. This means that every true twin-set in $G\backslash S$ is a true twin-set in G as well.

We claim that for each $i \in \{1, \ldots, k+4\}$, $V(D_i) \cap N_G(B)$ is a true twin-set in $G\backslash S$. Suppose that $V(D_i) \cap N_G(B)$ contains two vertices x and y such that x is non-adjacent to y. Let P be an induced path in D_i from x to y. We may assume that P has length exactly 2 by Lemma 2.7. Let z be a common neighbor of x and y in $V(P)$. We may assume that $z \in N_G(B)$, because otherwise $V(P)$ with a vertex in B induces a hole of length 4. Then for a vertex v in B, and v' in $V(D_j) \cap N_G(B)$ for some $j \in \{1, \ldots, k+4\}\backslash\{i\}$, $G[\{v, v', x, y, z\}]$ is isomorphic to the dart, contradicting the assumption that S is a modulator of G. Therefore, $V(D_i) \cap N_G(B)$ is a clique. Now, suppose that $G\backslash S$ has a vertex w adjacent to a vertex $t_1 \in V(D_i) \cap N_G(B)$ and non-adjacent to a vertex $t_2 \in V(D_i) \cap N_G(B)$. Note that w is a vertex of $D_i\backslash N_G(B)$. Then for a vertex v in B and a vertex v' of $V(D_j) \cap N_G(B)$ for some $j \in \{1, \ldots, k+4\}\backslash\{i\}$, $G[\{v, v', w, t_1, t_2\}]$ is isomorphic to the bull, a contradiction, and this proves the claim.

Suppose that $V(D_1) \cup \cdots \cup V(D_{k+4}) \subseteq N_G(B)$. By (O1), for each $i \in \{1, \ldots, k+4\}$, D_i has at most one vertex of H. By (O2), at most three of $D_1, \ldots, D_{k+4}$ have vertices of H. Since $|S'| \leqslant k$, one of $D_2, \ldots, D_{k+4}$, say D_j, has no vertices in $S' \cup V(H)$. Let t be a vertex in D_j. Since $N_G(V(D_1)) = N_G(V(D_j))$, s and t have the same set of neighbors in $V(H)$. Then $G[(V(H)\backslash\{s\}) \cup \{t\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(V(D_1) \cup S')$.

Therefore, $\varnothing \neq V(D_i) \cap N_G(B) \neq V(D_i)$ for each $i \in \{1, \ldots, k+4\}$. We first show that $D_i\backslash N_G(B)$ has no neighbors of S. Suppose that $D_i\backslash N_G(B)$ has a neighbor p_i of some vertex v in S. Let $j \in$

$\{1,\ldots,k+4\}\setminus\{i\}$. Since $N_G(V(D_i)) = N_G(V(D_j))$, D_j has a neighbor p_j of v. Since some vertex in B has neighbors in both D_i and D_j, $G\backslash S$ has a path P from p_i to p_j. Note that the length of P is at least 3, because p_i is not in $N_G(B)$. Since v is adjacent to both ends of P, $G[V(P) \cup \{v\}]$ is not distance-hereditary by Lemma 2.7, a contradiction, because it is an induced subgraph of $G\backslash(S\setminus\{v\})$, and this proves the claim.

For each $i \in \{1,\ldots,k+4\}$, since $V(D_i)\cap N_G(B)$ is a true twin-set in G, H has at most one vertex in $V(D_i)\cap N_G(B)$ by (O1). Let $D_{i,1},\ldots,D_{i,m(i)}$ be the components of $D_i\backslash N_G(B)$ for each $i \in \{1,\ldots,k+4\}$. We claim that for each $j \in \{1,\ldots,m(i)\}$, if $|V(D_{i,j})| \geqslant 2$, then $(V(D_{i,j}), V(G)\backslash V(D_{i,j}))$ is a complete split of G. Since $V(D_i) \cap N_G(B)$ is a true twin-set, and $D_i\backslash N_G(B)$ has no neighbors of S, it suffices to show that $N_G(N_G(B)) \cap V(D_{i,j})$ is a clique. Suppose that $N_G(N_G(B)) \cap V(D_{i,j})$ contains vertices x and y such that x and y are non-adjacent. Let P be an induced path in $D_{i,j}$ from x to y. We may assume that P has length exactly 2 by Lemma 2.7. Let z be a common neighbor of x and y in $V(P)$. We may assume that $z \in N_G(N_G(B))$, because otherwise P with a vertex v in $N_G(B)\cap V(D_i)$ induces a hole of length 4. Then for a vertex v' in B, $G[\{v,v',x,y,z\}]$ is isomorphic to the dart, a contradiction, and this proves the claim.

Therefore, each component of $D_i\backslash N_G(B)$ has at most one vertex of H by Lemma 6.3. Each $V(D_i) \cap N_G(B)$ has at most one vertex of H, because $V(D_i) \cap N_G(B)$ is a true twin-set. Therefore, at most one component of $D_i\backslash N_G(B)$ has a vertex of H, because H cannot have false twins of degree at most 1. By (O2), at most three of $D_1,\ldots,D_{k+4}$ have vertices of H. Since $|S'| \leqslant k$, one of $D_2,\ldots,D_{k+4}$, say D_i, has no vertices in $S' \cup V(H)$. Note that H has a vertex s_1 in $V(D_1) \cap N_G(B)$, because $D_1\backslash N_G(B)$ has no neighbors of S, H is connected, and has vertices in both S and $V(D_1)$. Let t_1t_2 be an edge of D_i such that $t_1 \in V(D_i) \cap N_G(B)$ and $t_2 \in V(D_i)\backslash N_G(B)$. Since $N_G(V(D_1)) = N_G(V(D_i))$, and both $V(D_1) \cap N_G(B)$ and $V(D_i) \cap N_G(B)$ are true twin-sets, s_1 and t_1 have the same set of neighbors in $V(H)\backslash V(D_1)$. If H has a vertex s_2 in $V(D_1)\backslash N_G(B)$, then $V(D_1) \cap V(H) = \{s_1, s_2\}$, because both $V(D_1) \cap N_G(B)$ and $V(D_1)\backslash N_G(B)$ have at most one vertex of H. Then $G[(V(H)\setminus\{s_1,s_2\}) \cup \{t_1,t_2\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(V(D_1) \cup S')$. Therefore, H has no vertices in $V(D_1)\backslash N_G(B)$. Then $G[(V(H)\setminus\{s_1\}) \cup \{t_1\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G\backslash(V(D_1) \cup S')$. □

Reduction Rule 8 (R8)**.** *Given an instance (G,k) with $k > 0$ and a non-empty good modulator S of G, let $B_1,\ldots,B_m$ be pairwise disjoint non-empty true twin-sets in $G\backslash S$ for $m \geqslant 6$ such that $N_G(B_i) = B_{i-1} \cup B_{i+1}$ for each $i \in \{2,\ldots,m-1\}$. Let ℓ be an integer in $\{3,\ldots,m-2\}$ such that $|B_\ell| \leqslant |B_i|$ for each $i \in \{3,\ldots,m-2\}$, and G' be a graph obtained from $G\backslash((B_3 \cup \cdots \cup B_{m-2})\backslash B_\ell)$ by making every vertex in B_ℓ adjacent to all vertices in $B_2 \cup B_{m-1}$. Then replace (G,k) with (G',k).*

To show that (R8) is safe, we will use the following two lemmas.

Lemma 6.4. *Let G be a graph with disjoint true twin-sets $B_1,\ldots,B_m$ for $m \geqslant 5$ such that $N(B_i) = B_{i-1} \cup B_{i+1}$ for each $i \in \{2,\ldots,m-1\}$. Then G is a 3-leaf power if and only if $G\backslash(B_3 \cup \cdots \cup B_{m-2})$ is a 3-leaf power, and has no paths from a vertex in B_2 to a vertex in B_{m-1}.*

Proof. It is clear that if G is a 3-leaf power, then $G\backslash(B_3 \cup \cdots \cup B_{m-2})$ is a 3-leaf power, and has no paths from a vertex in B_2 to a vertex in B_{m-1}, because otherwise G has a hole.

Conversely, suppose that $G\backslash(B_3 \cup \cdots \cup B_{m-2})$ is a 3-leaf power, and has no paths from a vertex in B_2 to a vertex in B_{m-1}, and G has an obstruction H. Since $G\backslash(B_3 \cup \cdots \cup B_{m-2})$ is a 3-leaf power, H has at least one vertex in $B_3 \cup \cdots \cup B_{m-2}$. For each $i \in \{1,\ldots,m\}$, since B_i is a true twin-set in G, H

has at most one vertex in B_i by (O1). Then every vertex of H in $B_2 \cup \cdots \cup B_{m-1}$ has degree at most 2 in H. If H has a vertex v in B_j for some $j \in \{3, \ldots, m-2\}$, then both B_{j-1} and B_{j+1} have vertices of H by (O6). This means that B_i contains exactly one vertex of H for each $i \in \{2, \ldots, m-1\}$. Then H has vertices in each B_1 and B_m as well by (O6). Thus, $V(H) \cap (B_2 \cup \cdots \cup B_{m-1})$ contains at least three vertices of degree 2 in H. Then H is a hole by (O7), a contradiction, because $H \backslash (B_3 \cup \cdots \cup B_{m-2})$ is a path in $G \backslash (B_3 \cup \cdots \cup B_{m-2})$ from a vertex in B_2 to a vertex in B_{m-1}. Therefore, G is a 3-leaf power. □

Lemma 6.5. *If a graph G has a modulator S and a true twin-set X such that $X \backslash S$ is non-empty, then $S \backslash X$ is a modulator of G.*

Proof. We may assume that $S \cap X$ is non-empty. Suppose that $G \backslash (S \backslash X)$ has an obstruction H. Since $G \backslash S$ is a 3-leaf power, H has at least one vertex in $S \cap X$. Then H has exactly one vertex v in $S \cap X$ by (O1). Let w be a vertex in $X \backslash S$. Then $G[(V(H) \backslash \{v\}) \cup \{w\}]$ is isomorphic to H, a contradiction, because it is an induced subgraph of $G \backslash S$. □

Proof of Safeness for (R8). First, let us show that if (G, k) is a yes-instance, then so is (G', k). Suppose that G has a minimal modulator S' of size at most k. Since S' is minimal, $S' \cap B_i = \varnothing$ or $S' \cap B_i = B_i$ for each $i \in \{1, \ldots, m\}$ by Lemma 6.5. We claim that if $S' \cap (B_1 \cup \cdots \cup B_m)$ is empty, then S' is a modulator of G'. Since $G \backslash S'$ is a 3-leaf power, $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$ is a 3-leaf power, and has no paths from a vertex in B_2 to a vertex in B_{m-1} by Lemma 6.4. Since $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$ is isomorphic to $G' \backslash (B_\ell \cup S')$, $G' \backslash S'$ is a 3-leaf power by Lemma 6.4, and this proves the claim.

We claim that if $S' \cap (B_1 \cup B_2 \cup B_{m-1} \cup B_m)$ is non-empty, then $S' \cap V(G')$ is a modulator of G'. Suppose that $G' \backslash (S' \cap V(G'))$ has an obstruction H. Since $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$ is a 3-leaf power, and is isomorphic to $G' \backslash (B_\ell \cup (S' \cap V(G')))$, $G' \backslash (B_\ell \cup (S' \cap V(G')))$ is a 3-leaf power. Therefore, H has at least one vertex in B_ℓ. For each $i \in \{1, 2, \ell, m-1, m\}$, since B_i is a true twin-set in G', H has at most one vertex in B_i by (O1). Then every vertex of H in $B_2 \cup B_\ell \cup B_{m-1}$ has degree at most 2 in H. Thus, for each $i \in \{1, 2, \ell, m-1, m\}$, B_i contains exactly one vertex of H by (O6). Then $S' \cap V(G')$ contains at least one vertex of H, a contradiction, because H is an induced subgraph of $G' \backslash (S' \cap V(G'))$, and this proves the claim.

Thus, we may assume that $S' \cap (B_1 \cup B_2 \cup B_{m-1} \cup B_m)$ is empty, and $S' \cap (B_3 \cup \cdots \cup B_{m-2})$ is non-empty. Let $T := (S' \backslash (B_3 \cup \cdots \cup B_{m-2})) \cup B_\ell$. Since $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$ is a 3-leaf power, and is isomorphic to $G' \backslash T$, $G' \backslash T$ is a 3-leaf power. Then $B_i \subseteq S'$ for some $i \in \{3, \ldots, m-2\}$, and therefore $|T| = |T \backslash (B_3 \cup \cdots \cup B_{m-2})| + |B_\ell| \leqslant |S' \backslash (B_3 \cup \cdots \cup B_{m-2})| + |B_i| \leqslant |S'| \leqslant k$. Therefore, if (G, k) is a yes-instance, then so is (G', k).

Secondly, we will show that if (G', k) is a yes-instance, then so is (G, k). Suppose that G' has a minimal modulator S' of size at most k. Since S' is minimal, $S' \cap B_i = \varnothing$ or $S' \cap B_i = B_i$ for each $i \in \{1, 2, \ell, m-1, m\}$ by Lemma 6.5. We claim that S' is a modulator of G.

Since $G' \backslash S'$ is a 3-leaf power, if $S' \cap (B_1 \cup B_2 \cup B_{m-1} \cup B_m)$ is empty, then $G' \backslash (B_\ell \cup S')$ is a 3-leaf power, and has no paths from a vertex in B_2 to a vertex in B_{m-1} by Lemma 6.4. Since $G' \backslash (B_\ell \cup S')$ is isomorphic to $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$, $G \backslash S'$ is a 3-leaf power by Lemma 6.4.

Thus, we may assume that $S' \cap (B_1 \cup B_2 \cup B_{m-1} \cup B_m)$ is non-empty, and $G \backslash S'$ has an obstruction H. Since $G' \backslash (B_\ell \cup S')$ is a 3-leaf power, and is isomorphic to $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$, $G \backslash (B_3 \cup \cdots \cup B_{m-2} \cup S')$ is a 3-leaf power. Therefore, H has at least one vertex in $B_3 \cup \cdots \cup B_{m-2}$. For each $i \in \{1, \ldots, m\}$, since B_i is a true twin-set in G, H has at most one vertex in B_i by (O1). Then every vertex of H in

$B_2 \cup \cdots \cup B_{m-1}$ has degree at most 2 in H. If H has a vertex v in B_j for some $j \in \{3, \ldots, m-2\}$, then both B_{j-1} and B_{j+1} have vertices of H by (O6). This means that B_i contains exactly one vertex of H for each $i \in \{2, \ldots, m-1\}$ by (O6), and both B_1 and B_m have vertices of H as well by (O6). Thus, S' contains at least one vertex of H, a contradiction, and this proves the claim. Therefore, if (G', k) is a yes-instance, then so is (G, k). □

Now, we are ready to prove that after applying some reduction rules exhaustively to (G, k) with a good modulator S of G, each incomplete component of $G \backslash S$ has bounded size.

Proposition 6.6. *Given an instance (G, k) with $k > 0$ and a non-empty good modulator S of G, if none of (R2), (R5), (R6), (R7), and (R8) is applicable to (G, k), then each incomplete component of $G \backslash S$ has at most $(k+1)(k+4)|S|(|S| + 2k + 14)$ vertices.*

To prove Proposition 6.6, we will use the following lemma and its corollary.

Lemma 6.7. *Let G be a 3-leaf power. If G has a vertex v of degree at least 1 such that $G \backslash v$ is a tree, then $G \backslash v$ has a vertex u such that $N_G(v) = \{u\}$ or $N_G(v) = N_G[u] \backslash \{v\}$.*

Proof. We may assume that v has at least two neighbors, and $G \backslash v$ has at least three vertices, because otherwise the statement clearly holds.

If v has exactly two neighbors u_1 and u_2, then u_1 and u_2 are adjacent, because otherwise G has a hole. Since $G \backslash v$ has at least three vertices, one of u_1 and u_2, say u_2, is not a leaf of $G \backslash v$. If u_1 is not a leaf of $G \backslash v$, then for a neighbor u_1' of u_1 different from u_2 and a neighbor u_2' of u_2 different from u_1, $G[\{v, u_1, u_1', u_2, u_2'\}]$ is isomorphic to the bull, a contradiction. Therefore, u_1 is a leaf of $G \backslash v$, and $N_G(v) = N_G[u_1] \backslash \{v\}$.

If v has at least three neighbors, then $G[N_G(v)]$ is connected, because otherwise G has a hole. If v has distinct neighbors u_1, u_2, u_3, and u_4 inducing a path, then $G[\{v, u_1, u_2, u_3, u_4\}]$ is isomorphic to the gem, a contradiction. Therefore, $G \backslash v$ has a vertex u such that $N_G(v) \subseteq N_G(u)$. If u has a neighbor u_1 that is non-adjacent to v, then for distinct neighbors u_2 and u_3 of v different from u, $G[\{v, u, u_1, u_2, u_3\}]$ is isomorphic to the dart, a contradiction. Therefore, every neighbor of u different from v is adjacent to v, and $N_G(v) = N_G[u] \backslash \{v\}$. □

Corollary 6.8. *Let G be a 3-leaf power. If G has a vertex v of degree at least 1 such that $G \backslash v$ is connected, then $G \backslash v$ has a true twin-set B such that $N_G(v) = B$ or $N_G(v) = N_G[B] \backslash \{v\}$.*

Proof. We may assume that $G \backslash v$ is incomplete. Then every true twin-set in $G \backslash v$ is also a true twin-set in G by Lemma 6.2. We proceed by induction on $|G|$. Suppose that $G \backslash v$ has true twins u and u'. Since u and u' are true twins, $G \backslash \{v, u\}$ is connected, and v has a neighbor in $V(G) \backslash \{u\}$. Then by applying the induction hypothesis, $G \backslash \{u, v\}$ has a true twin-set B' such that $N_G(v) \backslash \{u\}$ is equal to B' or $N_G[B'] \backslash \{u, v\}$. Let

$$B = \begin{cases} B' \cup \{u\} & \text{if } u' \in B', \\ B' & \text{otherwise.} \end{cases}$$

Since u and u' are true twins in $G \backslash v$, B is a true twin-set of $G \backslash v$. It is easy to see that if $N_G(v) \backslash \{u\} = B'$, then $N_G(v) = B$, because u and u' are true twins in G. If $N_G(v)\{u\} = N_G[B'] \backslash \{u, v\}$, then $N_G[B'] \backslash \{v\} = N_G[B] \backslash \{v\}$ by the same reason.

Thus, we may assume that $G \backslash v$ has no true twins. By Theorem 2.4, since $G \backslash v$ is connected, $G \backslash v$ is a tree. Then by Lemma 6.7, the statement holds. □

Proof of Proposition 6.6. Let C be an incomplete component of $G\backslash S$ with a tree-clique decomposition $(F, \{B_u : u \in V(F)\})$. Since S is a good modulator of G, $G[V(C) \cup \{v\}]$ is a 3-leaf power for each vertex v in S. Thus, if S has a vertex w having a neighbor in a bag B of C, then $\{w\}$ is complete to B by Lemma 6.2. This means that every bag of C is a true twin-set in G. Since (R5) is not applicable to (G, k), each bag of C contains at most $k + 1$ vertices. Therefore, in the remaining of this proof, we are going to bound the number of bags of C.

Claim 1. We claim that the maximum degree of F is at most $|S| + 2k + 6$.

Suppose that F has a node u of degree at least $|S| + 2k + 7$ in F. For each vertex w in S, if at least two components of $C\backslash B_u$ have neighbors of w, then all components of $C\backslash B_u$ have neighbors of w by Corollary 6.8. Thus, for each vertex w in S, we can choose a component of $C\backslash B_u$ such that either all other components of $C\backslash B_u$ have neighbors of w, or no other components of $C\backslash B_u$ have neighbors of w. Since $C\backslash B_u$ has at least $|S|+2k+7$ components, $C\backslash B_u$ has distinct components $D_1, \ldots, D_{2k+7}$ such that for each vertex w in S, either all or none of them have neighbors of w. Thus, $N_G(V(D_1)) = \cdots = N_G(V(D_{2k+7}))$. By the pigeonhole principle, $V(D_i) \subseteq N_G(B_u)$ or $\varnothing \neq V(D_i) \cap N_G(B_u) \neq V(D_i)$ is satisfied by at least $k + 4$ values of i, contradicting the assumption that (R7) is not applicable to (G, k), and this proves the claim.

Let X be the set of leaves of F whose bags are anti-complete to S.

Claim 2. We claim that if u is a node of $F\backslash X$ having degree at most 1 in $F\backslash X$, then B_u contains a neighbor of S.

If $N_F(u) \subseteq X$, then B_u contains a neighbor of S, because otherwise C has no neighbors of S, and (R2) is applicable to (G, k). If $N_F(u)\backslash X$ is non-empty, then $N_F(u)\backslash X$ contains exactly one node u_1, because u has degree at most 1 in $F\backslash X$. If B_u contains no neighbors of S, then (R6) is applicable to (G, k) by taking B_{u_1} as B. Therefore, B_u contains a neighbor of S, and this proves the claim.

For each vertex v in S, let X_v be the set of nodes of $F\backslash X$ whose bags contain neighbors of v, S_1 be the set of vertices v in S such that X_v contains some leaf of $F\backslash X$, and $S_2 := S\backslash S_1$. Let F' be a tree obtained from $F\backslash X$ by contracting all edges in $F[X_v]$ for each vertex v in S. Note that F' has at most $|S_1|$ leaves, and therefore it has at most $\max(|S_1| - 2, 0)$ branching nodes. Let Y be the set of nodes of F' which come form X_v for some vertex $v \in S$, and Z be the set of branching nodes of F'. Then $|Y \cup Z| \leqslant |Y| + |Z| \leqslant |S| + \max(|S_1| - 2, 0) \leqslant 2|S|$. Since (R8) is not applicable to (G, k), each component of $F'\backslash(Y \cup Z)$ has at most three nodes. Therefore, $|V(F'\backslash(Y \cup Z))| \leqslant 6|S|$. Then $|V(F\backslash X)|$ is at most

$$\begin{aligned} |Y|(|S| + 2k + 7) + |Z| + |V(F'\backslash(Y \cup Z))| &\leqslant |S|(|S| + 2k + 7) + |S| + 6|S| \\ &= |S|(|S| + 2k + 14). \end{aligned}$$

Since (R7) is not applicable to (G, k), each node of $F\backslash X$ is adjacent to at most $k + 3$ nodes in X. Thus, $|V(F)| \leqslant (k + 4)|S|(|S| + 2k + 14)$. By (R5), each bag of C has at most $k + 1$ nodes. Therefore, $|V(C)| \leqslant (k + 1)(k + 4)|S|(|S| + 2k + 14)$. □

Chapter 7. A proof of the main theorem

We prove Theorem 1.2 by presenting a kernel with $O(k^{14}\log^{12} k)$ vertices.

Algorithm 1 Kernelization for 3-Leaf Power Deletion

1: **function** Compress(G, k)
2: **if** $k = 0$ **then**
3: **if** G is a 3-leaf power **then return** $(K_1, 0)$.
4: **else return** $(K_{2,2}, 0)$.
5: **end if**
6: **else** Find an instance (G', k') equivalent to (G, k), and a good modulator S of G' having size $O(\min(k^3 \log k, k^2 \log^2 |V(G)|))$ by Lemma 4.5.
7: **if** $|S| \leqslant k$ **then return** $(K_1, 0)$.
8: **else if** $(G', k') \neq (G, k)$ **then return** Compress(G', k').
9: **else if** Rule (Ri) for some $i \in \{2, \ldots, 8\}$ is applicable to (G, k) **then return** Compress(G'', k'') where (G'', k'') is the resulting instance obtained from (G, k) with S by applying the rule (Ri).
10: **else return** (G, k).
11: **end if**
12: **end if**
13: **end function**

Proof of Theorem 1.2. By Theorem 2.2, we can find all maximal true twin-sets in a 3-leaf power in linear time. Thus, we can apply (R2), ..., (R8) in polynomial time to an input instance (G, k) with a good modulator S of G by investigating small subsets of $V(G)$ or true twin-sets in $G \backslash S$. Therefore, Algorithm 1 is a polynomial-time algorithm.

We claim that for the instance (G, k) obtained in Line 10, G has at most $O(k^2|S|^6)$ vertices. Note that $|S| \geqslant k + 1$. By Proposition 5.6, $G \backslash S$ has at most $2(k + 2)|S|^2$ non-trivial components. By Proposition 6.1, each complete component of $G \backslash S$ has at most $2(k+3)|S|^4/3$ vertices. By Proposition 6.6, each incomplete component of $G \backslash S$ has at most $(k+1)(k+4)|S|(|S|+2k+14)$ vertices. Therefore, each non-trivial component of $G \backslash S$ has at most $O(k|S|^4)$ vertices. Then the union of all non-trivial components of $G \backslash S$ has at most $2(k + 2)|S|^2 \cdot O(k|S|^4) = O(k^2|S|^6)$ vertices. By Proposition 5.7, $G \backslash S$ has at most $2(k + 3)|S|^4/3$ isolated vertices. Therefore, $|V(G)| \leqslant |S| + 2(k + 3)|S|^4/3 + O(k^2|S|^6) = O(k^2|S|^6)$, and this proves the claim.

For an instance (G, k) obtained from Line 10, $|S|$ is at most

$$
\begin{aligned}
O(\min(k^3 \log k, k^2 \log^2 |V(G)|)) &\leqslant O(\min(k^3 \log k, k^2 \log^2 (k^2|S|^6))) \\
&\leqslant O(\min(k^3 \log k, k^2 \log^2 (k^2(k^3 \log k)^6))) \\
&\leqslant O(\min(k^3 \log k, k^2 \log^2 k)) = O(k^2 \log^2 k)
\end{aligned}
$$

by Lemma 4.5 and the claim. Therefore, G has at most $O(k^{14}\log^{12} k)$ vertices. □

Chapter 8. Conclusions

In this paper, we show that 3-Leaf Power Deletion admits a kernel with $O(k^{14} \log^{12} k)$ vertices. It would be an interesting problem to reduce the size of the kernel for 3-Leaf Power Deletion.

For an integer $\ell \geqslant 4$, one may investigate about ℓ-Leaf Power Deletion, that is a problem of deciding whether there is a vertex set of size at most k whose deletion makes an input graph an ℓ-leaf power. There are linear-time algorithms to recognize 4- and 5-leaf powers [10, 12], and there is a polynomial-time algorithm to recognize 6-leaf powers [21]. Moreover, there are linear-time algorithms to recognize ℓ-leaf powers for $\ell \geqslant 7$ on graphs of bounded degeneracy [23].

Gurski and Wanke [25] stated that for each ℓ, ℓ-leaf powers have bounded clique-width. Rautenbach [41] presented a characterization of 4-leaf powers with no true twins as chordal graphs with ten forbidden induced subgraphs. This can be used to express, in monadic second-order logic, whether a graph is a 4-leaf power and whether there is a vertex set of size at most k whose deletion makes the graph a 4-leaf power. Therefore, by using the algorithm in [14], we deduce that 4-Leaf Power Deletion is fixed-parameter tractable when parameterized by k, and therefore it admits a kernel. It is natural to ask whether 4-Leaf Power Deletion admits a polynomial kernel. For $\ell \geqslant 5$, we do not know whether we can express ℓ-leaf powers in monadic second-order logic. If it is true for some ℓ, then not only ℓ-Leaf Power Deletion is fixed-parameter tractable, but also ℓ-Leaf Power Recognition can be solved in polynomial time, which is still open for $\ell \geqslant 7$.

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항상 수학적 조언과 따뜻한 격려를 아끼지 않으시고 물심양면으로 지원해주시는 엄상일 교수님께 진심으로 감사드립니다. 교수님의 추천으로 매개화 알고리즘 이론을 공부하게 되었고, 이를 계기로 연구의 방향을 잡게 되었습니다. 논문을 준비하는 과정에서 교수님과 늦은 시간까지 토론을 하곤 했는데, 그 덕에 석사 논문을 잘 마무리할 수 있었습니다. 교수님과의 토론 속에서 어떻게 사고를 논리적이며 동시에 창의적으로 전개해야 하는지에 대해 배울 수 있었습니다. 또한, 교수님은 제가 부족한 모습을 보일 때마다 고쳐야 할 점과 나아가야 할 방향을 가르쳐주셨습니다. 제가 더 나은 수학자가 될 수 있게 지도해주시는 엄상일 교수님께 다시 한 번 감사드립니다.

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