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계수가 주어진 델타매트로이드와 쌍선형 형식이 주어진 선형공간

Delta-matroids with coefficients and linear spaces equipped with a bilinear form

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한 국 과 학 기 술 원

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Delta-matroids with coefficients and linear spaces equipped with a bilinear form

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Co-Advisor: Sang-il Oum

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초 록

계수가 주어진 매트로이드에 대한 이론을 라그랑지안 직교/심플렉틱 그래스만니안과 관련하여 두 가지 방법으로 확장한다. 첫째는 계수가 주어진 직교 매트로이드이고, 다른 하나는 계수가 주어진 반대칭 매트로이드이다.

직교 매트로이드는 반대칭 행렬과 닫힌 유향 곡면 위에서 정의된 그래프의 조합적 성질을 포착한다. 우리는 계수가 주어진 직교 매트로이드에 대한 여러 동등한 정의를 제공하며, 이는 라그랑지안 직교 그래스마니안의 사영 공간으로의 매개화인 Wick 매장을 일반화한다. 응용으로 직교 매트로이드에 대한 여러 표현성 정리를 얻으며, Farkas 보조정리의 유향 직교 매트로이드로의 확장 또한 얻는다. 추가적으로 우리는 3-연결성을 유지하면서 이진 직교 매트로이드의 크기를 줄이는 필요충분 조건을 제시한다.

우리는 반대칭 매트로이드라는 매트로이드를 확장하는 새로운 개념을 도입한다. 계수가 주어진 반대칭 매트로이드를 두 가지 동등한 방법으로 제시하며, 두 정의의 동등성은 라그랑지안 심플렉틱 그래스만니안을 매개화하는 방법을 내포한다. 증명은 반대칭 매트로이드에 대한 호모토피 정리를 수반하는데, 해당 정리는 Maurer의 매트로이드에 대한 호모토피 정리를 일반화한다.

핵심낱말 그래프, 매트로이드, 계수가 주어진 매트로이드, Baker-Bowler 이론, 델타매트로이드, (라그랑지안) 직교 매트로이드, 그래스마니안, 라그랑지안 직교/심플렉틱 그래스마니안, Maurer의 호모토피 정리, Tutte의 바퀴 정리

Abstract

We extend theory on matroids with coefficients in two different manners regarding the Lagrangian orthogonal/symplectic Grassmannians, which are (1) orthogonal matroids with coefficients and (2) antisymmetric matroids with coefficients. Our theory is closely related to delta-matroids, which capture combinatorial properties of symmetric and skew-symmetric matrices and graphs embedded in closed surfaces, whereas matroids capture common properties of linear spaces and graphs.

Orthogonal matroids, equivalent to even delta-matroids, are the most important subclass of delta-matroids, which capture combinatorial properties of skew-symmetric matrices and graphs embedded in closed orientable surfaces. Wenzel, in the 1990s, introduced orthogonal matroids with coefficients in fuzzy rings by making use of the Wick relations. The Wick relations are the quadratic identities for pfaffians of skew-symmetric matrices, which generalize Grassmann-Plücker relations. We extend Wenzel's idea and present several cryptomorphic definitions of orthogonal matroids with coefficients. This cryptomorphism generalizes the Wick embedding, a parameterization of the Lagrangian orthogonal Grassmannian into a projective space. As applications, we show several theorems on the representability of orthogonal matroids and we present Farkas' Lemma for oriented orthogonal matroids. Additionally, we investigate necessary and sufficient conditions to reduce the size of binary orthogonal matroids under preserving 3-connectivity.

We introduce a new matroid-like object called antisymmetric matroids, which generalizes matroids and orthogonal matroids and is compatible with delta-matroids. We establish antisymmetric matroids with coefficients in two different ways, extending a parameterization of the Lagrangian symplectic Grassmannian by Boege et al. in 2019. Our proof of the cryptomorphism involves the homotopy theorem for graphs associated with antisymmetric matroids, which generalizes both Maurer's Homotopy Theorem for matroids and Wenzel's Homotopy Theorem for orthogonal matroids.

Keywords Graph, Matroid, Matroids with coefficients, Baker-Bowler theory, Delta-matroid, (Lagrangian) orthogonal matroid, Grassmannian, Lagrangian orthogonal/symplectic Grassmannian, Maurer's homotopy theorem, Tutte's wheel theorem

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Chapter 1. Introduction

Whitney [121] introduced matroids in 1935 as a combinatorial abstraction of both linear spaces and graphs. Matroids have been extensively studied and are closely related to various fields, such as graph theory, combinatorial optimization, and algebraic geometry. They possess many equivalent definitions in terms of bases, circuits, rank function, greedy algorithm, and many more. In this thesis, we study delta-matroids. Delta-matroids generalize matroids and capture common properties of following objects:

Matchings in graphs For a given graph, the family of vertex sets that are exactly the endvertices of some matching forms a delta-matroid.

(Skew-)symmetric matrices Let A be an $n \times n$ symmetric or skew-symmetric matrix. Then the family of subsets $X \subseteq [n]$ such that the principal submatrix $A[X]$ indexed by X is nonsingular forms a delta-matroid.

Graphs embedded on closed surfaces Let $G = (V, E)$ be a graph cellularly embedded on a closed surface Σ. Then the family of edge sets $Q \subseteq E$ such that a small ϵ-neighborhood of a subgraph (V, Q) in Σ has a single boundary component forms a delta-matroid.

Delta-matroids are defined several equivalent ways in terms of bases, greedy algorithms [20], circuits [56, 17], base polyotope [18, 19], and so on, which are the notions extending those in matroids. Delta-matroid theory has several analogies of classic results in matroid theory, which sometimes imply the original results on matroids. We see examples below:

Delta-matroid parity problem Geelen, Iwata, and Murota [64] presented an efficient algorithm solving the delta-matroid parity problem for delta-matroids associated with skew-symmetric matrices. This problem generalizes the matroid parity problem that was posed by Lawler [81, Chapter 9] as an extension of the matroid intersection problem and was solved by Lovász [83] for linear matroids.

Excluded-minor characterization of binary delta-matroids Bouchet and Duchamp [37] acomplsihed to characterize the excluded minors of binary delta-matroids. This extends Tutte's excluded-minor characterization of binary matroids [108].

Delta-matroids [20] and equivalent concepts were defined in the late 1980s independently by several researchers under various names such as 'symmetric matroids' [20], 'metroids' [49], and 'pseudomatroids' [41].

Two main topics in this thesis are to extend matroids with coefficients [48, 51, 4] in two directions regarding delta-matroids:

1. Orthogonal matroids with coefficients. (Section 1.2 and Chapter 5)

2. Antisymmetric matroids with coefficients. (Section 1.3 and Chapter 6)

We have another topic to introduce:

3. Variants of Tutte's Wheel Theorem for binary even delta-matroids. (Section 1.4 and Chapter 7)

Here we present some glimpses of these topics.

1. We investigate orthogonal matroids with coefficients in tracts extending both results of Baker and Bowler [4] and Wenzel [116, 119]. An orthogonal matroid is equivalent to an even delta-matroid that is a delta-matroid, all of whose bases have the same parity, whereas a matroid is a delta-matroid, all of whose bases have the same cardinality. Tracts are field-like objects introduced in [4], and the reader may consider fields instead of tracts if they are unfamiliar with those. We present several cryptomorphic definitions of orthogonal matroids with coefficients in a tract F. If $F = \mathbb{F}$ is a field, this cryptomorphism implies the Wick embedding, that is, the parameterization of the Lagrangian orthogonal Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ into the projective space $\mathbb{P}(\mathbb{F}^{2^n})$. As applications, we provide proofs of old and new theorems on the representability of orthogonal matroids. We also generalize Farkas' Lemma for oriented orthogonal matroids, which would be helpful for future studies on orthogonal matroids and ribbon graphs.

2. We introduce a new matroid-like object, called 'antisymmetric matroids,' and develop the theory of antisymmetric matroids with coefficients in tracts. Antisymmetric matroids capture combinatorial properties of nonsingular principal and almost-principal minors of a symmetric matrix, compared to delta-matroids that capture combinatorial properties of nonsingular principal minors of a symmetric matrix, and therefore each antisymmetric matroid naturally induces a delta-matroid. We show two cryptomorphic definitions of antisymmetric matroids with coefficients in tracts, which generalizes the parameterization of the Lagrangian symplectic Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ into the projective space $\mathbb{P}(\mathbb{F}^{2^n+\binom{n}{2}2^{n-2}})$ [13]. We also show that antisymmetric matroids with coefficients are compatible with several other concepts, such as the Lagrangian symplectic Dressian [9] and oriented gaussoids [13], depending on the choice of tracts.

3. Tutte's Wheel Theorem [110] states that every simple 3-connected graph has an edge e such that the deletion $G \setminus e$ or the contraction G/e is simple 3-connected unless G is a wheel graph. We show the following analgous result for binary even delta-matroids, which indeed implies the original result by Tutte.

- Every 3-connected binary even delta-matroid has two distinct elements x_1, x_2 such that the deletion $M \setminus x_i$ or the contraction M/x_i is 3-connected for each $i = 1, 2$, except for trivial cases.

Our theorem is motivated by an open problem: The excluded-minor characterization of regular even delta-matroids. We will describe how our results might potentially be used to address this problem.

We present main theorems of the three topics in Sections 1.2–1.4. Ahead of this, in Section 1.1.1, we look at preliminaries to describe our results. We will see the organization of this thesis in Section 1.5.

1.1 Brief review on matroids and delta-matroids

We arrange minimal terminologies and notions to introduce our results in Sections 1.2–1.4. We will see basics on matroids, delta-matroids, Grassmannian, Lagrangian orthogonal/symplectic Grassmannian, and tracts in order through Subsections 1.1.1–1.1.6. The readers who are familiar with these concepts may skip this section.

Let $[n] := \{1, 2, \dots, n\}$ for a positive integer n. We denote the symmetric difference of two sets X and Y by $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$. For a set E and an integer $0 \le r \le E$, we denote by 2^E the family of all subsets of E and denote by $\binom{E}{r}$ the family of all r-element subsets of E. Let $E^* := \{x^* : x \in E\}$ be a disjoint copy of E. We denote by $(x^*)^* = x$. A *skew pair* is a 2-element subset of $E \cup E^*$ of the form $\{x, x^*\}$. A *transversal* is an $|E|$-element subset of $E \cup E^*$ which intersects each skew pair in exactly one element, and a *subtransversal* is a subset of a transversal. An *almost-transversal* is an $|E|$-element subset of $E \cup E^*$ which contains exaclty one skew pair. We denote by $\mathcal{T}_n$ (resp. $\mathcal{A}_n$) the family of transversals (resp. almost-transversals) in $[n] \cup [n]^*$.

We usually denote a field by $\mathbb{F}$. We denote the transpose of a matrix A by A^t. For an $r \times n$ matrix A with $r \leq n$ and $S \in \binom{[n]}{r}$, we denote by $A[S]$ the $r \times r$ submatrix of A induced by the r columns indexed by S. For an $n \times n$ square matrix A and $S \subseteq [n]$, we denote by $A[S]$ the principal submatrix of A induced by the rows and columns indexed by S. A square matrix is *principally unimodular* (in short, *PU*) if it is real and the determinants of all principal submatrices are 0 or ± 1.

1.1.1 Matroids

Definition 1.1.1 (Matroids)**.** A *matroid* is a pair $M = (E, \mathcal{B})$ such that E is a finite set and $\mathcal{B}$ is a nonempty family of subsets of E satisfying the *base exchange axiom*:

(B) For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

We call $E(M) := E$ the *ground set* of M and call each element of $\mathcal{B}$ a *base* of M. A *circuit* is a minimal subset of E that is not a subset of any base.

Brualdi [39] showed that every matroid satisfies the following *strong* base exchange property:

(B′) For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 \setminus \{x\} \cup \{y\}$ and $B_2 \setminus \{y\} \cup \{x\}$ are in $\mathcal{B}$.

Hence, the *dual* $M^\perp := (E, \mathcal{B}^\perp)$ of a matroid M is also a matroid, where $\mathcal{B}^\perp := \{E \setminus B : B \in \mathcal{B}\}$. A *cocircuit* of M is a circuit of the dual $M^\perp$.

The *deletion* of an element $e \in E$ from a matroid $M = (E, \mathcal{B})$ is a pair $M \setminus e := (E \setminus \{e\}, \mathcal{B} \setminus e)$ where

$$\mathcal{B} \setminus e := \begin{cases} \{B : e \notin B \in \mathcal{B}\} & \text{if there is a base not containing } e, \\ \{B \setminus \{e\} : B \in \mathcal{B}\} & \text{otherwise.} \end{cases}$$

Then $M \setminus e$ is a matroid. The *contraction* of e from M is $M/e := (M^\perp \setminus e)^\perp$. A matroid M is a *minor* of another matroid N if M can be obtained from N by a sequence of deletions and contractions.

Matroids have a cryptomorphic definition in terms of circuits; see [98, Lemma 1.1.3 and Theorem 1.1.4].

Proposition 1.1.2 (see [98])**.** *Let $\mathcal{C}$ be a family of subsets of a finite set. Then $\mathcal{C}$ is the family of circuits of a matroid if and only if it satisfies the following conditions:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *For $C_1, C_2 \in \mathcal{C}$, $C_1 \subseteq C_2$ implies $C_1 = C_2$.*

(Elim) *For $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$.*

We call the last condition (Elim) the *circuit elimination axiom*.

Matroids capture common properties of both graphs and linear spaces.

Example 1.1.3 (Matroids from graphs)**.** For a graph $G = (V, E)$, the family of maximal acyclic edge sets of G is the family of bases of a matroid on E. This matroid is called the *cycle matroid* of G and is denoted by $M(G)$. The circuits of $M(G)$ are exactly the cycles of G.

Example 1.1.4 (Matroids from linear spaces)**.** Let $0 \leq r \leq n$ be integers and let A be a full-rank $r \times n$ matrix over a field $\mathbb{F}$, whose columns are indexed by $[n] = \{1, 2, \dots, n\}$. Then the family of sets of linearly independent r columns forms the family of bases of a matroid on $[n]$. We denote such a matroid by $M(A)$. Then the circuits of the dual matroid $M(A)^{\perp}$ are exactly the minimal supports of nonzero vectors in the row-space of A.

A matroid is *representable over a field* $\mathbb{F}$ or $\mathbb{F}$*-representable* if it is isomorphic to $M(A)$ for some matrix defined over $\mathbb{F}$. A matroid is *regular* if it is representable over all fields.

1.1.2 Delta-matroids

Definition 1.1.5 (Delta-matroids)**.** A *delta-matroid* is a pair $M = (E, \mathcal{B})$ such that E is a finite set and $\mathcal{B}$ is a nonempty family of subsets of E satisfying the *base exchange axiom*:

($\triangle$B) For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \triangle B_2$, there exists $y \in B_2 \triangle B_1$ (possibly, $x = y$) such that $B_1 \triangle \{x, y\} \in \mathcal{B}$.

The *ground set* of M is $E(M) := E$. Each element in $\mathcal{B}$ is called a *base*.

We note that a matroid is exactly a delta-matroid whose bases have the same cardinality. A delta-matroid is *even* if all bases have the same parity. In contrast to matroids, delta-matroid does not satisfy the strong base exchange property; see Example 1.1.8, but even delta-matroids do by Wenzel [115].

Proposition 1.1.6 ([115])**.** *Let $M = (E, \mathcal{B})$ be a pair of a finite set E and a nonempty family $\mathcal{B}$ of subsets of E. Then M is an even delta-matroid if and only if the following condition holds:*

($\triangle$B′) *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \triangle B_2$, then there is $y \in (B_1 \triangle B_2) \setminus \{x\}$ such that both $B_1 \triangle \{x, y\}$ and $B_2 \triangle \{x, y\}$ are in $\mathcal{B}$.*

For a delta-matroid $M = (E, \mathcal{B})$ and a subset $X \subseteq E$, let $\mathcal{B} \triangle X := \{B \triangle X : B \in \mathcal{B}\}$ and then $M \triangle X := (E, \mathcal{B} \triangle X)$ is a delta-matroid. We call this operation *twisting on* X. For $e \in E$, the *deletion of e from M* is a pair $M \setminus e := (E \setminus \{e\}, \mathcal{B} \setminus e)$ where $\mathcal{B} \setminus e := \{B : e \notin B \in \mathcal{B}\}$ if there is a base not containing e, and $\mathcal{B} \setminus e := \{B \setminus \{e\} : B \in \mathcal{B}\}$ otherwise. Then $M \setminus e$ is a delta-matroid. The *contraction of e from M* is a delta-matroid $M/e := (M \triangle \{e\}) \setminus e$. A delta-matroid M is a *strong-minor* of another delta-matroid N if M can be obtained from N by a sequence of deletions and contractions, and M is a *minor* of N if it is a twist of a strong-minor of N.

A delta-matroid $M = (E, \mathcal{B})$ is *connected* if there is no pair of delta-matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ such that E is partitioned into nonempty sets E_1 and E_2, and $\mathcal{B} = \{B_1 \cup B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\}$. A delta-matroid $M = (E, \mathcal{B})$ is 3*-connected* if it is connected and there is no pair of delta-matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ such that $|E_1 \cap E_2| = 1$, $\min\{|E_1 \setminus E_2|, |E_2 \setminus E_1|\} \geq 2$, $E = E_1 \triangle E_2$, and $\mathcal{B} = \{B_1 \triangle B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2 \text{ such that } B_1 \cap E_2 = B_2 \cap E_2\}$; see [61, Page 28].

The *fundamental graph* of a delta-matroid $M = (E, \mathcal{B})$ with respect to a base B is a graph whose vertex set is E, a vertex x has a loop if $B \triangle \{x\} \in \mathcal{B}$, and two distinct vertices x and y are adjacent if $B \triangle \{x, y\} \in \mathcal{B}$.

We give examples of delta-matroids arising from matrices and graphs.

Example 1.1.7 (Delta-matroids from (skew-)symmetric matrices)**.** Let A be an $n \times n$ symmetric or skew-symmetric matrix over a field $\mathbb{F}$ and let $\mathcal{B} := \{X \subseteq [n] : \det(A[X]) \neq 0\}$. Then $M(A) := ([n], \mathcal{B})$ is a delta-matroid [25].

A delta-matroid is *representable over* $\mathbb{F}$ or $\mathbb{F}$*-representable* if it isomorphic to $M(A)\triangle X$ for some symmetric or skew-symmetric matrix A over $\mathbb{F}$ and a set X. We simply call $\mathbb{F}_2$-representable delta-matroids *binary* delta-matroids, where $\mathbb{F}_2$ is the field with two elements. An even delta-matroid is *regular* if it is representable over all fields.

Example 1.1.8 (A delta-matroid with no strong base exchange)**.** Let $M = ([3], \{\emptyset, \{1\}, \{2\}, \{3\}, [3]\})$. Then it is a non-even delta-matroid. It does not satisfy the strong exchange property: For bases $B_1 = \emptyset$ and $B_2 = [3]$ and an element $e = 1 \in B_1 \triangle B_2$, either $B_1 \triangle \{e, f\} \notin \mathcal{B}$ or $B_2 \triangle \{e, f\} \notin \mathcal{B}$ for each $f \in B_1 \triangle B_2$. We note that M is non-binary and is representable over any field $\mathbb{F}$ of characteristic not two. More precisely, $M = M(A)$ where A is the 3×3 symmetric matrix over $\mathbb{F}$ whose diagonal entries are 1 and non-diagonal entries are -1.

Example 1.1.9 (Matching delta-matroids)**.** Let $G = (V, E)$ be a graph. A matching is an edge set in which no two edges share a vertex, and a vertex set $W \subseteq V$ is *matchable* if there is a matching of which endvertices are exactly W. Then a pair $(V, \{W \subseteq V : W \text{ is matchable in } G\})$ is an even delta-matroid [28].

Example 1.1.10 (Delta-matroids from embedded graphs)**.** Let G be a connected graph cellularly embedded on a closed surface Σ and let G^* be the dual graph whose edge set is $E(G)^*$. A subgraph Q of G is a *quasi-tree* if a small ϵ-neighborhood of Q in Σ has a single boundary. Note that every tree of G is a quasi-tree; conversely, if Σ is the sphere, then every quasi-tree of G is a tree. We say a subgraph Q is *spanning* if it contains all vertices of G. Then $M(G; \Sigma) := (E(G), \mathcal{B})$ is a delta-matroid [27], where $\mathcal{B}$ is the family of the edge sets of spanning quasi-trees of G. By the previous observation, $M(G; \Sigma) = M(G)$ if the surface Σ is the sphere.

The *lift* of a delta-matroid $M = (E, \mathcal{B})$ is a pair $\operatorname{lift}(M) := (E \cup E^*, \operatorname{lift}(\mathcal{B}))$ where $\operatorname{lift}(\mathcal{B}) := \{B \cup (E \setminus B)^* : B \in \mathcal{B}\}$ [33, Construction 2.13]. We call each element of $\operatorname{lift}(\mathcal{B})$ a *base* of the lift. A *circuit* of $\operatorname{lift}(M)$ is a minimal subtransversal that is not contained in any base. Booth, Moreira, and Pinto [17] presented the following circuit axiom for lifts of delta-matroids.

Proposition 1.1.11 ([17])**.** *Let E be a finite set and let $\mathcal{C}$ be a family of subtransversals of $E \cup E^*$. Then $\mathcal{C}$ is the family of circuits of the lift of a delta-matroid on E if and only if it satisfies* (C1), (C2), *and the following conditions:*

($\triangle$Elim) *For $C_1, C_2 \in \mathcal{C}$ such that $C_1 \cup C_2$ is a subtransversal and $e \in C_1 \cap C_2$, there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*

($\triangle$Orth) *For $C_1, C_2 \in \mathcal{C}$, if $C_1 \cup C_2$ is not a subtransversal, then $C_1 \cup C_2$ contains at least two skew pairs.*

Example 1.1.12. Let M be a matroid on E. Then the family of circuits of $\operatorname{lift}(M)$ is exactly the union of $\{C \subseteq E : C \text{ is a circuit of } M\}$ and $\{D^* \subseteq E^* : D \text{ is a cocircuit of } M\}$.

Example 1.1.13 (Example 1.1.7 continued)**.** Let V be the row-space of an $n \times 2n$ matrix $\Lambda := [A \mid I_n]$, where I_n is the $n \times n$ identity matrix and the first n columns are indexed by $1, 2, \ldots, n$ and the last n columns are indexed by $1^*, 2^*, \ldots, n^*$. Then the family of bases of $\operatorname{lift}(M(A))$ is $\{B \subseteq [n] \cup [n]^* : B \text{ is a transversal and } \det(\Lambda[B]) \neq \emptyset\}$. The family of circuits of $\operatorname{lift}(M(A))$ is the family of minimal subtransversals C^* such that the sets C are the supports of nonzero vectors in V.

Example 1.1.14 (Example 1.1.10 continued)**.** Each circuit of $\operatorname{lift}(M(G; \Sigma))$ is the edge set of a union X of cycles $C_1, \ldots, C_k$ in G and cycles $C_1^*, \ldots, C_\ell^*$ in the dual G^* such that $E(X)$ is a subtransversal and it minimally separates Σ, i.e., $\Sigma \setminus X$ consists of two connected components and $\Sigma \setminus X'$ is connected for every proper subgraph X' of X [5].

1.1.3 Grassmannian

Let $0 \leq r \leq n$ be integers. The *Gramssannian* $\mathrm{Gr}_{\mathbb{F}}(r,n)$ is the set of all r-dimensional linear subspaces of $\mathbb{F}^n$.

Theorem 1.1.15 (see [79, Section 4.1]). *The Grassmannian* $\mathrm{Gr}_{\mathbb{F}}(r,n)$ *is parameterized into the projective space* $\mathbb{P}(\mathbb{F}^{\binom{n}{r}})$ *by the* Plücker embedding*:*

$$V \mapsto \big(\det(A[B])\big)_{B \in \binom{[n]}{r}}$$

where A is an $r \times n$ matrix whose rows span V. The image is set-theoretically cut out by the Grassmann-Plücker relations*:*

$$\sum_{x \in S \setminus T} (-1)^{m_x} X_{S\setminus\{x\}} X_{T\cup\{x\}} = 0 \text{ for all } S \in \binom{[n]}{r+1} \text{ and } T \in \binom{[n]}{r-1}, \tag{GP}$$

where m_x means the number of elements in the symmetric difference $S \triangle T$ that are less than x.

The simplest nontrivial Grassmann-Plücker relations occur when $|S \setminus T| = 3$. These are called the 3*-term Grassmann-Plücker relations* and can be expressed as follows:

$$X_{R\cup\{i_1,i_2\}} X_{R\cup\{i_3,i_4\}} - X_{R\cup\{i_1,i_3\}} X_{R\cup\{i_2,i_4\}} + X_{R\cup\{i_1,i_4\}} X_{R\cup\{i_2,i_3\}} = 0 \tag{3GP}$$

with $R \in \binom{[n]}{r-2}$ and $i_1 < i_2 < i_3 < i_4$ in $[n] \setminus R$.

1.1.4 Pfaffian, Wick relations, and Lagrangian orthogonal Grassmannian

Definition 1.1.16. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ skew-symmetric matrix, which means that $a_{ij} = -a_{ji}$ and $a_{ii} = 0$ for all i, j. The *pfaffian* of A is defined as

$$\mathrm{pf}(A) := \sum_{\sigma} (-1)^{\mathrm{cr}(\sigma)} \prod_{i \in [n]\,:\, i < \sigma(i)} a_{i,\sigma(i)}$$

where the summation is taken over all fixed-point-free involutions σ on $[n]$ and $\mathrm{cr}(\sigma)$ is the number of pairs of $i, j \in [n]$ such that $i < j < \sigma(i) < \sigma(j)$.

By convention, $\mathrm{pf}(A) = 1$ if $n = 0$. If n is odd, then $\mathrm{pf}(A) = 0$ by definition. It is well known that $\det(A) = \mathrm{pf}(A)^2$, called Cayley's theorem; see [90, Proposition 7.3.3]. Taking $X_B := \mathrm{pf}(A[B])$ for $B \subseteq [n]$, we have

$$\sum_{k=1}^{m} (-1)^k X_{T_1 \triangle \{e_k\}} X_{T_2 \triangle \{e_k\}} = 0 \quad \text{for all } T_1, T_2 \subseteq [n],$$

where $\{e_1, \ldots, e_m\} = T_1 \triangle T_2$ with $e_1 < \ldots < e_m$ [116]. We call these quadratic relations the *Wick relations*; see [90, Subsection 7.3.2]. Conversely, we have the following.

Theorem 1.1.17 ([116]). *Let X be a vector in $\mathbb{F}^{2^{[n]}}$ such that $X_\emptyset = 1$ and it satisfies all Wick relations. Then there is a skew-symmetric matrix A such that $X_B = \mathrm{pf}(A[B])$ for all $B \subseteq [n]$.*

In the remaining of Subsection 1.1.4, we assume the field $\mathbb{F}$ has characteristic not two. Let $\mathbb{F}^{2n}$ be an $2n$-dimensional vector space of which coordinates are indexed by $1, \ldots, n, 1^*, \ldots, n^*$ and which is equipped with the standard symmetric bilinear form $\eta(X, Y) := \sum_{i=1}^{n}(X_i Y_{i^*} + X_{i^*} Y_i)$. A subspace V of $\mathbb{F}^{2n}$ is *Lagrangian* if $V = V^\perp$ where $V^\perp$ is the orthogonal complement of V with respect to $\eta(\cdot, \cdot)$, equivalently, $V^\perp := \{X \in \mathbb{F}^{2n} : \eta(X, Y) = 0 \text{ for all } Y \in V\}$. Note that every Lagrangian subspace has n-dimensional; see [80, Theorem 6.4]. The *Lagrangian orthogonal Grassmannian* $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ is the set of all Lagrangian subspaces in $\mathbb{F}^{2n}$ equipped with $\eta(\cdot, \cdot)$.

Lemma 1.1.18 (folklore; see [19, Lemma 3.4.1] for a proof idea)**.** *Let A_1 and A_2 be $n \times n$ matrices. Then the row-space of $[A_1 \mid A_2]$ is Lagrangian with respect to $\eta(\cdot, \cdot)$ if and only if $A_1 A_2^t$ is skew-symmetric.*

Theorem 1.1.19 (Wick embedding; [116, 50], see also [90, Section 7.3.2])**.** *Let $w : \mathrm{OGr}_{\mathbb{F}}(n, 2n) \to \mathbb{P}(\mathbb{F}^{2^n})$ be a map such that*

$$V \mapsto (\mathrm{pf}(A[B \setminus B_0]) : B \in \mathfrak{T}_n)$$

where $B_0 = \{b_1, \ldots, b_n\} \in \mathfrak{T}_n$ is a transversal such that $b_i \in \{i, i^*\}$ for all $i \in [n]$ and A is a skew-symmetric such that V is the row-space of $[I \mid A]$ of which columns are indexed in order $b_1, \ldots, b_n, b_1^*, \ldots, b_n^*$. Then w is injective and its image is set-theoretically cut out by the Wick relations*:*

$$\sum_{k=1}^{m}(-1)^k X_{T_1 \triangle \{e_k, e_k^*\}} X_{T_2 \triangle \{e_k, e_k^*\}} = 0 \quad \text{for all } T_1, T_2 \in \mathfrak{T}_n \tag{Wick}$$

where $\{e_1, \ldots, e_m\} = (T_1 \triangle T_2) \cap [n]$ with $e_1 < \cdots < e_m$.

1.1.5 Lagrangian symplectic Grassmannian

Let $\mathbb{F}^{2n}$ be an $2n$-dimensional vector space of which coordinates are indexed by $1, \ldots, n, 1^*, \ldots, n^*$ and which is equipped with the standard antisymmetric bilinear form $\omega(X, Y) := \sum_{i=1}^{n}(X_i Y_{i^*} - X_{i^*} Y_i)$. A subspace V of $\mathbb{F}^{2n}$ is *Lagrangian* if it coincide with its orthogonal complement $V^\perp$ with respect to $\omega(\cdot, \cdot)$. The *Lagrangian symplectic Grassmannian* $\mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ is the set of all Lagrangian subspaces in $\mathbb{F}^{2n}$ equipped with $\omega(\cdot, \cdot)$.

Lemma 1.1.20 (folklore; see [19, Lemma 3.4.1])**.** *Let A_1 and A_2 be $n \times n$ matrices. Then the row-space of $[A_1 \mid A_2]$ is Lagrangian with respect to $\omega(\cdot, \cdot)$ if and only if $A_1 A_2^t$ is symmetric.*

We index the cooridnates of the projective space $\mathbb{P}(\mathbb{F}^{2^n + 2^{n-2}\binom{n}{2}})$ by the transversals and almost-transversals of $[n] \cup [n]^*$ along with a condition that $X_{S \cup \{i, i^*\}} = (-1)^{i+j} X_{S \cup \{j, j^*\}}$ for each $(n-2)$-element subtransversal S and two distinct elements $i, j \in [n]$ such that $S \cap \{i, i^*\} = \emptyset = S \cap \{j, j^*\}$.

Theorem 1.1.21 ([13])**.** *Let $\Phi : \mathrm{SpGr}_{\mathbb{F}}(n, 2n) \to \mathbb{P}(\mathbb{F}^{2^n + \binom{n}{2} 2^{n-2}})$ be a map such that*

$$V \mapsto (\det(A[B]) : B \in \mathfrak{T}_n \cup \mathcal{A}_n)$$

where A is an $n \times 2n$-matrix of which row-space is V. Then Φ is well-defined and injective. Moreover, its image is set-theoretically cut out by the quadratic relations of its defining ideal.

1.1.6 Tracts

Definition 1.1.22 ([4])**.** A *tract* $F = (F^\times, N_F)$ is a multiplicatve abelian group $F^\times$ together with an additive relation structure N_F that is a subset of the group semiring $\mathbb{N}[F^\times]$ satisfying:

(T1) The zero element 0 of $\mathbb{N}[F^\times]$ belongs to N_F.

(T2) The identity element 1 of $F^\times$ does not belong to N_F.

(T3) There is a unique element ϵ of $F^\times$ such that $1 + \epsilon \in N_F$.

(T4) If $g \in F^\times$ and $x \in N_F$, then $gx \in N_F$.

We think of N_F as linear combinations of elements of $F^\times$ which 'sum to zero' and call it the *null set* of the tract F.

Lemma 1.1.23 ([4, Lemma 1.1]). *Let $F = (F^\times, N_F)$ be a tract. Then we have the following:*

(i) *If $x, y \in F^\times$ with $x + y \in N_F$, then $y = \epsilon x$.*

(ii) $\epsilon^2 = 1$.

(iii) $F^\times \cap N_F = \emptyset$.

Because of Lemma 1.1.23, we write F for the set $F^\times \cup \{0\}$ and write -1 instead of ϵ. A *tract homomorphism* $\varphi : F_1 \to F_2$ is a map such that $\varphi(0) = 0$, $\varphi(F_1^\times) \subseteq F_2^\times$, and $\varphi : F_1^\times \to F_2^\times$ is a group homomophism which induces the semiring homomorphism $\mathbb{N}[F_1^\times] \to \mathbb{N}[F_2^\times]$ satisfying $\varphi(N_{F_1}) \subseteq N_{F_2}$. The *product* of two tracts F_1 and F_2 is a tract $F_1 \times F_2 := (F_1^\times \oplus F_2^\times, N_{F_1 \times F_2})$ where

$$N_{F_1 \times F_2} := \left\{ \sum_{i=1}^{k} (x_i, y_i) \in \mathbb{N}[F_1^\times \oplus F_2^\times] : \sum_{i=1}^{k} x_i \in N_{F_1} \text{ and } \sum_{i=1}^{k} y_k \in N_{F_2} \right\}.$$

We depict how to understand fields and other related notions as tracts in the following. The readers can find more examples in Section 3.2.

Fields Given a field $\mathbb{F}$, the corresponding tract is a pair $(\mathbb{F}^\times, N_\mathbb{F})$ where $N_\mathbb{F}$ is the set of sums $\sum_{i=1}^{m} x_i \in \mathbb{N}[\mathbb{F}^\times]$ that are zero in $\mathbb{F}$. For instance, $N_{\mathbb{F}_2} = \{0, 1+1, 1+1+1+1, \cdots\}$ and $N_{\mathbb{F}_3} = \{0, 1+(-1), 1+1+1, (-1)+(-1)+(-1), \cdots\}$.

Regular partial field The *regular partial field* $\mathbb{U}_0$ is a set $\{0, \pm 1\}$ together with the addition and multiplication inherited from the integer ring $\mathbb{Z}$. Hence, a sum $(+1)+\ldots+(+1)+(-1)+\ldots+(-1)$ in $\mathbb{U}_0$ is only defined when the numbers of $+1$'s and -1's are only differed by at most one. The tract associated with $\mathbb{U}_0$ is a pair $(\{\pm 1\}, N_{\mathbb{U}_0})$ where $N_{\mathbb{U}_0} \subseteq \mathbb{N}[\{\pm 1\}]$ is the set of the zero element in $\mathbb{N}[\{\pm 1\}]$ and the sums $(+1)+\ldots+(+1)+(-1)+\ldots+(-1)$ such that the numbers of $+1$'s and -1's are the same. There is a natural tract homomorphism $\mathbb{U}_0$ to an arbitrary field $\mathbb{F}$.

Krasner hyperfield The Krasner hyperfield $\mathbb{K}$ is a set of two elements 0 and 1 with the hyperaddition $\boxplus$ (a binary operation that outputs a nonempty set) and the multplication $\odot$ satisfying the following tables:

$\boxplus$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

$\odot$	0	1
0	0	0
1	0	1

The tract associated with $\mathbb{K}$ is a pair $(\mathbb{K}^\times = \{1\}, N_\mathbb{K})$ where $N_\mathbb{K} := \mathbb{N}[\mathbb{K}^\times] \setminus \{1\} = \{0, 1+1, 1+1+1, \cdots\}$. For every tract F, there is a unique tract homomorphism $F \to \mathbb{K}$; that is, $x \mapsto 0$ if $x = 0$ and $x \mapsto 1$ otherwise.

Sign hyperfield The *sign hyperfield* $\mathbb{S}$ is a set of three elements 0, +, and − with the hyperaddition $\boxplus$ and the multiplication $\odot$ satisfying the following tables:

$\boxplus$	0	+	−
0	0	+	−
+	+	+	$\mathbb{S}$
−	−	$\mathbb{S}$	−

$\odot$	0	+	−
0	0	0	0
+	0	+	−
−	0	−	+

where a singleton x in the hyperaddition table means a 1-element set $\{x\}$. The tract associated with $\mathbb{S}$ is a pair $(\mathbb{S}^\times = \{+, -\}, N_\mathbb{S})$ where $N_\mathbb{S} \subseteq \mathbb{N}[\{+,-\}]$ consists of 0 and the sums $(+) + \cdots + (+) + (-) + \cdots + (-)$ such that both + and − appear at least once. We have a canonical tract homomorphism $\mathbb{R} \to \mathbb{S}$; that is, every positive number is mapped to + and every negative number is mapped to −.

Tropical hyperfield The *tropical hyperfield* $\mathbb{T}$ is a set $\mathbb{R}_{\geq 0}$ with the hyperaddition

$$a \boxplus b := \begin{cases} \{\max\{a,b\}\} & \text{if } a \neq b, \\ [0,a] & \text{otherwise,} \end{cases}$$

and the multiplication $a \odot b := ab$. The tract associated with $\mathbb{T}$ is a pair $(\mathbb{T}^\times = \mathbb{R}_{>0}, N_\mathbb{T})$ where a sum $\sum x_i \in \mathbb{N}[\mathbb{R}_{>0}]$ is in $N_\mathbb{T}$ if and only if it is 0 or the maximum of x_i's is achieved at least twice. For any field $\mathbb{F}$ with a valuation $\mathrm{val} : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$; see [84, Page 43] for definition, there is a tract homomorphism $\mathbb{F} \to \mathbb{T}$ by taking $x \mapsto e^{-\mathrm{val}(x)}$. It is due to the property that if $x_1 + \cdots + x_k = 0$ with $x_i \in \mathbb{F}^\times$ and $k \geq 2$, then the minimum $\mathrm{val}(x_i)$'s attains at least twice; which easily follows from [84, Lemma 2.1.1].

1.2 Orthogonal matroids with coefficients

We introduce orthogonal matroids with coefficients in a tract F and present their cryptomorphic definitions. We define *orthogonal matroids*[1] as lifts of even delta-matroids. We define an orthogonal F-matroid as a point satisfying the Wick relations over the tract F. Then, for a field $\mathbb{F}$, an orthogonal $\mathbb{F}$-matroid is equivalent to a skew-symmetric matrix due to Theorem 1.1.17.

For convenience, we will assume the ground set of every orthogonal matroid is $[n] \cup [n]^*$.

Definition 5.1.1. An *orthogonal matroid with coefficients in a tract F* or an *orthogonal F-matroid* is a point X in the projective space $\mathbb{P}(F^{\mathcal{T}_n}) := (F^{\mathcal{T}_n} \setminus \{\mathbf{0}\})/F^\times$ satisfying the Wick relations defined over F:

$$\sum_{k=1}^{m} (-1)^k X_{T_1 \triangle \{e_k, e_k^*\}} X_{T_2 \triangle \{e_k, e_k^*\}} \in N_F \quad \text{for all } T_1, T_2 \in \mathcal{T}_n,$$

where $\{e_1, \ldots, e_m\} = (T_1 \triangle T_2) \cap [n]$ with $e_1 < \ldots < e_m$.

An orthogonal matroid $\underline{M} = ([n], \mathcal{B})$ is *representable over a tract F* or *F-representable* if there is an orthogonal F-matroid M such that $\mathcal{B} = \{B \in \mathcal{T}_n : M_B \neq 0\}$. On the other hand, we call $\underline{M}$ the *underlying orthogonal matroid* of M.

Our main theorem is the following cryptomorphism on orthogonal matroids with coefficients.

[1] Some readers may confuse this terminology since the term 'orthogonal matroids' is used in [19] for a broader concept, and what we call orthogonal matroids in this thesis are exactly 'Lagrangian orthogonal matroids' in [19]. We, however, never use the concept 'orthogonal matroids' defined in [19], so we omit the adjective 'Lagrangian' for simplicity.

Theorem 5.1.18. *There are natural bijections between:*

(i) *Orthogonal F-matroids.*

(ii) *F-circuit sets of orthogonal matroids.*

(iii) *Orthogonal F-signatures.*

(iv) *Orthogonal F-vector sets.*

We remark that orthogonal F-matroids encompass several concepts as follows. First of all, orthogonal F-matroids M are equivalent to F-matroids introduced by Baker and Bowler [4] if $\underline{M}$ are the lifts of matroids. They are also tantamount to other concepts for various choices of F, which is summarized in Table 1.1.

A tract F	Orthogonal F-matroids
the Krasner hyperfiled $\mathbb{K}$	ordinary orthogonal matroids
a field $\mathbb{F}$	skew-symmetric matrices
the regular partial field $\mathbb{U}_0$	principally unimodular skew-symmetric matrices
the tropical hyperfield $\mathbb{T}$	valuated even delta-matroids [52, 116, 119, 89], tropical Wick vectors [102]
the sign hyperfield $\mathbb{S}$	oriented even delta-matroids [116, 119, 14]

Table 1.1: Equivalent concepts of orthogonal F-matroids for various tracts F.

Moreover, Theorem 5.1.18 is a common generalization of the following:

- The cryptomorphism on matroids with coefficients in tracts by Baker and Bowler [4] and Anderson [2].
- The Wick embedding of the Lagrangian orthogonal Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ if $F = \mathbb{F}$ is a field of characteristic not two. (Theorem 1.1.19)

The second item is evident from the definition of orthogonal F-signature along with Example 1.1.13.

Definition 5.1.8. An *orthogonal F-signature* is a set $\mathcal{C}$ of vectors in F^{2n} such that the family of supports of vectors X in $\mathcal{C}$ is the family of circuits of an orthogonal matroid on $[n] \cup [n]^*$, $X \in \mathcal{C}$ implies $c \cdot X \in \mathcal{C}$ for all $c \in F^\times$, and

(O) $\sum_{i=1}^n X_i Y_{i^*} + X_{i^*} Y_i \in N_F$ for all $X, Y \in \mathcal{C}$.

Each vector in an orthogonal F-signature is called an *F-circuit*.

We henceforth examine several applications of our results. First, we look at theorems on the representability of orthogonal matroids. The following observations are handy tools for this topic.

- (Proposition 5.1.21) If there is a tract homomorphism $F_1 \to F_2$, then every F_1-representable orthogonal matroid is F_2-representable.
- (Propositions 5.1.23) If F_1 and F_2 are tracts, then every orthogonal matroid representable over F_1 and F_2 is representable over $F_1 \times F_2$.

Note that there are homomorphisms from the regular partial field $\mathbb{U}_0$ to both the binary field $\mathbb{F}_2$ and the sign hyperfield $\mathbb{S}$, and therefore every $\mathbb{U}_0$-representable (regular) orthogonal matroid is both $\mathbb{F}_2$-representable (binary) and $\mathbb{S}$-representable (orientable). We show the converse holds if we exclude a specific orthogonal matroid as a minor, where the minor relation of orthogonal matroids is naturally induced by that of even delta-matroids. Let $M_4 := ([4], \{X \subseteq [n] : |X| \text{ is even}\})$, which is an even delta-matroid.

Theorem 5.3.2. *Let M be an orthogonal matroid without $\mathrm{lift}(M_4)$-minor. Then M is regular if and only if it is binary and orientable.*

Our result implies a similar result in matroid theory by Bland and Las Vergnas [12] since every matroid has no M_4-minor in the sense of delta-matroids.

Corollary 5.3.5 ([12])**.** *A matroid M is regular if and only if it is binary and orientable.* □

Our proof of Theorem 5.3.2 does not require any result on the excluded minors of regular orthogonal matroids, whereas the original proof of Corollary 5.3.5 in [12] relies on Tutte's excluded-minor characterization of regular matroids. Remark that the excluded-minor characterization of regular orthogonal matroid is a major open problem in delta-matroid theory; see [61, Pages 64–65].

Another application is Farkas' Lemma for oriented orthogonal matroids. Farkas' Lemma is a fundamental theorem in linear programming, which gives a necessary and sufficient condition for the existence of a solution to a finite system of linear inequalities. This lemma is extended for oriented matroids; see [11, Corollary 3.4.6]. We further extend it for oriented orthogonal matroids as follows.

Proposition 5.3.9. *Let M be an oriented orthogonal matroid and let $\{x, x^*\} \subseteq E(\underline{M})$ be a skew pair. Then there is an $\mathbb{S}$-circuit C of M such that $\mathrm{supp}(C) \cap \{x, x^*\} \neq \emptyset$ and $C(y) \in \{0, +\}$ for all $y \in E(\underline{M})$.*

1.3 Antisymmetric matroids with coefficients

We investigate the Baker-Bowler theory concerning symmetric matrices by introducing antisymmetric matroids, which are new combinatorial objects that capture the combinatorial properties of symmetric matrices. We first observe that the parameterization of the Lagrangian symplectic Grassmannian $\mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ described in Theorem 1.1.21 is cut out by certain quadratic relations that are equivalent to the Laplace expansion of symmetric matrices only concerning principal and almost-principal minors.

Theorem 4.2.2. *The image of Φ in Theorem 1.1.21 is set-theoretically cut out by the* restricted Grassmann-Plücker relations*:*

$$\sum_{x \in S \setminus T} (-1)^{m_x} X_{S \setminus \{x\}} X_{T \cup \{x\}} = 0 \qquad \text{(rGP)}$$

for all subsets S and T of E such that $|S| = n+1$, $|T| = n-1$, S contains exactly one $\{i, i^*\}$ for some $i \in [n]$, and T contains no $\{j, j^*\}$, where m_x is the number of elements in $S \triangle T$ less than x with respect to the linear ordering $1 < \cdots < n < 1^* < \cdots < n^*$.

We define antisymmetric matroids for which base exchange axiom captures a combinatorial property of the restricted Grassmann-Plücker relations, akin to that the strong base exchange axioms for matroids and orthogonal matroids are combinatorial counterparts of the Grassmann-Plücker relations and Wick relations, repsectively.

Definition 4.2.1. A pair $M = ([n] \cup [n]^*, \mathcal{B})$ is an *antisymmetric matroid* if $\mathcal{B}$ is a nonempty subfamily of $\mathcal{T}_n \cup \mathcal{A}_n$ and the following hold:

(Sym) For $T \in \mathcal{T}_n$ and distinct skew pairs p and q, $(T \cup p) \setminus q \in \mathcal{B} \cap \mathcal{A}_n$ if and only if $(T \cup q) \setminus p \in \mathcal{B} \cap \mathcal{A}_n$.

(Exch) For $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, if $B \setminus \{e\}$ has no skew pair and $B' \cup \{e\}$ has exactly one skew pair, then there is $f \in B' \setminus B$ such that both $(B \setminus \{e\}) \cup \{f\}$ and $(B' \cup \{e\}) \setminus \{f\}$ are in $\mathcal{B}$.

We call each element in $\mathcal{B}$ a *base* of M. A *circuit* is a minimal subset C of $[n] \cup [n]^*$ such that C contains at most one skew pair and C is not a subset of any base of M. We also have the circuit axiom for antisymmetric matroids (Theorem 4.2.9).

Example 1.3.1 (Antisymmetric matroids arising from symmetric matrices)**.** Let A be an $n \times n$ symmetric matrix and let $\Lambda := [A \mid I_n]$ be an $n \times 2n$ matrix whose columns are indexed by $1, \ldots, n, 1^*, \ldots, n^*$ in order. Then a pair $M = ([n] \cup [n]^*, \mathcal{B})$ is an antisymmetric matroid, where $\mathcal{B} = \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \det(\Lambda[B]) \neq 0\}$. The circuits of M are exactly the minimal subsets C^* of $[n] \cup [n]^*$ such that C contains at most one skew pair and C is the support of a nonzero vector in the row-space of Λ.

We provide two equivalent definitions of antisymmetric matroids with coefficients in tracts, which implies Theorems 1.1.21 and 4.2.2

Theorem 6.4.1. *There is a natural bijection between antisymmetric F-matroids and antisymmetric F-circuit sets.*

Definition 6.1.1. An *antisymmetric matroid with coefficient in a tract F* or an *antisymmetric F-matroid* is a point X in the projective space $\mathbb{F}(F^{2^n + \binom{n}{2} 2^{n-2}})$ satisfying the restricted Grassmann-Plücker relations over F:

$$\sum_{x \in S \setminus T} (-1)^{m_x} X_{S \setminus \{x\}} X_{T \cup \{x\}} \in N_F$$

for all subsets S and T of E such that $|S| = n+1$, $|T| = n-1$, S contains exactly one $\{i, i^*\}$ for some $i \in [n]$, and T contains no $\{j, j^*\}$, where m_x is the number of elements in $S \triangle T$ less than x with respect to the linear ordering $1 < \cdots < n < 1^* < \cdots < n^*$.

Definition 6.1.4. An *antisymmetric F-circuit set* is a set $\mathcal{C}$ of vectors in F^{2n} such that the family of supports of vectors in $\mathcal{C}$ is the family of circuits of an antisymmetric matroid on $[n] \cup [n]^*$, $X \in \mathcal{C}$ implies $c \cdot X \in \mathcal{C}$ for all $c \in F^\times$, and

(ΔOrth*) $\sum_{i=1}^n X_i Y_{i^*} - X_{i^*} Y_i \in N_F$ for all $X, Y \in \mathcal{C}$.

Antisymmetric matroids and those with coefficients are closely related to several other combinatorial objects. First, by collecting the transversal bases of an antisymmetric matroid, we obtain the lift of a delta-matroid. This is somewhat obvious since the bases of an antisymmetric matroid stand for nonsingular principal and almost-principal minors of a symmetric matrix, and the bases of a delta-matroid stand for nonsingular principal minors of a symmetric matrix as shown in Examples 1.1.7 and 1.3.1. Remarkably, the converse holds for the lift of an even delta-matroid, and therefore antisymmetric matroids are a concept generalizing both even delta-matroids and matroids.

Proposition 4.2.7. *Let $M = ([n] \cup [n]^*, \mathcal{B})$ be an antisymmetric matroid. Then the pair $([n] \cup [n]^*, \mathcal{B} \cap \mathcal{T}_n)$ is the lift of a delta-matroid on $[n]$.*

Theorem 4.2.23. *Let $M = ([n] \cup [n]^*, \mathcal{B})$ be an orthogonal matroid. There is a unique $\mathcal{B}' \subseteq \mathcal{A}_n$ such that $([n] \cup [n]^*, \mathcal{B} \cup \mathcal{B}')$ is an antisymmetric matroid.*

Lněnička and Matúš [82] defined *gaussoid* to understand which almost-principal submatrices of a positive definite symmetric matrix can be simultaneously singular. We show that if an antisymmetric matroid M contains all transversals as bases, then the family of the almost-transversal bases of M forms a gaussoid. In addition, if an antisymmetric $\mathbb{S}$-matroid (called an *oriented* antisymmetric matroid) satisfies a condition representing the positive definiteness, then it naturally induces an oriented gaussoid investigated in [13]. Balla and Olarte [9] introduced the *symplectic Dressian* $\mathrm{SpDr}(r, 2n)$ that is the tropical prevariety obtained by the Grassmann-Plücker relations together with certain linear relations parameterizing the symplectic Grassmannian $\mathrm{SpGr}_{\mathbb{F}}(r, 2n)$ into the projective space of dimension $\binom{2n}{r} - 1$ presented by De Concini [45]. We show that each point in the Lagrangian symplectic Dressian $\mathrm{SpDr}(n, 2n)$ canonically induces an antisymmetric $\mathbb{T}$-matroid (called a *valuated* antisymmetric matroid) by discarding all coordinates representing other than transversals and almost-transversals. We also show a connection between tropical Wick vectors [102, 9] and antisymmetric $\mathbb{T}$-circuit sets. Interestingly, every orthogonal F-matroid is an antisymmetric F-matroid whenever $1 = -1$ in a tract F, which is straightforward from Definitions 5.1.8 and 6.1.4. It generalizes a trivial observation that every skew-symmetric matrix over a field of characteristic two is symmetric. The readers can find detailed connections and more examples through Sections 4.2 and 6.2. We summarize them in Table 1.2.

A tract F	Related concepts to antisymmetric F-matroids	See
$\mathbb{K}$	delta-matroids, gaussoids [82, 13]	Prop. 4.2.7, Subsec. 4.2.6
a field $\mathbb{F}$	symmetric matrices	
$\mathbb{T}$	the symplectic Dressian [9], tropical Wick vectors [102]	Subsec. 6.2.3
$\mathbb{S}$	oriented gaussoids [13]	Subsec. 6.2.4
$1 = -1$ in F	orthogonal F-matroids	Subsec. 6.2.2

Table 1.2: Related concepts of antisymmetric F-matroids for various tracts F.

The proof of Theorem 6.4.1 involves the homotopy theorem for graphs associated with antisymmetric matroids, which generalizes both Maurer's Homotopy Theorem for matroids [85] (Theorem 3.1.13) and Wenzel's Homotopy Theorem for even delta-matroids [118] (Theorem 4.1.36). The *transversal base graph* of an antisymmetric matroid $M = ([n] \cup [n]^*, \mathcal{B})$ is a graph such that

- its vertex set is $\mathcal{B} \cap \mathcal{T}_n$, and
- two vertices B and B' are adjacent if and only if (i) $|B \setminus B'| = 1$ or (ii) $|B \setminus B'| = 2$ and there is $A \in \mathcal{B} \cap \mathcal{A}_n$ such that $|B \setminus A| = |B' \setminus A| = 1$.

The *weight* of an edge BB' is $|B \setminus B'|$ that is either 1 or 2.

Theorem 4.2.29 (Homotopy Theorem for Antisymmetric Matroids)**.** *Let G be the transversal base graph of an antisymmetric matroid. Then the (first) homology group of G is generated by the cycles of weights at most 8 in G.*

Theorem 4.2.29 can be restated in a combinatorial sense as follows. A walk in a graph is a sequence of vertices such that each pair of consecutive vertices is adjacent. For a given transversal base graph G, we consider a modification of a walk $v_1 \dots v_\ell$ by (i) replacing a subwalk $v_i v_{i+1} v_{i+2}$ with v_i, provided that $v_i = v_{i+2}$, (ii) applying the reverse process of (i), or (iii) replacing a subwalk $v_i P v_j$ with $v_i Q v_j$, provided that $v_i P v_j Q^{-1} v_i$ is a cycle of weight at most 8. We say that two walks are homotopic if one can be transformed into the other through such modifications. Then, Theorem 4.2.29 is equivalent that any two walks with the same endpoints are homotopic. This explains why Theorem 4.2.29 is referred to as the 'homotopy' theorem.

We finally show the following theorem that is an analog of a classic result bridging matroid theory and algebraic geometry: A point in $\mathbb{P}(\mathbb{F}^{\binom{n}{r}})$ satisfies all Grassmann-Plücker relations if and only if it satisfies all 3-term Grassmann-Plücker relations and its support forms the family of bases of a matroid.

Theorem 6.5.2. *Let $\mathbb{F}$ be a field. For $X \in \mathbb{P}(\mathbb{F}^{2^n+\binom{n}{2}2^{n-2}})$, the following are equivalent:*

(i) *X satisfies all restricted Grassmann-Plücker relations.*

(ii) *X satisfies all $3/4$-term restricted Grassmann-Plücker relations and the support of X forms the set of bases of an antisymmetric matroid.*

We remark that Tutte [108] implicitly proved the result for matroids, which was generalized by Baker and Bowler [4] replacing fields with perfect tracts. An analogous result for orthogonal matroids was shown by Baker and Jin [6].

1.4 Variants of Tutte's Wheel Theorem for binary even delta-matroids

We show the following theorem, which implies Tutte's Wheel Theorem [110].

Theorem 7.1.2. *Let M be a 3-connected binary even delta-matroid with $|E(M)| \geq 4$. Then M has two elements x_1, x_2 such that $M \setminus x_i$ or M/x_i is 3-connected for each i unless M has a cycle as a fundamental graph.*

We also characterize 3-connected binary even delta-matroids that have three elements each of which deletion or contraction preserves 3-connectivity. Let Θ be the set of graphs consisting of at least two internally-disjoint paths between two fixed distinct vertices having no common neighbor. Note that every cycle of length at least four is in Θ.

Theorem 7.1.4. *Let M be a 3-connected binary even delta-matroid with $|E(M)| \geq 4$. Then M has three elements x_1, x_2, x_3 such that $M \setminus x_i$ or M/x_i is 3-connected for each i unless M has a fundamental graph in Θ.*

We expect our results to be a stepping stone to the excluded-minor characterization of regular even delta-matroids. Since Bouchet and Duchamp [37] found the excluded minors for binary even delta-matroids (Corollary 4.1.19), it suffices to characterize the excluded minors for regular even delta-matroids among binary even delta-matroids. Geelen [61, Theorem 4.13] characterized regular even delta-matroids in terms of principally unimodular matrices as follows.

Theorem 1.4.1 ([61])**.** *An even delta-matroid is regular if and only if it is isomorphic to $M(A)\triangle X$ for some principally unimodular skew-symmetric matrix A and a set X.*[2]

[2] It is indeed the definition of regular even delta-matroid in [61], and Geelen showed that such an even delta-matroid is representable over all fields and vice versa.

Hence, we can reduce the problem to understanding principally unimodular matrices. Unfortunately, regular even delta-matroids can be represetned by two or more principally unimodular skew-symmetric matrices distinct up to some equivalence [36], in contrast to that every regular matroid is uniquely represented [40]. Bouchet, Cunningham, and Geelen showed that every 3-connected regular even delta-matroid is represented uniquely, and they also show that every minor-minimally non-regular binary even delta-matroid is 3-connected [36]. Therefore, we may use Theorem 7.1.2 or 7.1.4 to find a certain skew-symmetric matrix from a minor-minimally non-regular binary even delta-matroid M, which make us ease to analyze the structure of M.

Theorems 7.1.2 and 7.1.4 are described in terms of graphs and pivot-minors in Chapter 7. We examine connections between graphs with pivot-minors and delta-matroids in Section 4.1.4. We will also see more on the representations of regular even delta-matroids in the same chapter. We show that a regular even delta-matroid M is uniquely represented without any connectivity assumption if M has no minor isomorphic to $M_4 = ([4], \{X \subseteq [4] : |X| \text{ is even}\})$ (Proposition 7.2.2). It implies the same result for regular matroids shown by Camion [40].

1.5 Organization of the Thesis

In Chapter 2, we introduce basic notations and terminologies. Especially, all graph-theoretic notions will be defined in this chapter. We review the basics of matroid theory in the first half of Chapter 3. The remainder of this chapter is devoted to the Baker-Bowler theory, i.e., matroids with coefficients in tracts, and several representability theorems for matroids such as Tutte's regular matroid characterization (Theorems 3.3.2 and 3.3.5). In Chapter 4, we survey delta-matroid theory parallel to the first half of Chapter 3. We will see several equivalent concepts for delta-matroids and even delta-matroids. We introduce antisymmetric matroids in the same chapter and also discuss their relations to delta-matroids. Chapter 5 is the first main result of this thesis. We present the Baker-Bowler theory for even delta-matroids and prove Theorem 5.1.18, and we discuss its applications as well. In Chapter 6, we provide another extension of the Baker-Bowler theory via antisymmetric matroids and look into connections with several other concepts such as the symplectic Dressian [9] and the oriented gaussoids [13]. We also see that the Baker-Bowler theory for antisymmetric matroids extends that for orthogonal matroids over tracts with $1 = -1$. In Chapter 7, we prove several variants of Tutte's Wheel Theorem for binary delta-matroids, which will be described in terms of graphs and pivot-/vertex-minors. We also discuss some results closely related to regular even delta-matroids.

We provide a guideline to read this thesis in Figure 1.1. The readers who are interested in a specific topic may only read related content with reference to Figure 1.1.

We finally note that all results in Chapter 5 except for Section 5.3.2 are joint work with Tong Jin [71], and most results in Chapter 7 are joint work with Sang-il Oum [75]. Section 4.2 and Chapter 6 are based on a paper [72] available on the arXiv.

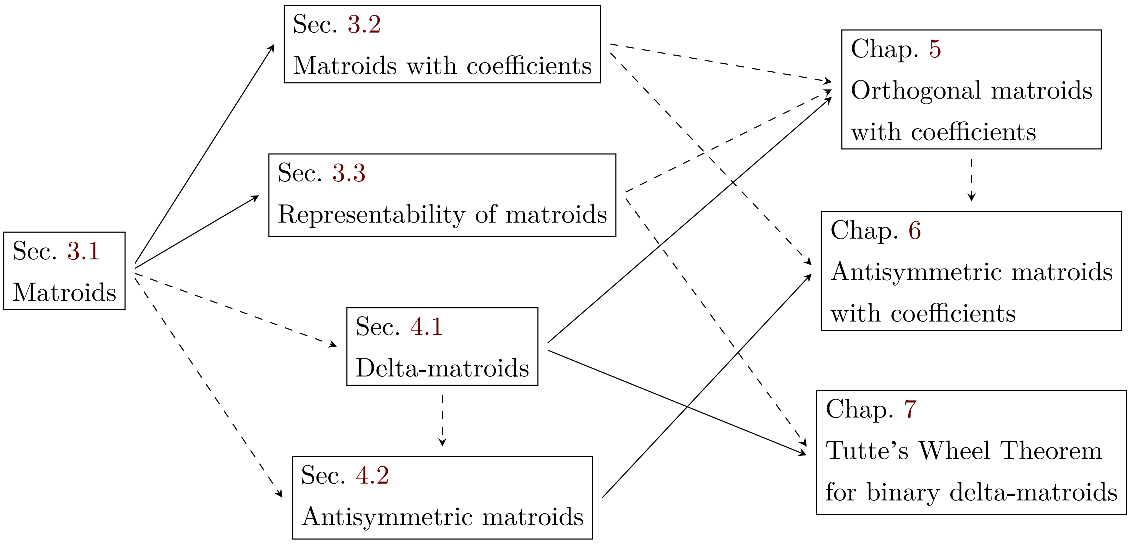


Figure 1.1: The organization of main contents. Each line represents the dependency between sections and chapters. For example, Chapter 6 requires an understanding of Section 4.2. We also recommend reading Section 3.2 and Chapter 5 ahead of Chapter 6 for full appreciation.

Chapter 2. Preliminaries

Basic notations and terminologies are introduced in this chapter. For convenience, we provide the index for symbols and terminologies at the end of the article.

Numbers We denote the sets of natural numbers and integers by $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}$, respectively. We usually denote by $\mathbb{F}$ an arbitrary field. We write the characteristic of $\mathbb{F}$ as $\mathrm{char}(\mathbb{F})$. For a prime power q, let $\mathbb{F}_q$ be the finite field of size q. We often call the field $\mathbb{F}_2$ with two elements the *binary field*, call the field $\mathbb{F}_3$ with three elements the *ternary field*, and so on. Let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ be the fields of rational numbers, real numbers, and complex numbers, respectively.

Sets Let $[n] := \{1, 2, \ldots, n\}$ for a positive integer n. For two sets S and T, we often write the union $S \cup T$ and the set difference $S \setminus T$ as $S + T$ and $S - T$, respectively. If $T = \{x\}$, we abuse the notation and write $S + x$ and $S - x$ rather than $S + \{x\}$ and $S - \{x\}$. When the symbols '+' and '−' are used more than once, we read them from left to right, such as $S - x + y = (S - x) + y$. We often omit brackets and commas while denoting a set, such as $abc = \{a, b, c\}$.

For a set E and an integer r, we denote by $\binom{E}{r}$ the set of all r-element subsets of E. We denote by 2^E the set of all subsets of E. Suppose E is equipped with a linear ordering $<$. Then for $S \subseteq E$ and $x \in E$, let $|S < x|$ be the number of elements $y \in S$ smaller than x. We similarly define $|S \leq x|$ as the number of elements $y \in S$ smaller than or equal to x.

For a family $\mathcal{S}$ of sets, let $\mathrm{Min}(\mathcal{S})$ denote the family of minimal elements of $\mathcal{C}$ with respect to inclusion. We similarly define $\mathrm{Max}(\mathcal{S})$ as the family of maximal elements of $\mathcal{S}$. A *clutter* is a set $\mathcal{C}$ of subsets of a finite set such that no element in $\mathcal{C}$ is a proper subset of another element in $\mathcal{C}$, i.e., it satisfies (C2). A clutter is *nontrivial* if it does not contain the empty set.

A *set system* is a pair $(E, \mathcal{S})$ of a set E and a family $\mathcal{S}$ of subsets of E. We call E the *ground set* of the set system. For two set system $S_1 = (E_1, \mathcal{S}_1)$ and $S_2 = (E_2, \mathcal{S}_2)$ with disjoint ground sets, the *direct sum* of S_1 and S_2 is the set system $S_1 \oplus S_2 := (E_1 \cup E_2, \mathcal{S}_1 \oplus \mathcal{S}_2)$ where $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{X \cup Y : X \in \mathcal{S}_1 \text{ and } Y \in \mathcal{S}_2\}$. We can also define the direct sum of more than two set systems, which is defined regardless of the order of the summands.

A binary relation $\leq$ on a set is *quasi-order* if it is reflexive and transitive, i.e., (Reflexivity) $x \leq x$ for all x, and (Transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$ for all x, y, z. A set X is *well-quasi-ordered* (in short, *WQO*) with respect to a quasi-order $\leq$ on X if every infinite sequence of elements $x_1, x_2, \ldots$ has an inifnite ascending chain $x_{i_1} \leq x_{i_2} \leq \cdots$ with $i_1 < i_2 < \cdots$. It is equivalent that there is neither infinite antichain nor infinite strictly descending chain (i.e., $x_1 > x_2 > \cdots$).

Vector spaces Let $\mathbf{e}_i$ in the vector space $\mathbb{F}^n$ be the i-th standard base vector, i.e., $\mathbf{e}_i(j) = 1$ if $i = j$ and $\mathbf{e}_i(j) = 0$ otherwise. The *indicator vector* of a set $S \subseteq [n]$ is $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$. We denote the zero vector in any vector space by $\mathbf{0}$. The *support* of a vector $X \in \mathbb{F}^E$, denoted by $\mathrm{supp}(X)$ or $\underline{X}$, is the set of $i \in E$ such that $X(i) \neq 0$. The *projectivization* of a linear space V over $\mathbb{F}$ is $\mathbb{P}(V) := (V \setminus \{\mathbf{0}\})/\mathbb{F}^\times$.

For a set X of vectors in a real vector space $\mathbb{R}^n$, the *convex hull* of X, denoted by $\mathrm{conv}(X)$, is the set of all convex combinations of vectors in X, i.e, $\mathrm{conv}(X) = \{\sum_{i=1}^m \lambda_i \mathbf{v}_i : \mathbf{v}_i \in X,\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1\}$. A 0/1-*polytope* is a convex hull of a (finite) set of indicator vectors in $\mathbb{R}^n$.

Matrices For a matrix A, we denote its *transpose* by A^t. The *row-space* (resp. the *column-space*) of a matrix is the linear span of its rows (resp. columns). A *minor* of a matrix is a determinant of a square submatrix. A minor is *maximal* if it is a determinant of a square submatrix of the largest possible size.

For an $R \times C$ matrix A, the submatrix of A whose rows and columns are indexed by $X \subseteq R$ and $Y \subseteq C$ is denoted by $A[X,Y]$. For simplicity, $A[R,Y]$ is often written as $A[Y]$. If A is a square matrix, we write $A[X]$ rather than $A[X,X]$.

For a square matrix A of which rows and columns are indexed by the same symbols, a *principal submatrix* is a square submatrix of A of which row and column index sets are the same. An *almost-principal submatrix* is a square submatrix of A of which row and column index sets differ by one element. We call their determinants a *principal minor* and an *almost-principal minor* of A, respectively. A square matrix A is *symmetric* if $A^t = A$, and it is *skew-symmetric* if $A^t = -A$ and all diagonal entries of A are zero. The *cut-rank function* ρ_A of an $V \times V$ square matrix A is a function from 2^V to $\mathbb{Z}$ such that $\rho_A(X) := \operatorname{rank}(A_G[X, V-X])$.

A matrix is *totally unimodular* (in short, *TU*) if it is real and each minor is 0 or ±1. A square matrix is *pricipally unimodular* (in short, *PU*) if if it is real and each principal minor is 0 or ±1.

Graphs A *graph* is a pair $G = (V,E)$ of a vertex set V and an edge set E. We denote by $V(G) := V$ and $E(G) := E$. A graph is *simple* if it has neither loops nor multiple edges. A vertex is a *neighbor* of another vertex if they are incident with the same edge. The set of neighbors of a vertex v in G is denoted by $N_G(v)$.

For a graph G, a vertex is *isolated* if it has no neighbors in G. A *pendant vertex* is a vertex of degree 1. Two vertices v, w are *twins* if their neighbors except for themselves are the same, i.e., $N_G(v) \setminus \{v,w\} = N_G(w) \setminus \{v,w\}$.

A *subgraph* of a graph G is a graph obtained by deleting some vertices and edges of G. A subgraph is *induced* if it can be obtained by only deleting vertices. We often say a graph G has another graph H if G has a subgraph isomorphic to H.

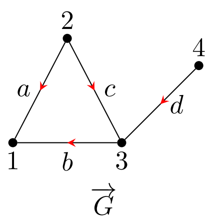
A *path* is a graph whose vertices can be enumerated as $v_1, \ldots, v_m$ so that it has exactly $m-1$ edges each of which is incident with v_i and v_{i+1}. We often represent a path as a sequence of vertices $v_1 v_2 \ldots v_m$. The length of a path is the number of its edges, and we denote the path with m vertices by P_m. A graph is *connected* if each pair of vertices can be connected by a path. A *cycle* is a graph whose vertices can be enumerated as $v_1, \ldots, v_m$ so that it has exactly m edges each of which is incident with v_i and v_{i+1}, where the subscripts are read modulo m. Especially, it is a single vertex attached with a loop if $m = 1$. We often represent a cycle as a sequence of vertices $v_1 v_2 \ldots v_m v_1$. The length of a cycle is the number of its edges, and we denote the cycle with m vertices by C_m. A graph is *acyclic* or is a *forest* if it does not have any cycle. A *tree* is a connected forest.

The *complete graph* on n vertices, denoted by K_n, is a graph with n vertices such that every pair of vertices is adjacent. The *complete bipartite graph* with a vertex bipartition with m and n vertices, denoted by $K_{m,n}$, is a graph with $m+n$ vertices partitioned into two sets V_1 and V_2 of size m and n such that every vertex in V_1 is adjacent to every vertex in V_2, and there is no other edge.

A *subdivision* of a graph is a new graph obtained by repalcing each edges with a path of length at least one. A subidivision is *odd* if each path replacing an edge has odd length.

Graphs and linear algebra The *adjacency matrix* of a graph G, denoted by A_G, is a $V(G) \times V(G)$ matrix whose (v,w)-entry is 1 if G has an edge incident with both v and w, and 0 otherwise. Then for each looped vertex v, the (v, v)-entry is 1. It can be defined over any field, but in this article, we usually consider it as a matrix over the binary field $\mathbb{F}_2$. The *incidence matrix* of G, denoted by I_G, is a $V(G) \times E(G)$ matrix whose (v, e)-entry is 1 if v is incident with e, and 0 otherwise. It also can be defined over any field.

An *oriented graph* is a graph along with a direction on each edge. An *orientation* $\overrightarrow{G}$ of a graph G is an oriented graph obtained by replacing each edge of G with an oriented edge. The (*oriented*) *adjacency matrix* of an oriented graph $\overrightarrow{G}$, denoted by $A_{\overrightarrow{G}}$, is a $V(G) \times V(G)$ skew-symmetric matrix whose non-diagonal (v, w)-entry is 1 if vw is an edge and w is its head, -1 if vw is an edge and w is its tail, and 0 otherwise. The (*oriented*) *incidence matrix* of $\overrightarrow{G}$, denoted by $I_{\overrightarrow{G}}$, is a $V(G) \times E(G)$ matrix such that the (v, e)-entry is 1 if v is the head of e, -1 if v is the tail of e, and 0 otherwise. Both matrices $A_{\overrightarrow{G}}$ and $I_{\overrightarrow{G}}$ are defined over the real field $\mathbb{R}$. See Figure 2.1 for an example. Depending on the context, we also call $A_{\overrightarrow{G}}$ and $I_{\overrightarrow{G}}$ the (oriented) adjacency and incidence matrix of a graph G, whenever $\overrightarrow{G}$ is an arbitrary orientation of G.



$$
A_{\overrightarrow{G}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}
\qquad
I_{\overrightarrow{G}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}
$$

Figure 2.1: An oriented graph $\overrightarrow{G}$ (left), its adjacency matrix (middle), and its incidence matrix (right).

Let $G = (V, E)$ be a graph and $\overrightarrow{G}$ be its orientation. Then $I_{\overrightarrow{G}}$ is identified with a linear map ∂_1 from $\mathbb{Z}^E$ to $\mathbb{Z}^V$ such that $\partial_1(e) = v - w$ if an edge e orient from w to v in $\overrightarrow{G}$. We note that for different choice of orientations, the kernel of ∂_1 is unique up to linear isomorphism. The (*first*) *homology group* $H(G)$ of G is the kernal $\ker(\partial_1)$. We will often identify a cycle $v_1 e_1 v_2 e_2 v_3 \ldots v_k e_k v_1$ of length $k \geq 2$ in G with an element $\sum_{i=1}^{k} \epsilon_i e_i \in H(G)$ where $\epsilon_i = \begin{cases} 1 & \text{if } e_i \text{ orients from } v_i \text{ to } v_{i+1} \\ -1 & \text{otherwise} \end{cases}$ and $v_{k+1} := v_1$. Then the cycles of G generate the homology group $H(G)$, i.e., each element in $H(G)$ is a linear combination of cycles of G. We say cycles $C_1, \ldots, C_m$ of G generates another cycle C if they generate C as elements in $H(G)$; see Figure 2.2.

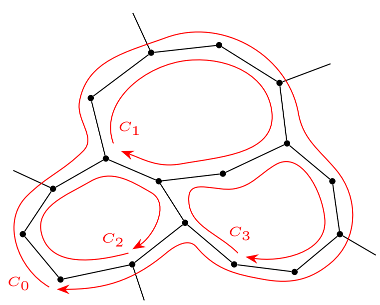


Figure 2.2: C_0 is generated by C_1, C_2, C_3.

Vertex-minors and pivot-minors We review vertex-/pivot-minors of simple graphs. In this part and the rest of the article, all graphs are assumed to be simple whenever we discuss vertex-/pivot-minors of graphs.

For a vertex v of a graph G, let $G*v$ be the graph obtained from G by deleting all edges joining two neighbors of v and adding edges joining non-adjacent pairs of two neighbors of v. This operation is called the *local complementation* at v to G. Two graphs are *locally equivalent* if one can be obtained from the other by applying a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph locally equivalent to G. For every edge vw, $G*v*w*v = G*w*v*w$ by Bouchet [24, (8.2)]. For an edge vw, let $G \wedge vw := G*v*w*v$. This operation is called the *pivoting* vw to G. We note that $G \wedge vw$ is equal to a graph obtained from G by toggling the adjacency between each pair of vertices in different sets $N_G(v) \setminus (N_G(w) \cup \{w\})$, $N_G(w) \setminus (N_G(v) \cup \{v\})$, and $N_G(v) \cap N_G(w)$, and swapping the labels of v and w. See Figure 2.3 for example. Two graphs are *pivot-equivalent* if one can be obtained from the other by applying a sequence of pivotings. A graph H is a *pivot-minor* of a graph G if H is an induced subgraph of a graph pivot-equivalent to G. Graphs with the pivot-minor relation are closely related to binary even delta-matroids with the minor relation, which will be explained in Section 4.1.4.

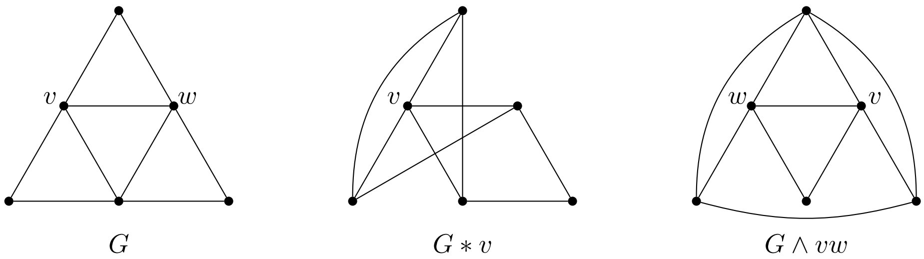


Figure 2.3: Local complementation and pivoting.

The *cut-rank function* ρ_G of G is the cut-rank function of its binary adjaency matrix A_G.

A *split* of a graph G is a partition (A, B) of $V(G)$ such that $\min\{|A|, |B|\} \geq 2$ and for some $A' \subseteq A$ and $B' \subseteq B$, two vertices $x \in A$ and $y \in B$ are adjacent if and only if $x \in A'$ and $y \in B'$. Equivalently, the vertex partition (A, B) is a split if and only if $\min\{|A|, |B|\} \geq 2$ and the $A \times B$ submatrix of the adjacency matrix of the given graph has rank at most 1, i.e., $\rho_G(A) \leq 1$. The 1-*join* of two graphs H_1 and H_2 with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$ is a graph obtained from the union of disjoint copies of $H_1 \setminus v_1$ and $H_2 \setminus v_2$ by adding all possible edges between $N_{H_1}(v_1)$ and $N_{H_2}(v_2)$. Then a vertex bipartition $(V(H_1) \setminus v_1, V(H_2) \setminus v_2)$ of the resulting graph is a split. A graph is *prime* if it has no split or, equivalently, it is not a 1-join of two smaller graphs. Bouchet [26, Corollary 3] showed that locally equivalent graphs have the same set of splits. Thus, if a graph G is prime, then every graph locally equivalent to G is prime.

We refer the readers to [76] for an extensive survey on vertex-minors by Oum and the author.

Chapter 3. Matroids with coefficients

We review basic definitions and properties of matroids in Section 3.1. The reader may refer to [98] for more background on matroid theory. In Section 3.2, we briefly survey the Baker-Bowler theory, which gives several cryptomorphic definitions of matroids with coefficients and examines their functoriality, duality, and minors. In Section 3.3, we discuss several theorems on the representability of matroids, including Tutte's excluded-minor characterizations of the classes of binary, regular, and graphic matroids.

3.1 Matroids

A *matroid* is a pair $M = (E, \mathcal{B})$ of a finite set E and a nonempty set $\mathcal{B}$ of subsets of E satisfying the *base exchange axiom*:

(B) For all $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, there is $f \in B' \setminus B$ such that $B - e + f \in \mathcal{B}$.

We call $E(M) :=$ the *ground set* of M. Each element in $\mathcal{B}$ is called a *base* of M, and we denote by $\mathcal{B}(M) := \mathcal{B}$ the family of bases. A subset S of E is *independent* if it is a subset of some base, and it is *dependent* otherwise. The *rank* of S is the maximum size of an independent subset of S. The *rank* of M is the size of a base.

A matroid is a combinatorial abstraction of linear independence in a vector space, which is revealed clearly from the Grassmann-Plücker relations. For a field $\mathbb{F}$ and integers $0 \le r \le n$, the *Grassmannian* $\mathrm{Gr}_{\mathbb{F}}(r, n)$ is the set of r-dimensional vector spaces in the n-dimensional vector space k^n. It is parameterized into the projective space of dimension $\binom{n}{r} - 1$ by the Grassmann-Plücker embedding and is exactly the solution of the *Grassmann-Plücker relations*:

$$\sum_{x \in S \setminus T} (-1)^{|S<x|+|T<x|} X_{S-x} X_{T+x} = 0 \text{ for all } S \in \binom{[n]}{r+1} \text{ and } T \in \binom{[n]}{r-1}. \tag{GP}$$

For a given point X satisfying all Grassmann-Plücker relations, let $\mathcal{B}$ be the set of $B \in \binom{[n]}{r}$ such that $X_B \neq 0$. Then for any $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, we have

$$\sum_{x \in (B'+e) \setminus (B-e)} (-1)^{|(B'+e)<x|+|(B-e)<x|} X_{B'+e-x} X_{B-e+x} = 0.$$

Thus, there is $f \in B' \setminus B$ such that $B' + e - f$ and $B - e + f$ are in $\mathcal{B}$, implying that $\mathcal{B}$ is the set of bases of a rank-r matroid on $[n]$. A matroid is *representable over* $\mathbb{F}$ or $\mathbb{F}$*-representable* if it is obtainable by the previous construction up to isomorphism. Then every representable matroid M satisfies the *strong base exchange property*:

(B′) For all $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, there is $f \in B' \setminus B$ such that $B - e + f \in \mathcal{B}$ and $B' + e - f \in \mathcal{B}$.

It holds for general matroids as well.

Theorem 3.1.1 (Brualdi [39]). *Every matroid satisfies the strong base exchange property.*

There are more ways to understand a matroid as an underlying combinatorial structure of a linear vector space. We first recall two cryptomorphic definitions of a matroid. A *circuit* of a matroid $M = (E, \mathcal{B})$ is a minimal depedent subset of E. We denote the family of circuits of M by $\mathcal{C}(M)$. The *dual* of M is a matroid $M^\perp := (E, \mathcal{B}^\perp)$ where $\mathcal{B}^\perp := \{E \setminus B : B \in \mathcal{B}\}$. A *cocircuit* of M is a circuit of the dual $M^\perp$. The circuit axiom for matroids was given in Proposition 1.1.2: Let $\mathcal{C}$ be a set of subsets of E. Then $\mathcal{C}$ is the set of circuits of a matroid if and only if $\mathcal{C}$ is a nontrivial clutter and satisfies the *circuit elimination axiom*:

(Elim) For distinct $C, C' \in \mathcal{C}$ and $e \in C \cap C'$, there is $C'' \in \mathcal{C}$ such that $C'' \subseteq (C \cup C') - e$.

We have another cryptomorphic definition of matroids, which is called *Minty's painting axiom* or the *self-dual axiom*.

Lemma 3.1.2 ([87])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sets of subsets of E. Then $\mathcal{C}$ is the set of circuits of a matroid and $\mathcal{D}$ is the set of cocircuits of the same matroid if and only if both $\mathcal{C}$ and $\mathcal{D}$ are nontrivial clutters and they satisfy the following:*

(Orth) $|C \cap D| \neq 1$ *for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

(Max) *For every tripartition $(P, Q, \{e\})$ of E, either there is $C \in \mathcal{C}$ such that $e \in C \subseteq P \cup \{e\}$ or there is $D \in \mathcal{D}$ such that $e \in D \subseteq Q \cup \{e\}$.*

We often call (Orth) the *orthogonality* of matroids.

Let V be an r-dimensional linear space in $\mathbb{F}^n$, and let $\mathcal{C}$ be the set of minimal supports of nonzero vectors in V. Then $\mathcal{C}$ is the set of circuits of a matroid M, because for any $X, Y \in \mathcal{C}$ with $X(e) = Y(e) \neq 0$, we have $\operatorname{supp}(X - Y) \subseteq (\operatorname{supp}(X) + \operatorname{supp}(Y)) - e$. We note that the rank of M is $n - r$ and $\mathcal{C}$ satisfies the *strong circuit elimination axiom*: For all $C, C' \in \mathcal{C}$, $e \in C \cap C'$, and $f \in C \setminus C'$, there is $C'' \in \mathcal{C}$ such that $f \in C'' \subseteq (C \cup C') - e$. We further remark that if we let N be the matroid induced by the supports of $p(V)$, where p is the Grassmann-Plücker embedding, then $N = M^\perp$. The linear space V also induces the sets of circuits and cocircuits of a matroid as follows.

Lemma 3.1.3 (folklore)**.** *Let V be a linear space in $\mathbb{F}^n$. Let $\mathcal{C}$ be the set of minimal supports of nonzero vectors in V, and let $\mathcal{D}$ be the set of minimal supports of nonzero vectors in the orthogonal complement $V^\perp$. Then $\mathcal{C}$ and $\mathcal{D}$ are the sets of circuits and cocircuits, respectively, of a matroid.*

Proof. It suffices to show that $\mathcal{C}$ and $\mathcal{D}$ fulfill (Orth) and (Max) in Lemma 3.1.2. Because $\sum_{i=1}^n X(i)Y(i) = 0$ for each $X \in V$ and $Y \in V^\perp$, two sets $\mathcal{C}$ and $\mathcal{D}$ satisfies (Orth), the orthogonality.

Suppose to the contrary that $(P, Q, \{e\})$ is a tripartition of $[n]$ violating (Max), i.e., there is no $C \in \mathcal{C}$ such that $e \in C \subseteq P + e$ and there is no $D \in \mathcal{D}$ such that $e \in D \subseteq Q + e$. Let P' a minimal subset of P such that $P \setminus P'$ does not contain any $C \in \mathcal{C}$, and let Q' be a minimal subset of Q such that $Q \setminus Q'$ does not contain any $D \in \mathcal{D}$. We denote by $P'' := P - P' + Q'$ and $Q'' := Q - Q' + P'$. Then $(P'', Q'', \{e\})$ is a tripartition of $[n]$. Note that for each $x \in Q'$, there is $D_x \in \mathcal{D}$ such that $D_x \subseteq Q$ and $D_x \cap Q' = \{x\}$. If $P'' + e$ contains some $C \in \mathcal{C}$, then $C \cap Q' \neq \emptyset$ and thus $C \cap D_x = \{x\}$ for $x \in C \cap Q'$, contradicting the orthogonality. Hence $P'' + e$ does not contain any $C \in \mathcal{C}$, and thus $\dim V \leq |[n] - (P'' + e)|$. Similarly, $Q'' + e$ does not contain any $D \in \mathcal{D}$, and so $\dim V^\perp \leq |[n] - (Q'' + e)|$. Therefore, $\dim V + \dim V^\perp \leq n - 1$, a contradiction. □

The following theorem is a fundamental result in matroid theory and linear algebra. To our best knowledge, it was first proved by Tutte [108] in terms of circuits and chain-group representations of matroids. We refer to [6] for the following statement and its proof.

Theorem 3.1.4. *For a field $\mathbb{F}$, let $X \in \mathbb{P}(\mathbb{F}^{\binom{n}{r}})$ be a point. Then the following are equivalent.*

(i) *X satisfies all Grassmann-Plücker relations* (GP).

(ii) *X satisfies all 3-term Grassmann-Plücker relations* (3GP) *and the support of X forms a matroid.*

(iii) *There is an $r \times n$ matrix A over k such that* $X = \left(\det(A[r, B]) : B \in \binom{[n]}{r}\right)$.

3.1.1 Examples

Two matroids $M = (E, \mathcal{B})$ and $M' = (E', \mathcal{B}')$ are *isomorphic*, denoted by $M \cong M'$, if there is a bijection between E and E' which induces a canonical bijection between $\mathcal{B}$ and $\mathcal{B}'$.

Example 3.1.5 (Representable matroids)**.** Let A be an $r \times n$ matrix over a field $\mathbb{F}$, and let $\mathcal{B}$ be the family of sets of column vectors in A such that those are a base of the column space of A. Then a pair $M(A) := (E, \mathcal{B})$ is a matroid, where E is the set of column vectors of A. If the rank of A is r, then $\mathcal{B}$ equals to $\{B \subseteq E : \det(A[B]) \neq 0\}$.

We say a matroid M is *representable over* $\mathbb{F}$ or $\mathbb{F}$*-representable* if M is isomorphic to $M(A)$ for some matrix A over $\mathbb{F}$, and we call A an $\mathbb{F}$*-representation* or, simply, a *representation* of M. We also say that M is *represented* by A. Note that by row-equivalence, deleting zero-rows, and permuting columns, we can modify A to another matrix $A' = [I_r \mid X]$ where r is the rank of M and X is an $r \times (n-r)$ matrix. Then A' is a representation of M as well, and we call such a *standard* representation. Let $A^* := [-X^t \mid I_{n-r}]$. Then the row-space of A' (that equals to the row-space of A) and the row-space of A^* are orthogonal complements of each other with respect to the standard inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{w}_i$. Moreover, A^* is a representation of the dual matroid M^* [98, Theorem 2.2.8]. Therefore, the representability of matroids is closed under taking duals.

Example 3.1.6 (Regular matroids)**.** A matroid is *regular* if it is representable over all fields. By Tutte [108], a matroid is regular if and only if it is represented by a TU matrix. More equivalent definitions of regular matroids are written in Theorem 3.3.5.

Example 3.1.7 (The Fano matroid)**.** The *Fano matroid* F_7 is a binary matroid obtained from the following representation

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

of which column vectors are the poins of the Fano plane in Figure 3.1.

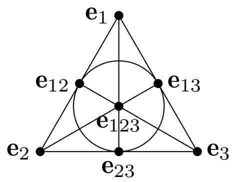


Figure 3.1: The Fano plane, where $\mathbf{e}_{ij} = \mathbf{e}_i + \mathbf{e}_j$ with distinct $i, j \in [3]$ and $\mathbf{e}_{123} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

Example 3.1.8 (Uniform matroids)**.** The *uniform matroid* of rank r on n elements is $U_{r,n} = ([n], \binom{[n]}{r})$. The rank-2 matroid $U_{2,4}$ on four elements is non-binary but representable over any field with at least three elements witnessed by a representation $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}$ with $x \neq 1$.

Example 3.1.9 (Transversal matroids)**.** Let G be a bipartite graph with a vertex bipartition (X, Y), and let $\mathcal{I}$ be the family of subsets of X that are covered by matchings in G. Then a pair $(X, \mathrm{Max}(\mathcal{I}))$ is a matroid, which follows from the alternating path argument for a union of two matchings. We call this matroid a *transversal matroid*. If $|X| = n$ and $|Y| = r$, it induces the uniform matroid $U_{r,n}$. We note that every transversal matroid is representable over a field with sufficiently many elements [98, Corollary 11.2.17].

Example 3.1.10 (Graphic and cographic matroids)**.** For a graph $G = (V, E)$, let $M(G)$ be a pair $(E, \mathcal{B})$ where $\mathcal{B}$ is the family of maiximal acyclic edge sets. Then $M(G)$ is a matroid [98, Proposition 1.1.7] and is called the *cycle matroid* or the *graphic matroid* of G. In particular, $\mathcal{B}$ is the family of the edge sets of spanning trees in G. By definition, the set of circuits of $M(G)$ equals to the family of cycles of G. We say a matroid is *graphic* if it is isomorphic to $M(G)$ for some graph G. A matroid is *cographic* if it is the dual of a graphic matroid, and $M^*(G) := M(G)^*$ is called the *bond matroid* or the *cographic matroid* of G. The family of circuits of $M^*(G)$ is exactly the family of bonds of G.

In cotrast to the representable matroids, the class of graphic matroids is not closed under taking duals. For instance, neither $M^*(K_5)$ nor $M^*(K_{3,3})$ is graphic [98, Proposition 2.3.3].

3.1.2 Minors

Let $M = (E, \mathcal{B})$ be a matroid and let $e \in E$. We call e a *loop* if $\{e\}$ is not contained in any bases, and we call e a *coloop* if e is contained in all bases. Two elements e and f are *parallel* if $\{e, f\}$ is a circuit. A matroid is *simple* if it has no loops and no parallel elements. The *deletion* of e from M is a pair $M \setminus e := (E \setminus \{e\}, \mathcal{B} \setminus e)$ where

$$\mathcal{B} \setminus e := \begin{cases} \{B : e \notin B \in \mathcal{B}\} & \text{if } e \text{ is not a coloop,} \\ \{B \setminus \{e\} : e \in B \in \mathcal{B}\} & \text{if } e \text{ is a coloop.} \end{cases}$$

The *contraction* of e from M is a pair $M/e := (E \setminus \{e\}, \mathcal{B}/e)$ where

$$\mathcal{B}/e := \begin{cases} \{B \setminus \{e\} : e \in B \in \mathcal{B}\} & \text{if } e \text{ is not a loop,} \\ \mathcal{B} & \text{if } e \text{ is a loop.} \end{cases}$$

It is easy to see that $M \setminus e$ and M/e are matroids, and $M/e = (M^* \setminus e)^*$.

A matroid M is a *minor* of another matroid M' if M can be obtained from M' by a sequence of deletions and contractions. A minor is *proper* if it has less elements than the original matroid. We note that $M \setminus e/f = M/f \setminus e$ for distinct elements $e, f \in E(M)$. Therefore, we can write $M = M' \setminus X/Y$ for some disjoint subsets X and Y of $E(M')$ whenever M is a minor of M'. Here, the choice of X and Y is not unique, but we can always choose X and Y are independent and coindependent in M' [98, Lemma 3.3.2]. We say a matroid M' has an *M-minor* if M' has a minor isomorphic to M.

We assume that a *class* of matroids is closed under taking isomorphism. A class of matroids is *minor-closed* if every minor of a matroid in the class is also in the class. For instance, $U_{r,n} \setminus e = U_{\min\{r,n-1\},n-1}$ and $U_{r,n}/e = U_{\max\{r-1,0\},n-1}$, and therefore the class of uniform matroids of rank at most r is minor closed. If G is a graph and $e \in E(G)$, then $M(G) \setminus e = M(G \setminus e)$ and $M(G/e) = M(G)/e$ [98, 3.1.2 and Proposition 3.2.1], and thus the class of graphic matroids is minor-closed. The class of $\mathbb{F}$-representable matroids is also minor-closed [98, Proposition 3.2.4], and hence the class of regular matroids is minor-closed.

The *excluded minors* for a minor-closed class $\mathcal{C}$ of matroids are the matroids M (up to isomorphism) such that M is not in $\mathcal{C}$ but every proper minor of M is in $\mathcal{C}$. For example, $U_{r+1,r+1}$ is the excluded minor for the class of matroids of rank at most r because every rank-r' matroid has a $U_{r',r'}$-minor. We will further discuss the excluded minors for several classes of matroids in Section 3.3.

3.1.3 Connectedness

We define the *direct sum* $M_1 \oplus M_2$ of two matroids M_1 and M_2 as the direct sum as set systems. It is obvious that $M_1 \oplus M_2$ is a matroid. A matroid M is *connected* if M is not the direct sum of two matroids of which ground sets are neither $\emptyset$ nor $E(M)$. A *component* of M is a connected matroid N such that $E(N) \neq \emptyset$ and $M = N \oplus M'$ for some matroid M'. Then every matroid can be written as the direct sum of its components. Analogous to graphs, higher connectivity of matroids can be defined, and we refer the readers to Chapter 8 in [98].

3.1.4 Fundamental graphs

Let $M = (E, \mathcal{B})$ a be matroid. Then for each base B and an element $e \in E \setminus B$, there is a circuit C such that $e \in C \subseteq B + e$. It follows from the circuit elimination axiom (Elim) that such a circuit C is unique, and we call $C_M(B, e) := C$ the *fundamental circuit* of M with respect to B and e. For each element $f \in B$, $B + e - f$ is a base if and only if $f \in C_M(B, e)$. Dually, we deifne the *fundamental cocircuit* of M with respect to a base B and an element $e \in B$ as the unique cocircuit $C^*_M(B, e)$ contained in $(E \setminus B) + e$.

For a base B, let $F_M(B)$ be the bipartite graph on the vertex set E with a bipartition $(B, E \setminus B)$ such that $e \in B$ and $f \in E \setminus B$ are adjacent if and only if $f \in C_M(B, e)$. Two matroids may have the same fundamental graph. It is however that if two connected binary matroids M_1 and M_2 have the same fundamental graph, then $M_1 \cong M_2$ or M_2^* because the fundamental graph of a binary matroid of a base B determines its standard binary representation with respect to B.

Now we let M be a binary matroid. Let B and $B' = B - e + f$ be its bases. We denote by G and G' the fundamental graphs with respect to B and B'. Then it is easy to observe that G' is obtained from G by pivoting ef, that is, toggling the adjacency between each pair of vertices $e' \in N_G(f) - e$ and $f' \in N_G(e) - f$. See Figure 3.2 for example. For a base B and an element e of M, one can check that

$$F_M(B) \setminus e = \begin{cases} F_{M \setminus e}(B) & \text{if } e \in E \setminus B, \\ F_{M/e}(B - e) & \text{otherwise,} \end{cases}$$

and therefore, binary matroids with the minor relation are compatible with bipartite graphs with the pivot-minor relation. See [91, Chapter 3.5] for more details. This observation is extended to general graphs and binary even delta-matroids in Section 4.1.4.

3.1.5 Base polytopes and base graphs

The *base polytope* P_M of a matroid $M = (E, \mathcal{B})$ is the convex hull of the indicator vectors $\mathbf{e}_B \in \mathbb{R}^E$ of the bases B of M, i.e., $P_M = \text{conv}\{\mathbf{e}_B \in \mathbb{R}^E : B \in \mathcal{B}\}$. Gelfand, Goresky, MacPerson, and Serganova [67] presented a criterion for checking whether a given 0/1-polytope is the base polytope of a matroid.

Theorem 3.1.11 ([67, Theorem 4.1])**.** *Let P be a 0/1-polytope. Then P is the base polytope of a matroid if and only if each edge of P is a translate of $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$ with $i \neq j$.*

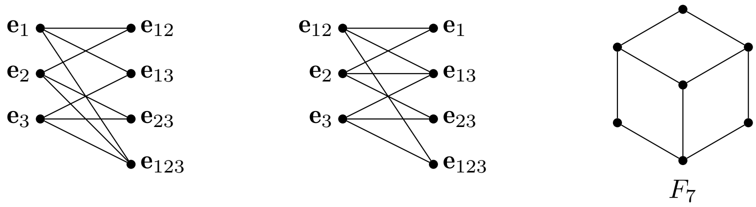


Figure 3.2: Two fundamental graphs of the Fano matroid F_7 defined in Example 3.1.5. The leftmost one is with respect to the base $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the midle one is with respect to the base $B' = \{\mathbf{e}_{12}, \mathbf{e}_2, \mathbf{e}_3\}$, where the indices follows Figure 3.1. These two graphs are isomorphic to each other, and moreover every fundamental graph of F_7 is isomorphic to the rightmost graph, which is called the *Fano graph* and is also denoted by F_7.

The *base graph* G_M of a matroid $M = (E, \mathcal{B})$ is a graph such that its vertex set $\mathcal{B}$, and two vertices B and B' are adjacent if and only if $|B \cap B'| = 1$. By Theorem 3.1.11, the base graph G_M is the 1-skeleton of the base polytope P_M. Base graphs determine matroids up to isomorphism, component-wise dual, and adding/deleting loops and coloops due to the following result by Holzmann, Norton, and Tobey [69, Theorem 5.3].

Theorem 3.1.12 ([69])**.** *Let M and N be connected matroids. If G_M and G_N are isomorphic, then M is isormophic to N or $N^{\perp}$.*

Maurer [85] showed the following theorem, which is a key tool in a proof of the cryptomorphism on matroids with coefficients (Theorem 3.2.12 and 3.2.13).

Theorem 3.1.13 (Maurer's Homotopy Theorem [85, Theorem 5.1])**.** *Let M be a matroid. Then the homology group of the base graph G_M is generated by the cycles of length at most four.*

Maurer's Homotopy Theorem can be restated in a combinatorial sense as follows. Given a matroid M, we define a modification of a walk $v_1 \dots v_\ell$ of G_M by replacing a subwalk $v_i P v_j$ with $v_i Q v_j$, provided that a closed walk $v_i P v_j Q^{-1} v_i$ has length at most four. Two walks are said to be homotopic if one can be transformed into the other by a sequence of such modifications. Then, Maurer's Homotopy Theorem is equivalent to the following: Any two walks in G_M with the same endpoints are homotopic. This is a reason why Theorem 3.1.13 is called the 'homotopy' theorem. Wenzel [117] extended Maurer's Homotopy Theorem to even delta-matroids (Theorem 4.1.36), and we further extend it to antisymmetric matroids (Theorem 4.2.29).

3.2 Matroids with coefficients

Matroids with coefficient were founded by Dress and Wenzel [48, 51, 53] offering a unified approach to theories of representable matroids, oriented matroids [11], valuated matroids [54], and ordinary matroids. It was based on comprehending Grassmann-Plücker relations over *fuzzy ring* and was culminated by Baker and Bowler [4] introducing tracts. Tracts are more tractable and extensive field-like structures than fuzzy rings, which we reviewed in Subsection 1.1.6. Baker and Bowler's approach encompasses partial field representations of matroids [105] previously not covered by fuzzy rings, and it is generalized to flag matroids [70], even delta-matroids [71], and antisymmetric matroids [72].

In Subsections 3.2.2–3.2.6, we briefly survey Baker-Bowler theory, that is, matroids with coefficients in tracts. Beforehand, we see more examples of tracts in Subsection 3.2.1.

3.2.1 More examples of tracts

The readers can find the definition of tracts in Subsection 1.1.6. Here, we look at several examples including tracts already seen in Subsection 1.1.6. Note that the tracts along with tract homomorphisms form a category. We depict some tract homomorphisms in Figure 3.3.

Example 3.2.1. The *initial tract* is $\mathbb{I} = (\{1, -1\}, \{0, 1 + (-1)\})$, where the multiplication on $\mathbb{I}^\times$ is the usual one. For any tract F, there is a unique tract homomorphism $\mathbb{I} \to F$ sending 1 to 1 and -1 to -1.

Example 3.2.2. A *partial field* P is a pair (G, R) of a commutative ring R with 1 and a subgroup $G \leq R^\times$ such that -1 belongs to G and G generates the ring R. Hence, P is regarded as a set $G \cup \{0\}$ along with the addition and mutiplication are inherited from R. The tract associated with the partial field P is a pair (G, N_P) where the null set N_P is the set of all formal sums $\sum_{i=1}^k x_i \in \mathbb{N}[G]$ such that $\sum_{i=1}^k x_i = 0 \in R$.

Here are some examples of partial fields.

- The *regular partial field* is $\mathbb{U}_0 := (\{1, -1\}, \mathbb{Z})$.
- The *dyadic partial field* is $\mathbb{D} := (\langle -1, \frac{1}{2} \rangle, \mathbb{Z}[\frac{1}{2}])$.
- The *sixth-root-of-unity partial field* is $R_6 := (\langle \zeta \rangle, \mathbb{Z}[\zeta])$ where $\zeta \in \mathbb{C}^\times$ is a root of $x^2 - x + 1 = 0$.

Example 3.2.3. Let $\mathbb{F}$ be a field and let G be a subgroup of $\mathbb{F}^\times$. Then the multiplicative monoid $F = \mathbb{F}/G = (\mathbb{F}^\times/G) \cup \{0\}$ can be endowed with a natural tract structure by setting the null set N_F as $\{\sum_{i=1}^k x_i \in \mathbb{N}[\mathbb{F}^\times/G] : 0 \in \sum_{i=1}^k x_i\}$. We call tracts of this form *quotient hyperfields*.

- The *Krasner hyperfield* $\mathbb{K}$ is the quotient hyperfield $\mathbb{R}/\mathbb{R}^\times$. Identically, it is the quotient hyperfield associated with a field $\mathbb{F} \neq \mathbb{F}_2$ and its multiplicative group $\mathbb{F}^\times$. As a tract, it can be written as $\mathbb{K} = (\{1\}, N_\mathbb{K})$ where $N_\mathbb{K} := \mathbb{N}[\mathbb{K}^\times] \setminus \{1\} = \{0, 1 + 1, 1 + 1 + 1, \cdots\}$.
- The *sign hyperfield* $\mathbb{S}$ is a quotient hyperfield $\mathbb{R}/\mathbb{R}_{>0}$. It is also identified with the quotient hyperfield associated with any ordered field along with the group consisting of positive elements. As a tract, it can be written as $\mathbb{S} = (\{+, -\}, N_\mathbb{S})$ where $N_\mathbb{S}$ is the set of the zero element $0 \in \mathbb{N}[\{+, -\}]$ and the sums $\sum x_i$ in which not all x_i's are identical. We sometimes write elements $+$ and $-$ as 1 and -1, respectively.

Example 3.2.4. Recall that the *tropical hyperfield* $\mathbb{T}$ is a tract $(\mathbb{T}^\times = \mathbb{R}_{>0}, N_\mathbb{T})$ where a sum $\sum x_i \in \mathbb{N}[\mathbb{R}_{>0}]$ is in $N_\mathbb{T}$ if and only if it is 0 or the maximum of x_i's is achieved at least twice. For an arbitrary tract F, we have a tract homomorphism $F \to \mathbb{T}$ by mapping every nonzero element to 1, and therefore $\mathbb{K} \to \mathbb{T}$. We, however, note that tract homomorphisms from F to $\mathbb{T}$ may not be unique. Recall that if $\mathbb{F}$ is a field with a valuation $\mathrm{val} : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$, then it naturally induces a tract homomorphism $\mathbb{F} \to \mathbb{T}$ taking $x \mapsto e^{-\mathrm{val}(x)}$. Therefore, if the valuation is nontrivial, i.e., there are distinct element $a, b \in \mathbb{F}^\times$ such that $\mathrm{val}(a) \neq \mathrm{val}(b)$, then the corresonding homomorphism is nontrivial as well. One can find several examples of nontrivial valuations in [84, Section 2.1].

3.2.2 Grassmann-Plücker relations over tracts

The Grassmann-Plücker relations (GP) can be considered over tracts as follows. The following definition is a slight modification of the original definition in [4], which will be reviewed later in the same section.

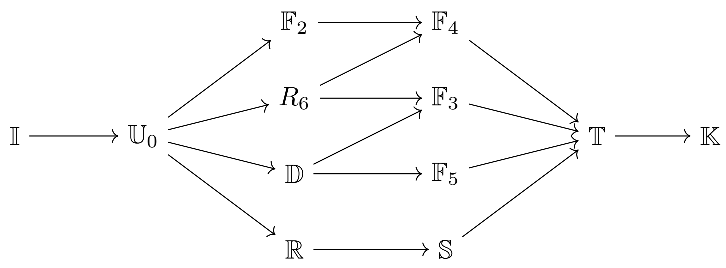


Figure 3.3: Examples of tracts and tract homomorphisms.

Definition 3.2.5 ([4])**.** Let F be a tract, E be a finite set, and r be an integer with $0 \leq r \leq |E|$. A (*strong*) *matroid of rank* r *on* E *over* F or a (*strong*) F*-matroid* is a point X in the projective space $\mathbb{P}(F^{\binom{n}{r}}) := (F^{\binom{E}{r}} \setminus \{\mathbf{0}\})/F^{\times}$ satisfying the Grassmann-Plücker relations:

$$\sum_{x \in S \setminus T} (-1)^{|S<x|+|T<x|} X_{S-x} X_{T+x} \in N_F$$

for all $S \in \binom{E}{r+1}$ and $T \in \binom{E}{r-1}$.

For an F-matroid M of rank r on E, let $\underline{M} := (E, \{B \in \binom{E}{r} : M_B \neq 0\})$. Then it is a matroid, and we call it the *underlying matroid* of M.

Remarkably, F-matroids are equivalent to several well-known notions, depending on the choice of a tract F. If $F = \mathbb{K}$ is the Krasner hyperfield, then a point X in $\mathbb{P}(\mathbb{K}^{\binom{E}{r}})$ is identified with the nonempty family $\mathcal{B} := \{B \in \binom{E}{r} : X_B \neq 0\}$ of r-subsets of E and the Grassmann-Plücker relations over $\mathbb{K}$ are exactly the strong base exchange property of matroids. More precisely, let $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$. Then by the Grassmann-Plücker relation applied to $S = B' + e$ and $T = B - e$, we have

$$1 + \sum_{x \in B' \setminus B} X_{B'+e-x} X_{B-e+x} = \sum_{x \in S \setminus T} X_{S-e} X_{T+e} \in N_{\mathbb{K}},$$

implying that $B - e + f$, $B' + e - f \in \mathcal{B}$ for some $f \in B' \setminus B$. Therefore, we can identify $\mathbb{K}$-matroids as ordinary matroids.

Oriented matroids [11] are equivalent to $\mathbb{S}$-matroids, and valuated matroids [54] are equivalent to $\mathbb{T}$-matroids. If P is a partial field, then P-matroids correpond to partial field representations of matroids [105]. Especially, $\mathbb{U}_0$-matroids are identified with regular matroids.

For a field $\mathbb{F}$, an $\mathbb{F}$-matroid is exactly a Plücker vector by definition. Theorem 3.1.4 shows that if the support of $X \in \mathbb{P}(\mathbb{F}^{\binom{E}{r}})$ forms the set of bases of a matroid, then it suffices to check the 3-term Grassmann-Plücker relations for guaranteeing that X is a $\mathbb{F}$-matroid. Hence, we can consider a weaker notion of matroids over tracts as follows.

Definition 3.2.6 ([4])**.** Let F be a tract, E be a finite set, and r be an integer with $0 \leq r \leq |E|$. A *weak matroid of rank* r *on* E *over* F or a *weak* F*-matroid* is a point X in the projective space $\mathbb{P}(F^{\binom{n}{r}}) := (F^{\binom{E}{r}} \setminus \{\mathbf{0}\})/F^{\times}$ such that $(E, \{B \in \binom{E}{r} : X_B \neq 0\})$ is a matroid and X satisfies the 3-term Grassmann-Plücker relations:

$$X_{R\cup\{i_1,i_2\}} X_{R\cup\{i_3,i_4\}} - X_{R\cup\{i_1,i_3\}} X_{R\cup\{i_2,i_4\}} + X_{R\cup\{i_1,i_4\}} X_{R\cup\{i_2,i_3\}} \in N_F$$

with $R \in \binom{[n]}{r-3}$ and $i_1 < i_2 < i_3 < i_4$ in $[n] \setminus R$.

Every strong F-matroid is a weak F-matroid, but the converse fails in general; see [4, Section 3.11]. Baker and Bowler extended Theorem 3.1.4 by showing that every weak F-matroid is strong if F is a *perfect* tract [4, Theorem 3.46]. We note that all fields, all partial fields, the sign hyperfield $\mathbb{S}$, the tropical hyperfield $\mathbb{T}$, and the Krasner hyperfield $\mathbb{K}$ are perfect tracts.

Here is the original definitions of strong and weak F-matroids in [4]

Definition 3.2.7 ([4])**.** Let F be a tract, E be a finite set, and r be an integer with $0 \le r \le |E|$. A (*strong*) *Grassmann-Plücker function of rank r on E with coefficients in F* is a function $\varphi : E^r \to F$ satisfying (GP1)–(GP3):

(GP1) φ is not identically zero.

(GP2) φ is alternating, i.e., for all $x_1, \dots x_r \in E$, $\varphi(x_1, \dots, x_r) = 0$ if $x_i = x_j$ for some $i \neq j$, and $\varphi(x_1, \dots, x_i, \dots, x_j, \dots, x_r) = -\varphi(x_1, \dots, x_j, \dots, x_i, \dots, x_r)$.

(GP3) For any two subsets $\{x_1, \dots, x_{r+1}\}$ and $\{y_1, \dots, y_{r-1}\}$ of E, we have the *Grassmann-Plücker relations*:

$$\sum_{k=1}^{r+1} (-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{r+1}) \varphi(x_k, y_1, \dots, y_{r-1}) \in N_F.$$

A *weak Grassmann-Plücker function of rank r on E with coefficients in F* is a function $\varphi : E^r \to F$ such that the support $\{\{x_1, \dots, x_r\} \in \binom{E}{r} : \varphi(x_1, \dots, x_r) \neq 0\}$ of φ is the set of bases of a rank r matroid on E, and φ satisfies (GP1), (GP2), and the next weaker replacement of (GP3).

(GP3′) For any two subsets $J_1 = \{x_1, \dots, x_{r+1}\}$ and $J_2 = \{y_1, \dots, y_{r-1}\}$ of E with $|J_1| = r+1$, $|J_2| = r-1$, and $|J_1 \setminus J_2| = 3$, we have the 3-*term Grassmann-Plücker relations*:

$$\sum_{k=1}^{r+1} (-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{r+1}) \varphi(x_k, y_1, \dots, y_{r-1}) \in N_F.$$

Two strong (resp. weak) Grassmann-Plücker functions φ_1 and φ_2 are *equivalent* if $\varphi_1 = c \cdot \varphi_2$ for some $c \in F^\times$, and we call an equivalence class $M_\varphi := [\varphi]$ of strong (resp. strong) Grassmann-Plücker functions a *strong* (resp. *weak*) *matroid over the tract F*, or simply a *strong* (resp. *weak*) *F-matroid*.

The *underlying matroid* of a strong or weak Grassmann-Plücker function φ is a matroid $\underline{M}_\varphi$ on E, whose set of bases is $\{\{x_1, \dots, x_r\} \in \binom{E}{r} : \varphi(x_1, \dots, x_r) \neq 0\}$.

3.2.3 F-circuits and dual pairs

We now review two cryptomorphic definitions of matroids over a tract F in terms of F-circuits and dual pairs of F-signatures.

We denote by F^E the set of all functions from E to F, which we call vectors. The *support* of $X \in F^E$ is the set of elements e in E such that $X(e) \neq 0$, and is denoted by $\underline{X}$ or supp$X$. Given two functions $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n) \in F^E$, where F is endowed with an involution $x \mapsto \overline{x}$, where an *involution* is a tract ismorphism $\tau : F \to F$ such that τ^2 is the identity map. The *inner product* of X and Y is $X \cdot Y := \sum_{k=1}^n x_k \overline{y_k}$. We say that two functions X and Y are *orthogonal*, denoted by $X \perp Y$, if $X \cdot Y \in N_F$.

When F is the field $\mathbb{C}$ of complex numbers or the sixth-root-of-unity partial field R_6, the involution $x \mapsto \overline{x}$ should be taken to be the complex conjugation. For $F \in \{\mathbb{K}, \mathbb{S}, \mathbb{T}\}$, the involution should be taken to be the identity map.

The *linear span* of $X_1, \ldots, X_k \in F^E$ is defined to be the set of all functions $X \in F^E$ such that

$$c_1X_1 + c_2X_2 + \cdots + c_kX_k - X \in (N_F)^E$$

for some $c_1, \ldots, c_k \in F$.

Definition 3.2.8. Let F be a tract and let $\underline{M}$ be an ordinary matroid on E. An *F-signature* of $\underline{M}$ is a subset $\mathcal{C} \subseteq F^E$ such that the following hold:

(i) The support $\underline{\mathcal{C}} := \{\underline{X} : X \in \mathcal{C}\}$ of $\mathcal{C}$ is the set of circuits of $\underline{M}$.

(ii) For all $X \in \mathcal{C}$ and $\alpha \in F^\times$, we have $\alpha X \in \mathcal{C}$.

(iii) If $X, Y \in \mathcal{C}$ and $\underline{X} \subseteq \underline{Y}$, then $X = \alpha Y$ for some $\alpha \in F^\times$.

For an ordinary matroid $\underline{M}$ on E, recall that we denote by $C_{\underline{M}}(B, e)$ the fundamental circuit of $\underline{M}$ with respect to a base B and an element $e \in E \setminus B$. The subscript will be omitted if no confusion arises.

Definition 3.2.9. Let F be a tract and let $\underline{M}$ be an ordinary matroid on E. A subset $\mathcal{C}$ of F^E is called a (*strong*) *F-circuit set of $\underline{M}$* if it satisfies the following axioms:

(CS1) $\mathcal{C}$ is an F-signature of $\underline{M}$.

(CS2) For every base B of $\underline{M}$ and for each $X \in \mathcal{C}$, X is in the linear span of $\{X_e : e \in E \setminus B\}$ where $X_e \in \mathcal{C}$ whose support is $C(B, e)$.

We call $\mathcal{C}$ a *weak F-circuit set of $\underline{M}$* if it satisfies (CS1) and the following replacement:

(CS2)′ For every base B of $\underline{M}$ and distinct elements $e_1, e_2 \in E \setminus B$, if X_1 and X_2 in $\mathcal{C}$ have supports $C(B, e_1)$ and $C(B, e_2)$, respectively, and f is a common element of the two supports, then there exists Y in $\mathcal{C}$ such that $Y(f) = 0$ and Y is in the linear span of X_1 and X_2.

Definition 3.2.10. Let F be a tract and let $\underline{M}$ be an ordinary matroid on E. A pair $(\mathcal{C}, \mathcal{D})$ of subsets of F^E is called a (*strong*) *dual pair of F-signatures of $\underline{M}$* if

(DP1) $\mathcal{C}$ is an F-signature of $\underline{M}$.

(DP2) $\mathcal{D}$ is an F-signature of the dual matroid $\underline{M}^*$.

(DP3) $X \perp Y$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

A pair $(\mathcal{C}, \mathcal{D})$ is called a *weak dual pair of F-signatures of $\underline{M}$* if it satisfies (DP1), (DP2), and the following weakening of (DP1):

(DP3)′ $X \perp Y$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ with $|\underline{X} \cap \underline{Y}| \leq 3$.

3.2.4 F-vectors

It would be natural to ask whether the (co)vectors axiom for oriented matroids can be generalized for F-matroids. Anderson [2] answered this question affirmatively by giving another cryptomorphic definition of strong F-matroids in terms of F-vectors.

For a subset $\mathcal{W} \subseteq F^E$, a *support base* for $\mathcal{W}$ is a minimal subset of E meeting every element of $\mathrm{supp}(\mathcal{W} \setminus \{0\})$. A *reduced row-echelon form* of $\mathcal{W}$ with respect to a support base B is a subset $\Phi_B = \{w_i^B\}_{i \in B} \subseteq \mathcal{W}$ such that $w_i^B(j) = \delta_{ij}$ for each $i, j \in B$, and every $w \in \mathcal{W}$ is in the linear span of Φ_B. It is not difficult to see that if Φ_B exists, then it is unique. We say a collection $\Phi = \{\Phi_B\}$ of reduced row-echelon forms is *tight* if $\mathcal{W}$ is precisely the set of elements of F^E which are in the linear span of Φ_B for all $\Phi_B \in \Phi$.

Definition 3.2.11. A subset $\mathcal{W}$ of F^E is an *F-vector set on E* if the following hold:

(V1) Every support base has a reduced row-echelon form.

(V2) The collection of all such reduced row-echelon forms is tight.

3.2.5 Crypotomorphisms

The main results of [4, 2] are the following theorems. Let E be a finite set and let F be a tract endowed with an involution $x \mapsto \overline{x}$.

Theorem 3.2.12 ([4, Theorem 4.17] and [2, Theorem 2.18])**.** *There are natural bijections between any pair of the following:*

(i) *Strong F-matroids on E.*

(ii) *Strong F-circuit sets of matroids on E.*

(iii) *Ordinary matroids on E endowed with a strong dual pair of F-signatures.*

(iv) *F-vector sets on E.*

Theorem 3.2.13 ([4, Theorem 4.18])**.** *There are natural bijections between any pair of the following:*

(i) *Weak F-matroids on E.*

(ii) *Weak F-circuit sets of matroids on E.*

(iii) *Ordinary matroids on E endowed with a weak dual pair of F-signatures.*

3.2.6 Functoriality, duality, and minors

Let F be a tract with an involution $x \mapsto \overline{x}$. The theory of functoriality, duality, and minors for matroids over tracts generalizes the corresponding classical theory for matroids. For simplicity, here we only give the descriptions via the strong Grassmann-Plücker functions.

Given a strong Grassmann-Plücker function $\varphi : E^r \to F$ and a tract homomorphism $f : F \to F'$, we define the *pushforword* $f_*\varphi : E^r \to F'$ as

$$(f_*\varphi)(x_1, \ldots, x_r) = f(\varphi(x_1, \ldots, x_r)).$$

It is not hard to see that $f_*\varphi$ is a strong Grassmann-Plücker function. Notice that pushforwards are functorial: If $F_1 \xrightarrow{f} F_2 \xrightarrow{g} F_3$ are tract homomorphisms, then $(g \circ f)_* = g_* \circ f_*$.

The *dual Grassmann-Plücker function* $\varphi^* : E^{n-r} \to F$ of φ is determined by (GP2) and

$$\varphi^*(x_1, \ldots, x_{n-r}) = \mathrm{sign}(x_1, \ldots, x_{n-r}, x'_1, \ldots, x'_r) \cdot \overline{\varphi(x'_1, \ldots, x'_r)},$$

where $x'_1, \ldots, x'_r$ is any ordering of $E \setminus \{x_1, \ldots, x_{n-r}\}$, and $\mathrm{sign}(x_1, \ldots, x_{n-r}, x'_1, \ldots, x'_r) \in \{\pm 1\}$ is the permutation sign taken as an element of F. This notion of dual Grassmann-Plücker functions satisfies $\varphi^{**} = \varphi$, and the underlying matroid of φ^* is the dual matroid of the underlying matroid of φ.

Let $\varphi : E^r \to F$ be a strong Grassmann-Plücker function with the underlying matroid $\underline{M} := \underline{M}_\varphi$ and let $A \subseteq E$. We denote by ℓ and k the ranks of A and $E \setminus A$ in $\underline{M}$, respectively. Choose $\{a_1, \dots, a_\ell\}$ be a maximal independent subset of A in $\underline{M}$. The *contraction* $\varphi/A : (E \setminus A)^{r-\ell} \to F$ is defined by

$$(\varphi/A)(x_1, \dots, x_{r-\ell}) = \varphi(x_1, \dots, x_{r-\ell}, a_1, \dots, a_\ell).$$

Choose $\{b_1, \dots, b_{r-k}\}$ such that $\{b_1, \dots, b_{r-k}\}$ is a base of $\underline{M}/(E \setminus A)$. Then the *deletion* $\varphi \setminus A : (E \setminus A)^k \to F$ is defined by

$$(\varphi \setminus A)(x_1, \dots, x_k) = \varphi(x_1, \dots, x_k, b_1, \dots, b_{r-k}).$$

The following lemma shows that the contractions and deletions are well-defined.

Lemma 3.2.14 ([4, Lemma 4.4])**.** *The following hold.*

(i) *Both φ/A and $\varphi \setminus A$ are strong Grassmann-Plücker functions, and they are independent of all choices $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_{r-k}\}$ up to a multiplication by an element in $F^\times$.*

(ii) $\underline{M}_{\varphi/A} = \underline{M}/A$ *and* $\underline{M}_{\varphi\setminus A} = \underline{M} \setminus A$.

(iii) $(\varphi \setminus A)^* = \varphi^*/A$.

3.3 Representability of matroids

Tutte [108, 109, 111] presented the excluded minors for binary, regular, and graphic matroids, which are arguably the most well-known results in matroid theory. By defintion, every regular matroid is binary. In addition, every graphic matroid is regular because, for a graph G and any field $\mathbb{F}$, the incidence $(0, \pm 1)$-matrix $I_{\overrightarrow{G}}$ over $\mathbb{F}$ regarding an arbitrary orientation $\overrightarrow{G}$ is an $\mathbb{F}$-representation of the cycle matroid $M(G)$. More precisely, each cycle of G corresponds to a set of minimally dependent columns in $I_{\overrightarrow{G}}$; see Figure 2.1 for example.

Theorem 3.3.1 (see [98, Theorem 9.1.3])**.** *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

Theorem 3.3.2 (see [98, Theorem 6.6.6])**.** *A matroid is regular if and only if it has no minor isomorphic to $U_{2,4}$, F_7, or F_7^*.*

Corollary 3.3.3. *A binary matroid is regular if and only if it has no minor isomorphic to F_7 or F_7^*.*

Theorem 3.3.4 (see [98, Theorem 6.6.7])**.** *A matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M(K_5)$, or $M(K_{3,3})$.*

The following equivalent conditions of regular matroids are easy consequences of Tutte's characterization on the excluded minors for binary and regular matroids, Theorems 3.3.1 and 3.3.2. More precisely, it is easy to see that the Fano matroid F_7 and its dual F_7^* are not representable over any field of characteristic other than two; see [98, Proposition 6.5.5], and also they are not orientable as well; see [12, Proposition 6.1] or [11, Examples 6.6.2(1)].

Theorem 3.3.5. *Let M be a matroid. Then the following are equivalent:*

(i) *M is regular, i.e., representable over all fields.*

(ii) *It is represented by a totally unimodular matrix.*

(iii) *It is* $\mathbb{U}_0$*-representable.*

(iv) *It is binary and ternary.*

(v) *It is binary and orientable.*

(vi) *It is binary and* $\mathbb{F}$*-represenable for some field* $\mathbb{F}$ *of characteristic other than two.*

This kind of representability theorem is extended to several other fields as follows. The next theorem follows from results by Whittle [122, Theorem 1.2] and van Zwam [113, Lemma 2.5.12].

Theorem 3.3.6. *Let* M *be a matroid. Then the following are equivalent:*

(i) M *is represented by a complex matrix of which maximal minors are* 0 *or powers of the sixth root* $e^{\pi/3}$ *of unity.*

(ii) *It is* R_6*-representable.*

(iii) *It is ternary and quaternary.*

(iv) *It is ternary and* $\mathbb{F}_{2^k}$*-representable for some positive even integer* k.

(v) *It is ternary,* $\mathbb{F}_{p^2}$*-representable for all primes* p*, and* $\mathbb{F}_q$*-representable for all primes* q *with* $q \equiv 1 \pmod 3$.

Whittle [122, Theorem 1.1] showed another result.

Theorem 3.3.7. *Let* M *be a matroid. Then the following are equivalent:*

(i) M *is dyadic, i.e., represented by a real matrix of which maximal minors are* 0 *or* $\pm 2^k$ *with* $k \in \mathbb{Z}$.

(ii) *It is* $\mathbb{D}$*-representable.*

(iii) *It is ternary and quintanary.*

(iv) *It is ternary and* $\mathbb{F}_q$*-representable for some odd prime power* q *such that* $q \equiv 2 \pmod 3$.

(v) *It is* $\mathbb{F}_p$*-representable over all odd primes* p.

(vi) *It is ternary and rational.*

(vii) *It is ternary and real.*

Chapter 4. Generalizations of matroids

We review several generalizations of matroids. In Section 4.1, we review delta-matroids that captures common properties of (skew-)symmetric matrices, matchings, and embedded graphs. This section is arranged parallel to Section 3.1, so it would be interesting to compare those two sections. We devote Subsection 4.1.7 to show that even delta-matroids are essentially equivalent to pfaffian structures introduced by Kung [77] earlier than delta-matroids [20, 49, 41]. In Section 4.2, we introduce a new concept, called 'antisymmetric matroids,' which generalize matroids and even delta-matroids. We present two equivalent defintions of antisymmetric matroids in terms of bases and circuits. We also prove that the Lagrangian symplectic Grassmannian is cut out by certain quadratic relations standing for the Laplace expansion of symmetric matrices only concerning prinicipal and alomst-principal minors, and we discuss the connection between these quadratic relations and antisymmetric matroids. We show that antisymmetric matroids encompasses both matroids and even delta-matroids, and we present the Homotopy Theorem for antisymmetric matroids (Theorem 4.2.29), which generalizes Maurer's Homotopy Theorem for matroids (Theorem 3.1.13) and Wenzel's Homotopy Theorem for even delta-matroids (Theorem 4.1.36).

4.1 Delta-matroids

We review delta-matroids which (i) are a generalization of matroids and (ii) capture combinatorial properties of symmetric/skew-symmetric matrices and graphs embedded on closed surfaces. In the late 1980s, Bouchet [20] introduced 'delta-matroid,' which can be defined by relaxing the base exchange axiom for matroids through the symmetric difference operation. He also introduced 'symmetric matroids' and '2-matroids' [32] that are equivalent to delta-matroids. Around the same time, independently, Dress and Havel [49] defined 'metroids' and Chandrasekaran and Kabadi [41] defined 'pseudomatroids,' which are tantamount to delta-matroids as well. In the sense of Coxeter matroids [19], delta-matroids exactly correspond to Lagrangian (symplectic) matroids that are Coxeter matroids of type C.

We will sometimes focus on even delta-matroids, which capture combinatorial properties of skew-symmetric matrices and graphs embedded on orientable surfaces. An even delta-matroid is defined as a delta-matroid whose bases have the same parity, whereas a matroid is exactly a delta-matroid whose bases have the same cardinality. Indeed, Kung [77] in 1978 introduced 'pfaffian structures' which are equivalent to even delta-matroids. It has, however, not been revealed that pfaffian structures and even delta-matroids are equivalent until we observed (Proposition 4.1.56); see [38, Page 56]. Note that even delta-matroids exactly correspond to Lagrangian orthogonal matroids that are Coxter matroids of type D [19].

Delta-matroids are defined by relaxing the base exchange axiom for matroids.

Definition 4.1.1. A *delta-matroid* is a pair $D = (E, \mathcal{B})$ of a finite set E and a nonempty set $\mathcal{B}$ of subsets of E satisfying the *symmetric exchange axiom*:

(△B) For all $B, B' \in \mathcal{B}$ and $e \in B \triangle B'$, there is $f \in B \triangle B'$ (possibly, $e = f$) such that $B \triangle \{e, f\} \in \mathcal{B}$.

Each element in $\mathcal{B}$ is called a *base* of D. A delta-matroid is *normal* if the empty set is a base. A delta-matroid is *even* if all bases have the same parity.

Example 4.1.2. A matroid is exactly a delta-matroid all of whose bases have the same cardinality.

For a delta-matroid $M = (E, \mathcal{B})$ and a subset $X \subseteq E$, let $\mathcal{B}\triangle X := \{B\triangle X : B \in \mathcal{B}\}$. Then $M\triangle X := (E, \mathcal{B}\triangle X)$ is a delta-matroid. We call $M\triangle X$ the *twist of* M *by* X and call the corresponding operation the *twisting by* X. The *dual* of M is $M^{\perp} := M\triangle E$.

For a set system $M = (E, \mathcal{B})$, let $M_{\max} := (E, \mathrm{Max}(\mathcal{B}))$ and $M_{\min} := (E, \mathrm{Min}(\mathcal{B}))$. If M is a delta-matroid, then $M_{\max}$ and $M_{\min}$ are matroids due to [20, (3.2)], and we call them the *upper matroid* and the *lower matroid* of M, respectively [27, Page 64]. We have the following alternative definition of delta-matroids in terms of upper matroids.

Proposition 4.1.3 ([37, Property 4.1])**.** *A set system $M = (E, \mathcal{B})$ with a finite set E is a delta-matroid if and only if $(M\triangle X)_{\max}$ is a matroid for every $X \subseteq E$.*

The same result holds even if we replace $(M\triangle X)_{\max}$ with $(M\triangle X)_{\min}$ in Proposition 4.1.3, and we note that $(M_{\max})^{\perp} = (M^{\perp})_{\min}$.

One of the primary examples of delta-matroids arises from symmetric and skew-symmetric matrices.

Example 4.1.4 (Representable delta-matroids)**.** Let A be an $E\times E$ symmetric or skew-symmetric matrix over a field $\mathbb{F}$. Then a pair $M(A) := (E, \mathcal{B})$ is a delta-matroid, where $\mathcal{B} := \{X \subseteq E : A[X] \text{ is nonsingular}\}$ by Bouchet [25]. Whenever A is skew-symmetric, the corresponding delta-matroid $M(A)$ is even.

We say a delta-matroid is *representable over* $\mathbb{F}$ or $\mathbb{F}$*-representable* if it is isomorphic to $M(A)\triangle X$ for some symmetric or skew-symmetric matrix A over $\mathbb{F}$ and a subset $X \subseteq E$. If A is skew-symmetric, then by Theorem 1.1.19 the associating delta-matroid $M(A) = (E, \mathcal{B})$ satisfies the following *strong exchange property*:

($\triangle$B′) If $B, B' \in \mathcal{B}$ and $e \in B\triangle B'$, then there is $f \in (B\triangle B') \setminus \{e\}$ such that both $B\triangle\{e, f\}$ and $B'\triangle\{e, f\}$ are in $\mathcal{B}$.

Wenzel [115] proved that every even delta-matroid satisfies the strong exchange property.

Proposition 4.1.5 (Wenzel [115])**.** *Let $M = (E, \mathcal{B})$ be a pair of a finite set E and a nonempty set $\mathcal{B}$ of subsets of E. Then the following are equivalent:*

(i) *M is an even delta-matroid.*

(ii) *If $B, B' \in \mathcal{B}$ and $e \in B\triangle B'$, then there is $f \in (B\triangle B') \setminus \{e\}$ such that both $B\triangle\{e, f\} \in \mathcal{B}$.*

(iii) *It satisfies the strong exchange property* ($\triangle$B′)*.*

Recall that the support of a Plücker vector is the set of bases of a matroid, and the strong base exchange property of matroids is exactly the Grassmann-Plücker relations over the Krasner hyperfield $\mathbb{K}$; Subsection 3.2.2 or [4]. Analogously, the strong exchange property of even delta-matroids is identified with the Wick relations over the Krasner hyperfield $\mathbb{K}$. We extend the Wick relations over general tracts in Section 5.1.1. In contrast to matroids and even delta-matroids, general delta-matroids do not satisfy a strong exchange property as shown in Example 1.1.8. To utilize delta-matroids as a combinatorial abstraction of the Lagrangian symplectic Grassmannian, it is natural to demand a representable delta-matroid as the support of a point parameterizing a Lagrangian subspace into the projective space. It is however impossible because the defining ideal of the Lagrangian symplectic Grassmannian in the projective space is homogeneous and some representable delta-matroids do not satisfy the strong base exchange property. We highlight that antisymmetric matroids overcome this problem and are combinatorial counterparts of the Lagrangian symplectic Grassmannian. We discuss it in the next Section 4.2.

Bouchet [25, (4.4)] showed that a matroid is representable over a field $\mathbb{F}$ in the sense of matroids if and only if it is representable over $\mathbb{F}$ in the sense of delta-matroids. The idea of the forward direction is easy. For a given matroid M of rank r on a ground set E, let $[I_r \mid A]$ be a matrix representation of M over a field $\mathbb{F}$. We denote by the first r columns by $B \subseteq E$, and thus B is a base of M. Then it is striahgtforward that $M = M(\Lambda)\triangle B = M(\Lambda')\triangle B$, where

$$\Lambda := \begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix} \quad \text{and} \quad \Lambda' := \begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix}.$$

An even delta-matroid represented by a skew-symmetric matrix is always even. In contrast, an even delta-matroid can be represented not only by a skew-symmetric matrix but also by a symmetric matrix. By the following observation by Geelen [61], we can always assume that a matrix witnessing that an even delta-matroid is $\mathbb{F}$-representable is skew-symmetric.

Proposition 4.1.6 ([61, Page 27]). *An even delta-matroid admits a symmetric matrix representation over a field $\mathbb{F}$ if and only if M is a matroid up to twisting or $\mathbb{F}$ has characteristic two.*

Corollary 4.1.7. *If an even delta-matroid M on E is representable over a field $\mathbb{F}$, then there are a subset X of E and an $E \times E$ skew-symmetric matrix A over F such that $M = M(A)\triangle X$.* □

In the remaining subsections, we review more properties and examples of delta-matroids. We give several examples in Subsection 4.1.1. In Subsection 4.1.2, we discuss minors of delta-matroids and exlcluded minors for several classes of delta-matroids. In Subsections 4.1.3–4.1.6, we discuss connectivity, fundamental graphs, base graphs, and circuits in order. We devote Subsection 4.1.7 to show that pfaffian structures [77, 78] are equivalent to normal even delta-matroids.

4.1.1 Examples

Example 4.1.8 (Matching delta-matroids). Bouchet [28] showed that for a graph G, all vertex subsets inducing subgraphs with perfect matchings form the bases of an even delta-matroid on $V(G)$. We call such a delta-matroid a *matching delta-matroid.*

Example 4.1.9 (Mader delta-matroids). Let G be a graph, let T be a subset of $V(G)$, and let $\mathcal{S}$ be a partition of T. An $\mathcal{S}$-*path* is a path in G whose end points are in distinct parts of $\mathcal{S}$ and which is internally disjoint from T. An $\mathcal{S}$-*path packing* is a set of parwise disjoint $\mathcal{S}$-paths. We say a subset F of T is *fesible* if there is an $\mathcal{S}$-path packing whose end points are exactly F. Then $(T, \mathcal{B})$ is an even delta-matroid and is representable over fields with sufficiently many elements, shown by Wahlström [114]. Such a delta-matroid is called a *Mader delta-matroid.* Note that if $T = V(G)$ and $\mathcal{S}$ is the partition into single vertices, then the corresponding Mader delta-matroid is a matching delta-matroid.

Example 4.1.10 (Γ-graphic delta-matroids). Let Γ be an abelian group. A (Γ, γ) is a pair of a graph G and a map $\gamma : V(G) \to \Gamma$. A subgraph H of G is γ-*nonzero* if, for each component C of H,

(i) $\sigma_{v\in V(C)}\gamma(v) \neq 0$ or $\gamma|_{V(C)} \equiv 0$, and

(ii) if $\gamma|_{V(C)} \equiv 0$, then $G[V(C)]$ is a component of G.

An edge set $F \subseteq E(G)$ is *feasible* if a spanning subgraph $(V(G), F)$ is acyclic and γ-nonzero. Then $\mathcal{G}(G,\gamma) := (E(G), \mathcal{F})$ is a delta-matroid, where $\mathcal{F}$ is the set of feasible edge sets by Kim, Lee, and Oum [73, Theorem 1.1].

A delta-matroid is Γ-*graphic* if it is isomorphic to a twisitng of some $\mathcal{G}(G,\gamma)$. Note that $\mathbb{Z}_2$-graphic delta-matroids are precisely *graphic delta-matroids* introduced by Oum [94], and it is even. We also note that if γ is the constant map to 0, then the corresponding graphic delta-matroid is equal to the cycle matroid $M(G)$.

We discuss another extension of graphic matroids below.

Example 4.1.11 (Ribbon-graphic delta-matroids)**.** Let G be a graph 2-cell-embedded on a closed surface Σ, i.e., each face is homeomorphic to an open disk. Let $\mathcal{B}$ be the set of edge sets F such that $\Sigma \setminus (V \cup F)$ is homeomorphic to an open disk. Then $\mathcal{B}$ satisfies the base exchange axiom ($\triangle$B) for delta-matroids, and we call $M(G;\Sigma) := (E(G), \mathcal{B})$ a *ribbon-graphic* delta-matroid. We remark that $M(G;\Sigma)$ is even if and only if Σ is orientable. Whenever Σ is the sphere, then G is a planar graph and $M(G;\Sigma) = M(G)$. Bouchet [20, 27] showed several fundamental results on ribbon-graphic delta-matroids, and we refer the readers to [88, Sections 5 and 6].

Recall that a square matrix is *principally unimodular* (in short, *PU*) if it is real and all principal minors are 0 or ± 1. We note that each principal minor of a PU skew-symmetric is either 0 or 1 by Cayley's theorem; see [90, Proposition 7.3.3]. Moreover, a skew-symmetric matrix A is PU if and only if the pfaffian of each principal submatrix of A is 0 or ± 1.

Example 4.1.12 (Regular even delta-matroids)**.** An even delta-matroid is *regular* if it can be represented by a principally unimodular skew-symmetric matrix, i.e., it is isomorphic to $M(A) \triangle X$ for some set X and PU skew-symmetric matrix A. Regular even delta-matroid are often called regular delta-matroids by omitting the word "even." Remark that every even ribbon-graphic delta-matroid is regular by [23, 15].

Regular delta-matroids generalize regular matroids. Let M be a regular matroid of rank r on n elements. Then it admits a TU matrix representation, and by rearraging columns and row-equivalence, such TU matrix can be written as $[I_r \mid A]$ where A is an $r \times (n-r)$ TU matrix. Then $\Lambda := \begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix}$ is skew-symmetric and PU. Therefore, M is a regular delta-matroid because it is a twist of a normal regular delta-matroid $M(\Lambda)$. Tutte [108] showed the following celebrated result for regular matroids, and Geelen [61, Theorem 4.13] extended it to regular delta-matroids.

Theorem 3.3.5 ([108])**.** *Let M be a matroid. Then the following are equivalent:*

(i) *M is represented by a TU matrix.*

(ii) *It is representable over all fields.*

(iii) *It is binary and ternary.*

Theorem 4.1.13 ([61])**.** *Let M be an even delta-matroid. Then the following are equivalent:*

(i) *M is represented by a PU skew-symmetric matrix.*

(ii) *It is representable over all fields.*

(iii) *It is binary and ternary.*

It is easy to see that (i) implies (ii) in Theorem 4.1.13, which follows from that a PU matrix A induces an $\mathbb{F}$-matrix A' such that for every index subset X, $\det(A[X]) = 0$ if and only if $\det(A'[X]) = 0$. Only nontrivial part is to show that (iii) implies (ii). In Section 5.3.1, we provide its alternative proof. Our proof demands an additional theorem about partial field representations of even delta-matroids [6, Theorem 4.2], but it is more systematical and extendable to representability over other fields.

Example 4.1.14 (Equable delta-matroids). A delta-matroid is *equable* if it is represented by a PU symmetric matrix, i.e., it is isomorphic to $M(A)\triangle X$ for some set X and PU symmetric matrix A.

Equable delta-matroids generalize regular matroids, analogous to that regular delta-matroids generalize regular matroids. We note that the class of regular delta-matroids and the class of equable delta-matroids does not contains each other. A delta-matroid $(\{x\}, \{\emptyset, \{x\}\})$ is equable and non-even, but not regular. An even delta matroid $([3], \{\emptyset, 12, 13, 23\})$ is regular but not equable.

Geelen [62, Theorem 2.3] showed another extension of Theorem 3.3.5.

Theorem 4.1.15 ([62]). *Let M be a delta-matroid. Then the following are equivalent:*

(i) *M is equable.*

(ii) *It is representable over all fields.*

(iii) *It is binary and ternary.*

Theorem 4.1.15 is a consequence of the excluded minors for equable delta-matroids [62, Theorem 1.1], which implies Tutte's characterization of the excluded minors for regular matroids [108]. We will review it in the next Subsection 4.1.2.

4.1.2 Minors

We review several ways to construct delta-matroids from a given one. Let $M = (E, \mathcal{B})$ be a delta-matroid. Recall that a twist $M\triangle X$ of M by $X \subseteq E$ is a delta-matroid whose bases are $\{B\triangle X : B \in \mathcal{B}\}$. Evidently, $M\triangle X$ is even whenever M is even. The *dual* of M is $M^* := M\triangle E$. If M is a matroid, then M^* is equal to the dual matroid of M. An element $e \in E$ is a *loop* if it is not in any base of M, and e is a *coloop* if it is in all bases of M. Note that e is a loop in M if and only if it is a coloop in $M\triangle\{e\}$. For $e \in E$, the *deletion of e from M* is a pair $M \setminus e := (E \setminus \{e\}, \mathcal{B} \setminus e)$ where

$$\mathcal{B} \setminus e := \begin{cases} \{B : e \notin B \in \mathcal{B}\} & \text{if } e \text{ is not a coloop of } M, \\ \{B \setminus \{e\} : B \in \mathcal{B}\} & \text{otherwise.} \end{cases}$$

It is easy to see that $M \setminus e$ is a delta-matroid, and if M is even, then $M \setminus e$ is also even. The *contraction of e from M* is a delta-matroid $M/e := (M\triangle\{e\}) \setminus e$.

A delta-matroid M is a *strong-minor* of another delta-matroid N if M can be obtained from N by a sequence of deletions and contractions. We say M is a *minor* of N if it is a twist of a strong-minor of N or, equivalently, is obtainable from N by a sequence of deletions, contractions, and twistings.

It is straightforward that every minor of an even delta-matroid is even. Furthermore, there is a unique exluded minor for even delta-matroids.

Theorem 4.1.16 ([27, Lemma 5.4]; see also [57, Proposition 4.1] and [88, Excercise 3.17]). *Let $M = (E, \mathcal{B})$ be a delta-matroid. Then the following are equivalent:*

(i) *M is not even.*

(ii) *There are $B \in \mathcal{B}$ and $e \in E \setminus B$ such that $B \cup \{e\} \in \mathcal{B}$.*

(iii) *M has a strong-minor isomorphic to $(\{x\}, \{\emptyset, \{x\}\})$.*

(iv) *M has a minor isomorphic to $(\{x\}, \{\emptyset, \{x\}\})$.*

The representability is also closed under taking minors.

Theorem 4.1.17 ([37, Property 2.2]). *Let M be a minor of a delta-matroid N. Then the following hold.*

- *If N is represented by a symmetric matrix over a field $\mathbb{F}$, then M is represented by a symmetric matrix over $\mathbb{F}$.*
- *If N is represented by a skew-symmetric matrix over a field $\mathbb{F}$, then M is represented by a skew-symmetric matrix over $\mathbb{F}$.*

In matroid theory, finding the excluded minors for a minor-closed class of matroids is a fundamental problem. Tutte [108] showed that a matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$ (Theorem 3.3.1). Bouchet and Duchamp [37] extended this result for delta-matroids.

Theorem 4.1.18 ([37]). *A delta-matroid M is binary if and only if M has no minor isomorphic to one of the of follwoing:*

(i) $S_1 = ([3], \{\emptyset, 12, 13, 23, 123\})$,

(ii) $S_2 = ([3], \{\emptyset, 1, 2, 3, 12, 13, 23\})$,

(iii) $S_3 = ([3], \{2, 3, 12, 13, 123\})$,

(iv) $S_4 = ([4], \{\emptyset, 12, 13, 14, 23, 24, 34\})$, *and*

(v) $U_{2,4} = ([4], \binom{[4]}{2})$.

Corollary 4.1.19. *An even delta-matroid is binary if and only if it has no minor isomorphic to S_4 or $U_{2,4}$.* □

Interestingly, the upper and lower matroids inherit the representability of an original delta-matroid.

Proposition 4.1.20 ([37, Property 4.2]). *Let M be a delta-matroid representable over a field $\mathbb{F}$. Then a matroid $(M \triangle X)_{\max}$ is representable over $\mathbb{F}$ for every $X \subseteq E$. In particular, the upper and lower matroids $M_{\max}$ and $M_{\min}$ are representable over $\mathbb{F}$.*

Bouchet and Duchamp [37] showed the above proposition using chain-group representations of matroids [108] and delta-matroids [20, 37]; see also [34]. We provide an alternative proof of Proposition 4.1.20 whenever M is even. It is easily deduced by relations between the Grassmann-Plücker relations (GP) and the Wick relations (Wick).

Proof of Proposition 4.1.20 if M is even. As all twistings of M are also $\mathbb{F}$-representable, it suffices to show that $M_{\max}$ is $\mathbb{F}$-representable. Since the even delta-matroid M is F-representable, there is a Wick vector $X \in \mathbb{P}(\mathbb{F}^{2^E})$ such that $\mathcal{B}(M) = \{B \subseteq E : X_B \neq 0\}$. Let r be the size of a base of the upper matroid $M_{\max}$, and let $Y := (X_B : B \in \binom{E}{r}) \in \mathbb{P}(\mathbb{F}^{\binom{E}{r}})$. Then $\mathcal{B}(M_{\max}) = \{B \subseteq E : Y_B \neq 0\}$. Therefore, it suffices to show that Y is a Plücker vector. For $S \in \binom{E}{r+1}$ and $T \in \binom{E}{r-1}$, we have the following Wick relation:

$$\sum_{e \in S \triangle T} (-1)^{|S<e|+|T<e|} X_{S \triangle \{e\}} X_{T \triangle \{e\}} = 0.$$

For each $e \in T \setminus S$, $X_{S\triangle\{e\}} = X_{S+e}$ vanishes because $|S+e| > r$. Therefore, Y satisfies the Grassmann-Plücker relation:

$$\sum_{e \in S \setminus T} (-1)^{|S<e|+|T<e|} Y_{S-e} Y_{T+e} = 0.$$

Thus, Y is a Plücker vector, and $M_{\max}$ is $\mathbb{F}$-representable. □

The converse of Proposition 4.1.20 fails. For instance, S_1, S_2, and S_3 in Theorem 4.1.18 are non-even binary delta-matroids, but all upper matroids of twists of S_1, S_2, S_3 are binary becuase their ground sets are of size three. Bouchet and Duchamp [37, Property 5.3] provided an example of a ternary even deta-matroid S such that $(S \triangle X)_{\max}$ is regular and thus ternary for every $X \subseteq E(S)$. We note that S is not representable over any field of characteristic not two. The converse of Proposition 4.1.20 holds for binary even delta-matroids.

Proposition 4.1.21 ([37, Property 5.2])**.** *Let M be an even delta-matroid. Then M is binary if and only if $(M\triangle X)_{\max}$ is binary for every $X \subseteq E$.*

The exluded minors for ribbon-graphic delta-matroids are characterized by Geelen and Oum [66]. It is based on the following result and computer search.

Theorem 4.1.22 ([66, Theorem 4.1])**.** *The excluded minors for ribbon-graphic delta-matroids have at most* 10 *elements.*

There are total 171 excluded minors for ribbon-graphic delta-matroids, which are distinct up to tiwisting and isomorphism; see [66, Pages 10–11]. Among 171 excluded minors, 166 of them are binary and 5 excluded minors are non-binary that are listed in Theorem 4.1.18. We remark that the exluded minor characterization of ribbon-graphic delta-matroids implies Kuratowski's theorem for planar graphs [46, Theorem 4.4.6] and Tutte's characterization of planar matroids [109, Theorem in page 534]; see [66, Page 2].

Bouchet [23] showed that every even ribbon-graphic delta-matroid is regular. It is however that the excluded minors for regular delta-matroids are not known yet. Remark that the characterization of excluded minors for regular delta-matroids imlies Tutte's characterization of excluded minors for regular matroids [108]. Geelen [61, Pages 64–66] provided seven binary even delta-matroid which are minor-minimally non-regular. Such delta-matroids are the delta-matroids represented by the adjacency matrices over $\mathbb{F}_2$ of the seven graphs depicted in Figure 4.1. We discuss more about this toipc in Chapter 7.

Geelen [62] characterized the full list of exlcuded minors for equable delta-matroids. His result implies the characterization of excluded minors for regular matroids [108].

Theorem 4.1.23 ([62, Theorem 1.1]; see also [61, Theorem 7.2])**.** *A binary delta-matroid is equable if and only if it does not have a minor isomorphic to a binary delta-matroid represented by an adjacency matrix over $\mathbb{F}_2$ of one of the five graphs in Figure 4.2.*

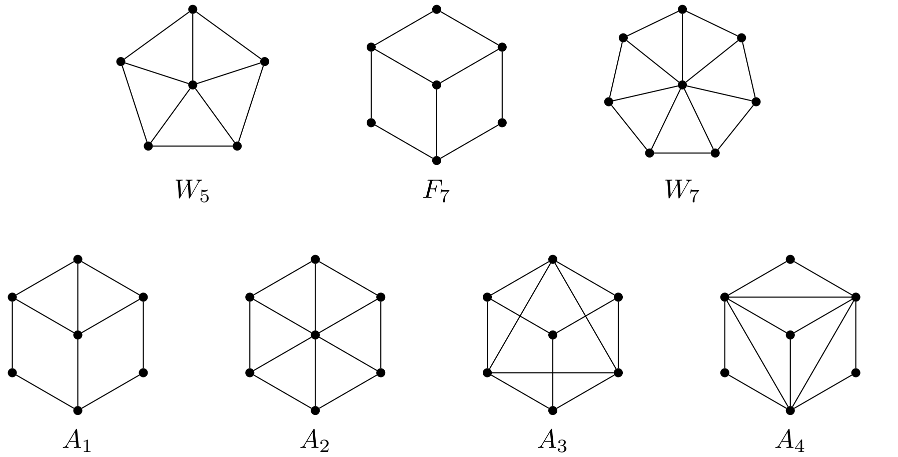


Figure 4.1: Seven graphs representing binary non-regular even delta-matroids.

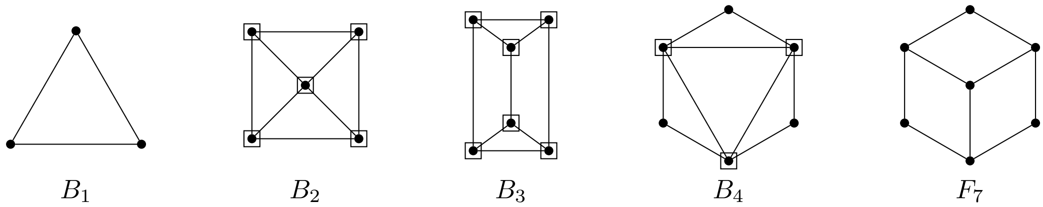


Figure 4.2: Five graphs representing binary non-equable delta-matroids, where each box enclosing a vertex indicates a loop.

For instance, the adjacency matrix of B_2 in Figure 4.2 is the following 5×5 matrix all of whose diagonals are nozero:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We finally remark that the binary delta-matroid associated with the above matrix is ribbon-graphic since the excluded minors for ribbon-graphic delta-matroids are of size at least six by [66], and therefore, a delta-matroid associated with a non-orientable ribbon graph may be non-equable.

4.1.3 Connectedness

We define the *direct sum* $M_1 \oplus M_2$ of two delta-matroids M_1 and M_2 as the direct sum as set systems. Obviously, the direct sum $M_1 \oplus M_2$ of two delta-matroids is a delta-matroid. A delta-matroid M is *connected* if M is not the direct sum of two delta-matroids of which ground sets are neither $\emptyset$ nor $E(M)$. A *component* of M is a connected delta-matroid N such that $E(N) \neq \emptyset$ and $M = N \oplus M'$ for some matroid M'. It is easy to see that a delta-matroid is even if and only if all of its components are even.

Tutte [112] proved that for every connected matroid M and every element $x \in E(M)$, $M \setminus x$ or M/x is connected. Bouchet [35] generalized this result to tight multimatroids. A special case of this result implies the following for even delta-matroids.

Theorem 4.1.24 (Bouchet [35])**.** *Let M be an even delta-matroid. If M is connected, then for every $x \in E(M)$, $M \setminus x$ or M/x is connected.*

The components of a delta-matroid form a partition of the ground set. For an even delta-matroid M, we say $x \sim_M y$ if $x = y$ or there are elements $x_0 = x, x_1, x_2 \ldots, x_k = y$ in E such that $\{x_{i-1}, x_i\} = B_i \triangle B'_i$ for every integer $1 \leq i \leq k$ and some bases B_i and B'_i of M. It is easy to see that $\sim_M$ is an equivalence relation on $E(M)$. Then the equivalence classes of $\sim_M$ are precisely the components of M as shown in the following lemma.

Lemma 4.1.25 (Wenzel [118] and Kim and Oum [74])**.** *Let M be an even delta-matroid. For $x, y \in E(M)$, the following are equivalent:*

(i) *x and y belong to the same component of M.*

(ii) *$x \sim_M y$.*

(iii) *$x = y$ or there are bases B_1, B_2 of M such that $B_1 \triangle B_2 = \{x, y\}$.*

We will see one more equivalent condition in Lemma 4.1.54.

Remark 4.1.26. We can definie higher connectivity for binary delta-matroids using the cut-rank functions of their representations. Let M be a binary delta-matroid and let A be its binary representation. Then the cut-rank function $\rho_A : 2^{E(M)} \to \mathbb{Z}$ behaves as the connectivity function of M since $0 \leq \rho_A(X) \leq |X|$, $\rho_A(X) = \rho_A(E(M) \setminus X)$, and $\rho_A(X) + \rho_A(Y) \geq \rho_A(X \cup Y) + \rho_A(X \cap Y)$ by [26]; see [1, Proposition 2.3.1]. By [26, Theorem 6], such cut-rank functions concerning M are identical regardless of the choice of representations A . Moreover, if M is a matroid, then ρ_A is identical with the connectivity function λ_M of the matroid M; see [98, Section 8.1] for definition. We will discuss more about this in Subsections 7.1.2–7.1.3 in terms of isotropic systems.

For non-binary cases, the choice of representations may give different cut-rank functions. Let $\mathbb{F}$ be a field with at least three elements and let $x \in \mathbb{F} \setminus \{0, 1\}$. Let

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & x \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -x & -1 & -1 & 0 \end{bmatrix}$$

be skew-symmetric matrices defined over $\mathbb{F}$. Then $M(A_1) = M(A_2) = ([4], \{X \subseteq [4] : |X| \text{ is even}\})$ and $\rho_{A_1}(\{1, 2\}) = 1 \neq 2 = \rho_{A_2}(\{1, 2\})$. For each $i \in \{1, 2\}$, let A'_i be a symmetric matrix obtained from A_i by negating the lower triangular part. Then $\rho_{A'_1}(\{1, 2\}) = 1 \neq 2 = \rho_{A'_2}(\{1, 2\})$. Note that $\det(A'_1) = -3$ and $\det(A'_2) = x^2 - 4x$. Hence, one can easily see that $M(A_1) = M(A_2)$ if

- the characteristic of $\mathbb{F}$ is two, or
- the characteristic of $\mathbb{F}$ is larger than three and $x = 2$

4.1.4 Fundamental graphs

Let $M = (E, \mathcal{B})$ be a delta-matroid and let B be a base. The *fundamental graph* of M with respect to B, denoted by $F_M(B)$, is a graph such that

- its vertex set is E,

- a vertex $x \in E$ has a loop if $B \triangle \{x\}$ is a base, and
- two distinct vertices x and y are adjacent if $B \triangle \{x, y\}$ is a base.

One can observe the following from definition.

Lemma 4.1.27. *Let M be a delta-matroid M and B be a base. We have $F_M(B) = F_{M \triangle X}(B \triangle X)$ for any $X \subseteq E$.* □

By Theorem 4.1.16, if a delta-matroid is not even, then it has a fundamental graph with a loop. The converse does not hold. A delta-matroid $([3], \{\emptyset, 1, 2, 3, 123\})$ in Example 1.1.8 is non-even and its fundamental graph with resect to a base 123 is a triangle with no loop.

Recall that every fundamental graph of a matroid is bipartite. Hence, if a delta-matroid is a twist of a matroid, then all of its fundalemtal graphs are bipartite. The converse holds by the following proposition extending the same result for even delta-matroids by Bouchet [28, Corollary 4.2].

Proposition 4.1.28. *Let M be a delta-matroid and B be a base of M. Then $F_M(B)$ is bipartite if and only if M is a twist of a matroid.*

Proof. Since we already know the backward direction, we only show the forward direction. Hence, we assume that $F_M(B)$ is bipartite. Let P and $E(M) \setminus P$ be two color classes of the bipartite graph $F_M(B)$. Let M' be a delta-matroid obtained from M by twisting by $B \triangle P$. Then P is a base of M' and $F_{M'}(P)$ is a bipartite graph with color classes P and $E(M) \setminus P$.

We claim that M' is a matroid. Suppose to the contrary that M' has a base Q such that $|Q| \neq |P|$. We choose such Q minimizing $|P \triangle Q|$. Let $X := P - Q$ and $Y := Q - P$. Then $Q = P - X + Y$ and $|X| \neq |Y|$. We first concern the case $|X| < |Y|$. If $X \neq \emptyset$, then by ($\triangle$B), for an element $x \in X$, there is $y \in X \cup Y$ such that $Q' := Q \triangle \{x, y\}$ is a base of M'. Then $|Q'| \neq |P|$ and $|P \triangle Q'| < |P \triangle Q|$, contradicting the minimality. Therefore, $X = \emptyset$. Similarly, by ($\triangle$B), we can deduce that $|Y| = 1$ or 2. It contradicts that $F_{M'}(P)$ has no edge with both endpoints in $E(M) \setminus P$. Therefore, we may assume that $|X| > |Y|$. By the same argument, we can show that $Y = \emptyset$ and $|X| = 1$ or 2, which contradicts that $F_{M'}(P)$ has no edge with both endpoints in P. Therefore, M' is a matroid. □

Remark 4.1.29. Duchamp [56] investigated conditions that the simplifications (deleting all loops) of fundamental graphs of a delta-matroid are bipartite.

We now discuss a connection between binary even delta-matroids with the minor relation and their fundamental graphs with the pivot-minor relation, which extends a connection between binary matroids and their fundamental graphs shown in Section 3.1.4.

Henceforth, let $M = (E, \mathcal{B})$ be a binary even delta-matroid and let F be a fundamental graph of M. Due to Lemma 4.1.27, it is harmless to assume that M is normal and F is the fundamental graph with respect to $\emptyset$, i.e., $F = F_M(\emptyset)$. Let A_F be the adjacency matrix of F over the binary field. Then $M = M(A_F)$, and thus a graph is a faithful object to understand binary even delta-matroids. Moreover, the pivot-minor relation of graphs is compatible with the minor relation of binary even delta-matroids as shown in the following lemmas.

Lemma 4.1.30. *$F \setminus x = F_{M \setminus x}(\emptyset)$ for each $x \in V(F) = E(M)$.* □

Lemma 4.1.31. *$F \wedge xy = F_M(\{x, y\})$ for each $xy \in E(F)$, i.e., $\{x, y\} \in \mathcal{B}(M)$.* □

We leave their proofs as exercises. The readers may refer to [88, Sections 7-8] for the proofs, which also describes the connection between graphs with vertex-minor and binary delta-matroids, not necessarily even. We will review isotropic systems in Section 7.1.2, which are closely related to both binary delta-matroids and graphs with vertex-minors.

Example 4.1.32. A 5-*wheel* W_5 is a graph depicted in Figure 4.3, which was already shown in Figure 4.1. Let $A := A_{W_5}$ be the adjacency matrix of W_5 over the binary field and let $M := M(A)$. Then $W_5 = F_M(\emptyset)$. We denote the center vertex by 0 and denote the remaining vertices on the rim by $1, 2, 3, 4, 5$ in cyclic order on the rim, as described in Figure 4.3. Now we check that a graph $W_5 \wedge 12$ is the fundamental graph $F_M(12)$. By definition, the edge set of $F_M(12)$ is equal to a 2-element set xy such that $12 \triangle xy$ is a base of M. We note that a subset X of $\{0, 1, 2, 3, 4, 5\}$ is a base of M if and only if the induced subgraph $W_5[X]$ has odd number of perfect matching. From this observation, one may easily see the following:

- The empty set $\emptyset$ is a base of M.
- The 2-element bases containing 1 are $01, 12, 15$.
- The 2-element bases containing 2 are $02, 12, 23$.
- The 4-element bases containing 12 are $0124, 1234, 1235, 1245$.

Therefore, $E(F_M(12)) = \{12\} \cup \{02, 25\} \cup \{01, 13\} \cup \{04, 34, 35, 45\}$, implying that $W_5 \wedge 12 = F_M(12)$. See Figure 4.3. We remark that every graph pivot-equivalent to W_5 is isomorphic to either W_5 or $W_5 \wedge 12$.

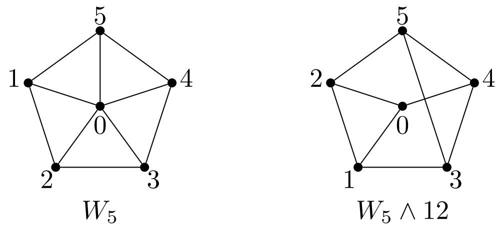


Figure 4.3: A 5-wheel W_5 and its pivoting.

4.1.5 Base polytopes and base graphs

Let $M = (E, \mathcal{B})$ be a delta-matroid. The *base polytope* P_M of M is the convex hull of the indicator vectors $\mathbf{e}_B \in \mathbb{R}^E$ of the bases B of M in $\mathbb{R}^E$. Borovik, Gelfand, and White [18] extended Theorem 3.1.11 as follows. A more general theorem for Coxeter matroids can be found in [19, Theorem 6.3.1].

Theorem 4.1.33 ([18, Theorem 10])**.** *Let P be a 0/1-polytope. Then P is the base polytope of a delta-matroid if and only if each edge of P is a translate of $\mathbf{e}_i$, $\mathbf{e}_i + \mathbf{e}_j$, or $\mathbf{e}_i - \mathbf{e}_j$ with $i \neq j$.*

Corollary 4.1.34. *Let P be a 0/1-polytope. Then P is the base polytope of an even delta-matroid if and only if each edge of P is a translate of $\mathbf{e}_i + \mathbf{e}_j$ or $\mathbf{e}_i - \mathbf{e}_j$ with $i \neq j$.*

The *base graph* of an even delta-matroid M is the graph whose vertices are the bases of M and two vertices B_1 and B_2 are adjacent if $|B_1 \triangle B_2| = 2$.

Remark 4.1.35. Holzmann, Norton, and Tobey [69] showed that if the base graphs of two connected matroids are isomorphic, then the two matroids are the same up to isomorphism and dual (Theorem 3.1.12). Hence, one may ask a similar question for even delta-matroids. Since the twisting preserves base graphs and is a notion extending the dual, we can ask the following question: If the base graphs of two connected even delta-matroids are isomorphic, then are the two even delta-matroids are the same up to isomorphism and twisting? This fails by the following counterexample. The base graph of the uniform matroud $U_{1,4}$ of rank 1 on four elements is isomorphic to the complete graph K_4 on four vertices. The base graph of an even delta-matroid $M_3 := ([3], \{\emptyset, \{12, 13, 23\}\})$ is also isomorphic to K_4. It is easy to check that $U_{1,4}$ and M_3 are connected. Evidently, $U_{1,4}$ is not isomorphic to any twist of M_3 because their ground sets have different sizes. We remark that if the base graphs of two connected even delta-matroids M and M' are isomorphic and they are not isomorphic to an induced subgraph of the base graph of $M_4 := ([4], \{X \subseteq [4] : |X| \text{ is even}\})$, then M is isomorphic to a twist of M' by Chepoi [42, Propositions 1 and 2].

Wenzel [117, 118] proved the following Homotopy Theorem for even delta-matroids, which generalizes Maurer's Homotopy Theorem (Theorem 3.1.13).

Theorem 4.1.36 (Wenzel's Homotopy Theorem [118, Theorem 1.12])**.** *Let M be an even delta-matroid. Then the homology group of the base graph G_M is generated by the cycles of length at most four.*

Remark 4.1.37. Wenzel, indeed, showed a more general version of the homotopy theorem for combinatorial (W, P)-geometries in [117, Theorem 5.7]. Combinatorial (W, P)-geometries generalize both matroids and even delta-matroids, and they are closely related to Coxeter matroids [19].

4.1.6 Symmetric matroids and circuits

Bases of a delta-matroid may have different sizes, so it makes nonsense to define a circuit as a minimal set not contained in any bases. We can overcome this problem by considering symmetric matroids, a concept equivalent to delta-matroids. Bouchet [20] introduced symmetric matroids in the same paper introducing delta-matroids.

Definition 4.1.38. A *symmetric matroid* is a pair $(E = [n] \cup [n]^*, \mathcal{B})$ such that $\mathcal{B}$ is a nonempty set of transversals such that

(ΔB$''$) for every $B_1, B_2 \in \mathcal{B}$ and a skew pair $\{e, e^*\} \subseteq B_1 \triangle B_2$, there is a skew pair $\{f, f^*\} \subseteq B_1 \triangle B_2$ such that $B_1 \triangle \{e, e^*, f, f^*\} \in \mathcal{B}$.

We call each member in $\mathcal{B}$ a *base*.

By definition, the bases of a symmetric matroid M on $[n] \cup [n]^*$ have the same size n. Every subset of a base is called an *independent* set. Subtransversals that are not independent are called *dependent*. A *circuit* is a minimal dependent set with respect to inclusion, i.e., a minimal subtransversal that is not contained in any base. We denote by $\mathcal{B}(M)$ the family of bases of M and denote by $\mathcal{C}(M)$ the family of circuits of M.

Let N be a delta-matroid on $[n]$. Then a pair $\text{lift}(N) := ([n] \cup [n]^*, \mathcal{B})$ is a symmetric matroid, where $\mathcal{B} := \{B \cup ([n] \setminus B)^* : B \in \mathcal{B}(N)\} \subseteq \mathcal{T}_n$. We call such a symmetric matroid the *lift* of N. Conversely, one can obtain a delta-matroid from a symmetric matroid M such that it lift is exactly M. Therefore, we can identify delta-matroids with symmetric matroids.

Remark 4.1.39. In some literature, the circuits of a delta-matroid $M = (E, \mathcal{B})$ are defined as pairs of disjoint subsets $C, D \subseteq E$ such that C is not contained in any base or D is not contained in any complement of base, and there is no pair $(C', D') \neq (C, D)$ with $C' \subseteq C$ and $D' \subseteq D$ satisfying the same property. Then a pair (C, D) of disjoint subsets of E is a circuits of M if and only if $C \cup D^*$ is a circuit of the symmetric matroid $\mathrm{lift}(M)$.

We have a cryptomorphic definition of symmetric matroids in terms of circuits.

Theorem 4.1.40 ([17, Theorem 8])**.** *Let $\mathcal{C}$ be a set of subsets C of $E = [n] \cup [n]^*$ such that C is a subtransversal. Then $\mathcal{C}$ is the set of circuits of a symmetric matroid if and only if it satisfies the following properties:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(△Orth) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \cup C_2$ is not a subtransversal, then $C_1 \cup C_2$ contains at least two skew pairs.*

(△Elim) *If $C_1, C_2 \in \mathcal{C}$ such that $C_1 \cup C_2$ is a subtransversal and $e \in C_1 \cap C_2$, then there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.*

Remark 4.1.41. The circuit axiom for symmetric matroids (Theorem 4.1.40) appeared several other papers earlier than [17]. For instance, Definition 2.2 in a paper [56] by Duchamp provides this cryptomorphic definition without proving that it is equivalent to the definition in terms of bases. He mentioned the same result in another paper [57, Theorem 1.1], and for its proof, he referred to his thesis [55] written in French.

Remark 4.1.42. Bouchet already presented a circuit characterization of symmetric matroid in his original paper [20] defining symmetric matroids and delta-matroids. His circuit axiom [20, (1.6)] is slightly different from Theorem 4.1.40 and concerns a transversal system.

Remark 4.1.43. Bouchet introduced *multimatroids* generalizing both matroids and symmetric matroids in his series of papers [32, 33, 35, 34]. In his terminology, matroids are exactly *1-matroids* and symmetric matroids are exactly *2-matroids*. He found the cryptomorphic definition of multimatroids in terms of circuits [32, Proposition 5.4], earlier than [17]. He further presented the cryptomorphic definitions of multimatroids in terms of rank functions and independent sets [32, Section 3 and Proposition 5.3].

Recall that if N is a matroid on $[n]$, then $\mathrm{lift}(N)$ is a symmetric matroid whose set of bases $\mathcal{B}(\mathrm{lift}(N))$ is given by $\{B \cup ([n] \setminus B)^* : B \in \mathcal{B}(N)\}$. The circuits of $\mathrm{lift}(N)$ are also given by circuits and cocircuits of N as follows.

Proposition 4.1.44 ([33, Proposition 4.1])**.** *Let M be a matroid. Then the set of circuits of the symmetric matroid* $\mathrm{lift}(M)$ *is*

$$\mathcal{C}(\mathrm{lift}(M)) = \{C : C \textit{ is a circuit of } M\} \cup \{D^* : D \textit{ is a cocircuit of } M\}.$$

It is straightforward that the orthogonality, Lemma 3.1.2(Orth), of a matroid N and the condition (△Orth) for the lift of N are tantamount. The condition (△Elim) for $\mathrm{lift}(N)$ is equivalent to the circuit elimination axiom (Elim) for both circuits and cocircuits of N. In addition, the lift of the dual matroid N^* can be obtained by taking the involution $*$ for all bases and circuits of the lift of the original matroid N. In other words, $\mathcal{B}(\mathrm{lift}(N^*)) = (\mathcal{B}(\mathrm{lift}(N)))^*$ and $\mathcal{C}(\mathrm{lift}(N^*)) = (\mathcal{C}(\mathrm{lift}(N)))^*$.

For a base B of a symmetric matroid M on $E = [n] \cup [n]^*$ and an element $e \in E \setminus B$, if $B \triangle \{e, e^*\}$ is not a base, then there is a circuit C contained in $B \triangle \{e, e^*\}$. Then $e \in C$ and such circuit is unique due to (△Elim); if there is another circuit C' contained in $B \triangle \{e, e^*\}$, then M has a circuit C'' contained in $(C \cup C') - e \subseteq B$, contradicting that B is a base. We call such a circuit $C = C_M(B, e)$ the *fundamental circuit* of M with respect to B and e. Booth, Moreira, and Pinto [17] observed a simple criterion for describing fundamental circuits.

Lemma 4.1.45 ([17, Lemma 4])**.** *Let $M = (E, \mathcal{B})$ be a symmetric matroid, $B \in \mathcal{B}$ be a base, and $e \in E$ be an element. Then either $B \triangle \{e, e^*\}$ is a base or there exists a unique circuit $C_M(B, e)$ of M such that $C_M(B, e) \subseteq B \cup \{e\}$. Furthermore, $C_M(B, e) = \{e\} \cup \{b \in B \setminus \{e^*\} : B \triangle \{b, b^*, e, e^*\} \in \mathcal{B}\}$.*

The following lemma is an immediate corollary.

Lemma 4.1.46. *Let C be a circuit of a symmetric matroid M. Then there exists a transversal T containing C such that for every $x \in C$, $T \triangle \{x, x^*\}$ is a base of M.*

Proof. We choose an arbitrary $y \in C$ and take a base B containing $C \setminus \{y\}$. Then $T = B \triangle \{y, y^*\}$ satisfies the desired property. □

As we already pointed out, there is a symmetric matroid of which a fundamental circuit with respect to a certain base and an element does not exist. Interestingly, the existence of fundamental circuits of a symmetric matroid M with respect to any choice of a base and an element are equivalent that M is the lift of an even delta-matroid. It directly follows from Theorem 4.1.16.

Proposition 4.1.47. *Let M be a symmetric matroid on $[n] \cup [n]^*$. Then the following are equivalent:*

(i) *M has the fundamental circuit with respect to every base B and every element $e \in B^*$.*

(ii) *M is the lift of an even delta-matroid on $[n]$.* □

A symmetric matroid is *even* if it is the lift of an even delta-matroid. Equivalently, a symmetric matroid M on $[n] \cup [n]^*$ is even if $B \cap [n]$ with bases B have the same parity. We often call an even symmetric matroid an *orthogonal matroid* in this thesis. We have the following characterization of even symmetric matroids in terms of circuits.

Theorem 4.1.48 ([17, Theorem 12])**.** *Let $\mathcal{C}$ be a family of subtransversals of $E = [n] \cup [n]^*$. Then $\mathcal{C}$ is the family of circuits of an even symmetric matroid if and only if $\mathcal{C}$ satisfies* (C1)*,* (C2)*,* (△Orth)*,* (△Elim)*, and the following:*

(ΔMax) *If T is a transversal and $x \notin T$, then $T \cup \{x\}$ contains an element in $\mathcal{C}$.*

Remark 4.1.49. *Tight* 2-matroids [35] are exactly even symmetric matroids; recall from Remark 4.1.43 that 2-matroids and symmetric matroids are the same. Combining Bouchet's two results, [32, Proposition 5.4] and [35, Theorem 4.2], we deduce a circuit axiom for tight k-matroids, which encompasses Theorem 4.1.48.

By the following lemma, we can omit the condition (△Elim) in the above theorem. We will prove a stronger Lemma 4.2.18 later in Section 4.2.

Lemma 4.1.50. *Let $\mathcal{C}$ be a family of subtransversals of $E = [n] \cup [n]^*$. If $\mathcal{C}$ satisfies* (△Orth) *and* (ΔMax)*, then it satisfies* (△Elim)*.*

Corollary 4.1.51. *Let $\mathcal{C}$ be a family of subtransversals of $E = [n] \cup [n]^*$. Then $\mathcal{C}$ is the family of circuits of an even symmetric matroid if and only if $\mathcal{C}$ satisfies* (C1), (C2), ($\triangle$Orth), *and* (ΔMax). $\square$

As the lift of a matroid N is an even symmetric matroid, its circuit set satisfies (ΔMax). It is easy to check that (ΔMax) for $\mathrm{lift}(N)$ is equivalent to Lemma 3.1.2(Max) for N.

We now discuss minors of symmetric matroids in the sense of [33], which are compatible with strong-minors of delta-matroids.

An element $x \in [n] \cup [n]^*$ of a symmetric matroid M is *singular* if M has no base containing x, or equivalently, $\{x\}$ is a circuit of M. Otherwise, we call the element x *nonsingular*. By ($\triangle$Orth), if an element x is singular, then x^* is nonsingular. Suppose that M is the lift of a delta-matroid N on $[n]$. Then an element $x \in [n] \cup [n]^*$ is singular in M if and only if either $x \in [n]$ and x is a loop in N, or $x \in [n]^*$ and x^* is a coloop in N.

Let M be a symmetric matroid on E and let $x \in E$. If x is nonsingular, then

$$\{B \setminus \{x\} : x \in B \in \mathcal{B}(M)\}$$

is the set of bases of an orthogonal matroid on $E \setminus \{x, x^*\}$. We denote this orthogonal matroid by $M|x$. If x is singular, then we define $M|x := M|x^*$. We call $M|x$ an *elementary minor* of M. In particular, if M is the lift of a delta-matroid N on $[n]$ and $x \in [n]$ (resp. $x \in [n]^*$), then $M|x$ corresponds to the contraction N/x (resp. the deletion $N \setminus x$). A symmetric matroid M' is a *minor* of another orthogonal matroid M if M' can be obtained from M by taking elementary minors sequentially. Note that $M|x|y = M|y|x$, and thus we write $M|x_1|x_2|\dots|x_k$ as $M|S$ where $S = \{x_1, \dots, x_k\}$.

The following proposition characterizes circuits of minors of symmetric matroids, which was proved in [71, Proposition 1.16] for even symmetric matroids but it is extendable to all symmetric matroids.

Proposition 4.1.52. *For a symmetric matroid M and an element $x \in E$, we have*

$$\mathcal{C}(M|x) = \mathrm{Min}\left(\{C \setminus \{x\} : x^* \notin C \in \mathcal{C}(M) \text{ and } C \neq \{x\}\}.\right)$$

Proof. By the definition of $M|x$, if x is nonsingular, then $\{x\}$ is not a circuit of M and

$$\begin{aligned}\mathcal{C}(M|x) &= \mathrm{Min}\{C \in \mathcal{A} : C \not\subseteq B \text{ for all bases } B \text{ of } M \text{ with } x \in B\}\\ &= \mathrm{Min}\{C \in \mathcal{A} : C \cup \{x\} \text{ is dependent in } M\}\\ &= \mathrm{Min}\{C \in \mathcal{A} : C \text{ or } C \cup \{x\} \text{ is a circuit of } M\}\\ &= \mathrm{Min}\{C \setminus \{x\} : x^* \notin C \in \mathcal{C}(M)\},\end{aligned}$$

where $\mathcal{A}$ is the set of all subtransversals in $E \setminus \{x, x^*\}$. Now we assume that x is singular. Then $\{x\}$ is the only circuit of M containing x, and M has no circuit containing x^*. Since $M|x = M|x^*$, by the previous result, we have

$$\begin{aligned}\mathcal{C}(M|x) = \mathcal{C}(M|x^*) &= \mathrm{Min}\{C \setminus \{x^*\} : x \notin C \in \mathcal{C}(M)\}\\ &= \mathrm{Min}\{C \setminus \{x\} : x^* \notin C \in \mathcal{C}(M) \text{ and } C \neq \{x\}\}.\end{aligned}$$ $\square$

We remark that minors of the lift of a matroid can be expressed as lifts of minors of the matroid.

Proposition 4.1.53 ([33, Corollary 5.3])**.** *Let N be a matroid on $[n]$ and let $x \in [n]$. Then $\mathrm{lift}(N)|x = \mathrm{lift}(N/x)$ and $\mathrm{lift}(N)|x^* = \mathrm{lift}(M \setminus x)$. As a consequence, we have $\mathcal{C}(\mathrm{lift}(M)|x) = \mathcal{C}(M/x) \cup (\mathcal{C}^*(M/x))^* = \mathcal{C}(M/x) \cup (\mathcal{C}(M^* \setminus x))^*$, where $\mathcal{C}^*(M/x)$ denotes the set of cocircuits of M/x.*

Lemma 4.1.25 has one more equivalent condition with respect to circuits of the lift as follows.

Lemma 4.1.54. *Let M be an even delta-matroid and let $x, y \in E(M)$. Then x and y belong to the same compnent of M if and only if* $\mathrm{lift}(M)$ *has a circuit C such that $C \cap \{x, x^*\} \neq \emptyset \neq C \cap \{y, y^*\}$.*

Proof. By Lemma 4.1.25 and the definition of $\mathrm{lift}(M)$, we can replace the first clause to the following:

- $\{x, x^*\} = \{y, y^*\}$ or there are bases B_1, B_2 of $\mathrm{lift}(M)$ such that $B_1 \triangle B_2 = \{x, x^*, y, y^*\}$.

Suppose that the above condition holds. Let $x' \in \{x, x^*\}$ be such that $x' \in B_2 \setminus B_1$, and let $y' \in \{y, y^*\}$ be such that $y' \in B_1 \setminus B_2$. Let C be the fundamental circuit of $\mathrm{lift}(M)$ with respect to B_1 and x'. Then $x' \in C$. Also, by Lemma 4.1.45, $y' \in C$.

We now suppose that $\mathrm{lift}(M)$ has a circuit C such that $C \cap \{x, x^*\} \neq \emptyset \neq C \cap \{y, y^*\}$. By Lemma 4.1.46, there is a transversal T containing C such that for every $z \in C$, $T \triangle \{z, z^*\}$ is a base of $\mathrm{lift}(M)$. Then we can take $B_1 := T \triangle \{x, x^*\}$ and $B_2 := T \triangle \{y, y^*\}$ so that $B_1 \triangle B_2 = \{x, x^*, y, y^*\}$. $\square$

4.1.7 Pfaffian structures

We devote the final subsection to reviewing Pfaffian structures and showing that they are essentially equivalent to even delta-matroids. Pfaffian structures were defined by Kung [77], and it has been known that every normal even delta-matroid is a Pfaffian structure [38, Page 56]; see also [43, Appendix B]. We will show that the two structures are equivalent. To the best of our knowledge, this equivalence has not been previously known.

Definition 4.1.55 (Kung [77]; see also [78, Section 3])**.** A *Pfaffian structure* $P(S)$ on a finite set S is defined by a family of $\mathcal{P}$ of subsets of S satisfying the following axioms, where the subsets in $\mathcal{P}$ are called *composite sets*:

(Pf 1) The empty set $\emptyset$ is a composite set.

(Pf 2) For every element x of S, the 1-element set $\{x\}$ is not composite.

(Pf 3) The *exchange-augmentation axiom*: If the sets C_1 and C_2 are composite and x is an element of C_1, then at least one of the following holds:

Exchange. There exists an element y in C_2 such that both $(C_1 \setminus \{x\}) \cup \{y\}$ and $(C_2 \setminus \{y\}) \cup \{x\}$ are composite.

Augmentation. There exists an element x' in C_1 such that both $C_1 \setminus \{x, x'\}$ and $C_2 \cup \{x, x'\}$ are composite.

Proposition 4.1.56. *A pair $D = (E, \mathcal{B})$ is a normal even delta-matroid if and only if $\mathcal{B}$ is a Pfaffian structure on E.*

Proof. The forward direction is obvious by the strong exchange axiom ($\triangle$B′), which was already mentioned in [38].

To prove the backward direction, suppose that D is a Pfaffian structure. By (Pf 1), the empty set is in $\mathcal{B}$.

We first claim that all elements of $\mathcal{B}$ are even-sized; this was also mentioned in [78, Page 161]. Suppose not, and let $B \in \mathcal{B}$ be an odd-sized set with the minimum size. Let $x \in B$ be an arbitrary element. Then by (Pf3), there is $x' \in B$ such that both $B \setminus \{x, x'\}$ and $\{x, x'\}$ are in $\mathcal{B}$. The size of each set is smaller than $|B|$ and either of them is odd-sized, contradicting the minimality of B.

We next claim that D satisfies the strong exchange property ($\triangle$B′). Let $B, B' \in \mathcal{B}$ and $x \in B \triangle B'$. By the earlier claim, B and B' are even-sized. By symmetry, we may assume that $x \in B \setminus B'$. Then by (Pf3), there is $y \in B'$ such that both $(B \setminus \{x\}) \cup \{y\}$ and $(B' \setminus \{y\}) \cup \{x\}$ are in $\mathcal{B}$, or there is $x' \in B$ such that both $B \setminus \{x, x'\}$ and $B' \cup \{x, x'\}$ are in $\mathcal{B}$. As all elements in $\mathcal{B}$ is even-sized, $y \in B' \setminus B$ in the former case and $x \neq x' \in B \setminus B'$ in the latter case. Therefore, D satisfies the strong exchange property, implying that D is a normal even delta-matroid. □

4.2 Antisymmetric matroids

We introduce a new combinatorial structure, called an *antisymmetric matroid*, which is a generalization of a matroid and captures properties of principal and almost-principal minors of a symmetric matrix. Let $E := [n] \cup [n]^*$ through this section.

Definition 4.2.1. A pair $M = ([n] \cup [n]^*, \mathcal{B})$ is an *antisymmetric matroid* if $\mathcal{B}$ is a nonempty subfamily of $\mathcal{T}_n \cup \mathcal{A}_n$ and satisfies the following conditions:

(Sym) For $T \in \mathcal{T}_n$ and distinct skew pairs p and q, $T + p - q \in \mathcal{B} \cap \mathcal{A}_n$ if and only if $T - p + q \in \mathcal{B} \cap \mathcal{A}_n$.

(Exch) For $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, if $B - e$ has no skew pair and $B' + e$ has exactly one skew pair, then there is $f \in B' \setminus B$ such that both $B - e + f$ and $B' + e - f$ are in $\mathcal{B}$.

We call each element in $\mathcal{B}(M) := \mathcal{B}$ a *base* of M.

The (strong) base exchange axiom for matroids is a combinatorial counterpart of the Grassmann-Plücker relations. Analogously, the base axiom for antisymmetric matroids captures a combinatorial property of the restricted Grassmann-Plücker relations that are quadratic relations standing for the Laplace expansion of symmetric matrices only concerning principal and almost-principal minors. In the following theorem, we explicitly describe the restricted Grassmann-Plücker relations and prove that these relations cut out the Lagrangian symplectic Grassmannian. Recall that $\Phi : \mathrm{SpGr}_{\mathbb{F}}(n, 2n) \to \mathbb{P}(\mathbb{F}^{2^n + \binom{n}{2} 2^{n-2}})$ is a map, defined in Theorem 1.1.21, such that

$$V \mapsto (\det(A[B]) : B \in \mathcal{T}_n \cup \mathcal{A}_n)$$

where A is an $n \times 2n$-matrix of which row-space is V. It is well-defined and injective.

Theorem 4.2.2. *The image of* Φ *is set-theoretically cut out by the* restricted Grassmann-Plücker relations (*in short,* restricted G-P relations):

$$\sum_{e \in S \setminus T} (-1)^{|S<e| + |T<e|} X_{S-e} X_{T+e} = 0 \tag{rGP}$$

for all subsets S *and* T *of* E *such that* $|S| = n+1$, $|T| = n-1$, S *contains exactly one* $\{i, i^*\}$ *for some* $i \in [n]$*, and* T *contains no* $\{j, j^*\}$.

The simplest restricted G-P relations consist of 3 terms, and they can be classified into the following two kinds. The first kind is

$$x_{Sab} x_{Sa^*b^*} + x_{Sab^*} x_{Sa^*b} - x_{Saa^*} x_{Sbb^*} = 0$$

where $Sab = S + \{a, b\}$ is a transversal with $|S| = n - 2$ and $a, b \in [n]$, which is a restricted G-P relation (rGP) applied to $S+\{a, a^*, b^*\}$ and $S+\{b\}$. It contains a square term $x_{Saa^*}x_{Sbb^*} = (-1)^{a+b}x^2_{Saa^*}$ and thus we call such relations *square relations*. The second kind is

$$(-1)^{|L<a|}x_{Sabc}x_{Sbb^*c^*} + (-1)^{|L<b^*|}x_{Sabc^*}x_{Sbb^*c} + (-1)^{|L<c^*|}x_{Sabb^*}x_{Sbcc^*} = 0$$

where $Sabc = S + \{a, b, c\}$ is a transversal with $|S| = n - 3$ and $L = \{a, c, b^*, c^*\}$, which is a restricted G-P relation applied to $S + \{a, b, b^*, c^*\}$ and $S + \{b, c\}$. Note that $Sabc$ and $Sabc^*$ are transversals, and Sbb^*c^*, Sbb^*c^*, $Sabb^*$, and $Sbcc^*$ are almost-transversals. We call these relations *edge relations*.[1]

Proof of Theorem 4.2.2. It suffices to show that for each point X satisfying all restricted G-P relations, there is a Lagrangian subspace W such that $\Phi(W) = X$. By the square relations, there is a transversal $T \in \mathcal{T}_n$ such that $X_T \neq 0$. For each $i \in T$, let $\mathbf{v}_i$ be a vector in $\mathbb{F}^E$ such that $\mathrm{supp}(\mathbf{v}_i) \subseteq T^* + i$ and $\mathbf{v}_i(j) = (-1)^{|T<i|+|(T-i)<j|}\frac{X_{T-i+j}}{X_T}$ for each $j \in T^* + i$. As $\mathbf{v}_i(i) = 1$, the n vectors $\mathbf{v}_i$'s are linearly independent.

We claim that the span of $\{\mathbf{v}_i : i \in T\}$, say W, is Lagrangian. It suffices to check that $\omega(\mathbf{v}_i, \mathbf{v}_j) = 0$ for all $i, j \in T$. Clearly, $\omega(\mathbf{v}_i, \mathbf{v}_i) = (-1)^{\chi(i)}\mathbf{v}_i(i)\mathbf{v}_i(i^*) + (-1)^{\chi(i^*)}\mathbf{v}_i(i^*)\mathbf{v}_i(i) = 0$ for each $i \in T$. For distinct $i, j \in T$, let $U = T - ij$. Then we have $|T < i| = |U < i| + 1_{j<i}$ and $|(T - i) < j^*| = |U < j^*| + \chi(j^*)$. Let $\underline{i} = \begin{cases} i & \text{if } i \in [n], \\ i^* & \text{otherwise.} \end{cases}$ We similarly define $\underline{j}$. Since U is a subtransversal of size $n - 2$ non-intersecting with $\{i, i^*\}$ and $\{j, j^*\}$, we deduce that $\sum_{e\in\{i,i^*,j,j^*\}} |U < e| \equiv 1 + \underline{i} + \underline{j} \pmod 2$. Hence $|T < i| + |(T - i) < j^*| + |T < j| + |(T - j) < i^*| \equiv \chi(i^*) + \chi(j^*) + \underline{i} + \underline{j} \pmod 2$. Note that $x_{T-i+j} = (-1)^{\underline{i}+\underline{j}}x_{T-j+i}$. Then we have

$$\mathbf{v}_i(j^*) = (-1)^{|T<i|+|(T-i)<j|}\frac{X_{T-i+j}}{X_T} = (-1)^{|T<j|+|(T-j)<i|+\chi(i^*)+\chi(j^*)}\frac{X_{T-j+i}}{X_T} = (-1)^{\chi(i^*)+\chi(j^*)}\mathbf{v}_j(i^*)$$

and so $\omega(\mathbf{v}_i, \mathbf{v}_j) = (-1)^{\chi(i)}\mathbf{v}_i(i)\mathbf{v}_j(i^*) + (-1)^{\chi(j^*)}\mathbf{v}_i(j^*)\mathbf{v}_j(j) = 0$. Thus, W is Lagrangian.

We finally show that $\Phi(W) = X$. Let Λ be an $n \times E$-matrix whose rows are $\mathbf{v}_i$'s ordered with respect to the linear ordering $1 < \cdots < n < 1^* < \cdots < n^*$. Then $\Lambda[n, T]$ is the identity matrix and for each $i \in T$ and $j \in T^*$, we have

$$\det(\Lambda[n, T - i + j]) = (-1)^{|T<i|+|(T-i)<j|}\mathbf{v}_i(j) = \frac{X_{T-i+j}}{X_T}.$$

Therefore, $\det(\Lambda[n, B]) = \frac{X_B}{X_T}$ for every $B \in \mathcal{T}_n \cup \mathcal{A}_n$ and thus $\Phi(W) = X$. □

Remark 4.2.3. The Lagrangian orthogonal Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ consists of two components. Ding and the author [47] show that, analogous to Theorem 1.1.21, each component of $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ is parameterized into the projective space $\mathbb{P}(\mathbb{F}^{2^n+\binom{n}{2}2^{n-2}})$ using principal and almost-principal minors of skew-symmetric matrices and is set-theoretically cut out by quadratic relations that are similar to the restricted G-P relations.

One can rewrite (Exch) as follows, which captures the zero and nonzero patterns of a point in the projective space of dimension $2^n + \binom{n}{2}2^{n-2} - 1$ satisfying all restricted Grassmann-Plücker relations (rGP).

(Exch′) For arbitrary transversals T, T' and $e, f \in T' \setminus T$ (possibly, $e = f$) there are no or at least two elements $g \in (T + e) \setminus (T' - f)$ such that $\{T + e - g, T' - f + g\} \subseteq \mathcal{B}$.

[1]The corresponding trinomials of square and edge relations are respectively called *square* and *edge* trinomials in [13].

Therefore, one can regard antisymmetric matroids as a point satisfying all restricted Grassmann-Plücker relations over the Krasner hyperfield $\mathbb{K}$.

Example 4.2.4. Let A be an $n \times n$ symmetric matrix. Let $\mathcal{B} := \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \det(A[B \cap [n], [n] \setminus B^*]) \neq 0\}$. Then $([n] \cup [n]^*, \mathcal{B})$ is an antisymmetric matroid by Theorem 1.1.21. Note that $([n] \cup [n]^*, \mathcal{B} \cap \mathcal{T}_n)$ is a symmetric matroid because $B \cap [n] = [n] \setminus B^*$ if $B \in \mathcal{T}_n$.

The following two lemmas are combinatorial versions of square relations and edge relations.

Lemma 4.2.5. *Let $M = (E, \mathcal{B})$ be an antisymmetric matroid. Let T be a transversal and p, q be distinct skew pairs. Then none or at least two of*

$$\{T + p - q, T - p + q\},\ \{T, T \triangle (p + q)\},\ \text{and}\ \{T \triangle p, T \triangle q\}$$

are contained in $\mathcal{B}$. In particular, if $T + p - q \in \mathcal{B} \cap \mathcal{A}_n$, then $\{T, T \triangle (p+q)\} \subseteq \mathcal{B}$ or $\{T \triangle p, T \triangle q\} \subseteq \mathcal{B}$.

Proof. We denote by $\{x\} = T \cap p$ and $\{y\} = T \cap q$. Applying (Exch′) to $T + p - q - x$ and $T - p + q + x$, there is no or at least two $g \in \{x, y, y^*\} = (T - p + q + x) \setminus (T + p - q - x)$ such that $T + p - q - x + g$ and $T - p + q + x - g$ are bases of M. Note that

$$T + p - q - x + g = \begin{cases} T + p - q & \text{if } g = x, \\ T \triangle (p + q) & \text{if } g = y^*, \\ T \triangle p & \text{otherwise,} \end{cases} \quad \text{and} \quad T - p + q + x - g = \begin{cases} T - p + q & \text{if } g = x, \\ T & \text{if } g = y^*, \\ T \triangle q & \text{otherwise.} \end{cases}$$

By (Sym), $T + p - q \in \mathcal{B}$ if and only if $T - p + q \in \mathcal{B}$. Hence the proof is completed. □

Lemma 4.2.6. *Let $M = (E, \mathcal{B})$ be an antisymmetric matroid. Let $T \in \mathcal{T}_n$ and distinct skew pairs p, q, r. Then none or at least two of $\{T, (T \triangle p) + q - r\}$, $\{T \triangle p, T + q - r\}$, $\{T + p - q, T + p - r\}$ are contained in $\mathcal{B}$.*

Proof. By (Exch′), there are no or at least two elements $e \in \{x^*, y^*, z\} = ((T \triangle p) + q) \setminus (T - r)$ such that $\{(T \triangle p) + q - e, T - r + e\} \subseteq \mathcal{B}$. Note that

$$T - r + e = \begin{cases} T + p - r & \text{if } e = x^*, \\ T + q - r & \text{if } e = y^*, \\ T & \text{otherwise,} \end{cases} \quad \text{and} \quad (T \triangle p) + q - e = \begin{cases} T - p + q & \text{if } e = x^*, \\ T \triangle p & \text{if } e = y^*, \\ (T \triangle p) + q - r & \text{otherwise.} \end{cases}$$

The proof is completed by (Sym). □

By definition, an antisymmetric matroid has two types of bases, one is a transversal and the other is an almost-transversal. They stand for principal and almost-principal minors of a symmetric matrix, respectively. As shown in Example 4.2.4, by collecting all transversal bases of a representable antisymmetric matroid, we obtain a (representable) symmetric matroid. This property is true in general.

Proposition 4.2.7. *Let $M = ([n] \cup [n]^*, \mathcal{B})$ is an antisymmetric matroid. Then $([n] \cup [n]^*, \mathcal{B} \cap \mathcal{T}_n)$ is a symmetric matroid.*

Proof. By Lemma 4.2.5, $\mathcal{B} \cap \mathcal{T}_n \neq \emptyset$. Let $B_1, B_2 \in \mathcal{B} \cap \mathcal{T}_n$ and let $x \in B_1 - B_2$. Then $x^* \in B_2 - B_1$. We may assume that $B_1 - x + x^* \notin \mathcal{B} \cap \mathcal{T}_n$. Then by (Exch), there is $y \in (B_1 - B_2) - x$ such that $B_1 + x^* - y \in \mathcal{B} \cap \mathcal{A}_n$. By Lemma 4.2.5, $(B_1 + x^* - y) - x + y^* = B_1 \triangle \{x, x^*, y, y^*\} \in \mathcal{B} \cap \mathcal{T}_n$. □

Proposition 4.2.7 induces the canonical map from the class of antisymmetric matroids to the class of symmetric matroids. This map is not injective by Example 4.2.8. It is open whether this map is surjective.

Example 4.2.8. Two antisymmetric matroids $M_1 = ([2] \cup [2]^*, \mathcal{T}_2)$ and $M_2 = ([2] \cup [2]^*, \mathcal{T}_2 \cup \mathcal{A}_2)$ induce the same symmetric matroid that is identified with M_1. It is easily seen that the antisymmetric matroid M_1 is representable over the binary field, but M_2 is not. If $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ are ternary matrices, then M_i is represented by $\begin{bmatrix} I \mid A_i \end{bmatrix}$ and thus both M_1 and M_2 are representable over the ternary field.

4.2.1 Circuits

A *circuit* of an antisymmetric matroid M on $E = [n] \cup [n]^*$ is a minimal subset C of E such that C contains at most one skew pair and C is not a subset of any base of M. We denote by $\mathcal{C}(M)$ the set of circuits of M. Note that every circuit is nonempty because $\mathcal{B}(M) \neq \emptyset$. The family $\mathcal{B}(M)$ of bases is equal to the set of $B \in \mathcal{T}_n \cup \mathcal{A}_n$ such that B is not a superset of any circuit of M. We present a cryptomorphic definition of antisymmetric matroids in terms of circuits, which is reminiscent of Minty's Painting Axiom (Lemma 3.1.2).

Theorem 4.2.9. *Let $\mathcal{C}$ be a family of subsets C of E such that C contains at most one skew pair. Then $\mathcal{C}$ is the family of circuits of an antisymmetric matroid on E if and only if it satisfies the following:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(△Orth) $|C_1 \cap C_2^*| \neq 1$ *for all $C_1, C_2 \in \mathcal{C}$.*

(ΔMax) *For every transversal $T \in \mathcal{T}_n$ and element $e \in T^*$, there is $C \in \mathcal{C}$ such that $C \subseteq T \cup \{e\}$.*

The following lemma implies that (ΔMax) can be replaced with a stronger condition written below:

(ΔMax′) For every transversal $T \in \mathcal{T}_n$ and element $e \in T^*$, there is $C \in \mathcal{C}$ such that $C \subseteq T \cup \{e\}$ and $C \cap \{e, e^*\} \neq \emptyset$.

Lemma 4.2.10. *Let $\mathcal{C}$ be a set of subsets C of E such that C contains at most one skew pair. If $\mathcal{C}$ satisfies (△Orth) and (ΔMax), then it satisfies (ΔMax′).*

Proof. Let $T \in \mathcal{T}_n$ and $e \in T^*$. Let S be a minimal subset of $T - e^*$ such that $T - e^* - S$ does not contain any $C \in \mathcal{C}$. Then for each $f \in S$, there is $C_f \in \mathcal{C}$ such that $C_f \subseteq T - e^*$ and $C_f \cap S = \{f\}$. Let $T' = T - S + S^*$. By (ΔMax), there is $D \in \mathcal{C}$ such that $D \subseteq T' + e$. If $D \cap S^* \neq \emptyset$, then $D \cap C_f^* = \{f^*\}$ for $f^* \in D \cap S^*$, contradicting (△Orth). Thus, $D \subseteq T' + e - S^* = T + e - S$ and $D \cap \{e, e^*\} \neq \emptyset$. □

The set $\bigsqcup_{0 \leq r \leq n} \mathrm{Gr}_{\mathbb{F}}(r, n)$ of linear spaces can be embedded into the Lagrangian symplectic Grassmannian $\mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ by mapping a linear space V to $V \oplus V^{\perp}$. Analogously, there is a natural injection from the set of matroids on $[n]$ to the set of antisymmetric matroids on $[n] \cup [n]^*$. For a matroid M on $[n]$, and let $\mathcal{D}$ be the set $\mathcal{C}(M) \oplus \mathcal{C}(M^{\perp}) := \mathcal{C}(M) \cup \{C^* \subseteq [n]^* : C \in \mathcal{C}(M^{\perp})\}$. Then $\mathcal{D}$ satisfies (C1), (C2), (△Orth), and (ΔMax) as follows. As $\mathcal{C}(M)$ and $\mathcal{C}(M^{\perp})$ are the sets of circuits and cocircuits of M, it obviously satisfies (C1) and (C2). To check (△Orth), we can assume that $C_1 \in \mathcal{C}(M)$ and

$C_2^* \in \mathcal{C}(M^\perp)$. Then $|C_1 \cap C_2^*| \neq 1$ is exactly the orthogonality of matroids. We finally examine (ΔMax) for a transversal T and an element $e \in T^*$. We may assume that $e \in [n]$. Applying Lemma 3.1.2(Max) to a tripartition $(P, Q, \{e\}) = \big((T-e) \cap [n], (T^*-e) \cap [n], \{e\}\big)$, the matroid M has a circuit contained in $T \cap [n]$ or has a cocircuit contained in $T^* \cap [n]$. Then there is $C \in \mathcal{D}$ that is a subset of $T+e$. Therefore, by Theorem 4.2.9, we conclude that $\mathcal{D}$ is the set of circuits of an antisymmetric matroid, denoted by $\mathrm{ant}(M)$, on $[n] \cup [n]^*$. Moreover, we deduce the following commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\mathbb{F}}(r,n) & \longrightarrow & \mathrm{SpGr}_{\mathbb{F}}(n,2n) \\ \downarrow & & \downarrow \\ \mathrm{Mat}_{r,n} & \longrightarrow & \mathrm{AntMat}_n \end{array} \qquad \begin{array}{ccc} V & \longmapsto & V \oplus V^\perp \\ \downarrow & & \downarrow \\ M & \longmapsto & \mathrm{ant}(M) \end{array}$$

where M is the rank-r matroid on $[n]$ whose set of cocircuits is the set of minimal supports of vectors in $V \setminus \{\mathbf{0}\}$. Here, $\mathrm{Mat}_{r,n}$ denotes the set of rank-r matroids on $[n]$ and AntMat_n denotes the set of antisymmetric matroids on $[n] \cup [n]^*$. The map $\mathrm{SpGr}_{\mathbb{F}}(n,2n) \to \mathrm{AntMat}_n$ is defined by sending a Lagrangian subspace W to an antisymmetric matroid M such that $\mathcal{C}(M)^* = \{C^* : C \in \mathcal{C}(M)\}$ equals the set of minimal supports D of nonzero vectors in W such that D contains at most one skew pair, of which well-definedness will be checked in Proposition 4.2.19. This diagram is further extended by replacing the domain $\mathrm{Mat}_{r,n}$ with the set of even delta-matroids whenever the field $\mathbb{F}$ has characteristic two.

Example 4.2.11. Let V be the row-space of a 2-by-3 matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ over an arbitrary field k. Then V is 2-dimenaional. A set $\{X \subseteq [3] : \det(A[2,X]) \neq 0\} = \{12, 13, 23\}$ is the set of bases of a uniform matroid $M = U_{2,3}$. The orthogonal complement $V^\perp$ is the span of $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, which induces the dual matroid $M^\perp = U_{1,3}$. Note that $\mathcal{C}(M) = \{123\}$ and $\mathcal{C}(M^\perp) = \{12, 13, 23\}$, which are the sets of minimal supports of nonzero vectors in $V^\perp$ and V, respectively. Then the set of circuits of an antisymmetric matroid $\mathrm{ant}(M)$ is $\mathcal{C}(M) \oplus \mathcal{C}(M^\perp) = \{123, 1^*2^*, 1^*3^*, 2^*3^*\}$ by definition, which is equal to the set of minimal supports of nonzero vectors in $V^\perp \oplus V$. The bases of an antisymmetric matroid $\mathrm{ant}(M)$ are 123^*, 12^*3, and 1^*23.

We show several properties of circuits to prove Theorem 4.2.9.

Lemma 4.2.12. *Let C be a circuit of an antisymmetric matroid on E and let $e \in C$. If $C - e$ has no skew pair, then there is a base B such that B is a transversal and $C \setminus B = \{e\}$.*

Proof. Since $C - e$ is not a circuit, there is a base B such that $C - e \subseteq B$ and $e \notin B$. We may assume that $B \in \mathcal{A}_n$ and let p and q be skew pairs such that $p \subseteq B$ and $q \cap B = \emptyset$. Then $p \neq \{e, e^*\}$. Let $f \in p \setminus C$. By Lemma 4.2.5, for some $g \in q$, $B' := B - f + g$ is a base. Then B' is a transversal and $C \setminus B' = \{e\}$. □

Lemma 4.2.13. *Let $M = (E, \mathcal{B})$ be an antisymmetric matroid and let S be a subset of E such that $|S| = n + 1$, S has exactly one skew pair, and $S - e \in \mathcal{B}$ for some $e \in S$. Then there is a unique circuit C contained in S. Moreover, $C = \{e \in S : S - e \in \mathcal{B}\}$.*

Proof. Let $C := \{e \in S : S - e \in \mathcal{B}\}$ and $\{z, z^*\} \subseteq S$. By the assumption, $C \neq \emptyset$. If $C \setminus \{z, z^*\} \neq \emptyset$, then $S - x$ is a base for $x \in C \setminus \{z, z^*\}$ and by Lemma 4.2.5, $S - z$ or $S - z^*$ is a base. Thus, $C \cap \{z, z^*\} \neq \emptyset$. By relabelling we can assume that a transversal $B := S - z$ is a base.

We claim that C is a circuit of M. For every $x \in C$, we have $C - x \subseteq S - x \in \mathcal{B}$. Hence every proper subset of C is not a circuit. Therefore, it suffices to check that C is not a subset of any base. Suppose to the contrary that there is a base B' containing C. Then $z \in C \setminus B \subseteq B' \setminus B$. By Lemma 4.2.5, we may assume that $B' - z$ has no skew pair. By (Exch), there is $y \in B \setminus B'$ such that $S - y = B + z - y \in \mathcal{B}$. Then $y \in C$ and thus $C \nsubseteq B'$, a contradiction. Therefore, C is a circuit.

Let D be a circuit of M such that $D \subseteq S$. If $e \in S \setminus D$, then $S - e \notin \mathcal{B}$ and hence $e \notin C$. Then $C \subseteq D$. This implies that C is a unique circuit contained in S. □

For an antisymmetric matroid M, a transversal base B, and an element $e \in B^*$, the unique circuit contained in $B + e$ is called the *fundamental circuit* of M with respect to B and e. Obviously, such a fundamental circuit contains e.

Remark 4.2.14. Let M be a matroid of rank r on n elements, and let A be a standard matrix representation of M with respect to a base B over an arbitrary field. Then the r fundamental cocircuits of A with respect to B are exactly the supports of r rows of A.

A similar observation holds for antisymmetric matroids. Let N be an antisymmetric matroid and let Λ be an n-by-$[n] \cup [n]^*$ matrix such that its row-space is Lagrangian and $\mathcal{B}(N) = \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \det(\Lambda[n, B]) \neq 0\}$. Let B_0 be a transversal base of N, and we relabel the columns of Λ in order $b_1, \ldots, b_n, b_1^*, \ldots, b_n^*$, where $b_1 < \ldots < b_n$ are elements of B_0. We denote by the supports $C_1, \ldots, C_n$ of the rows of the reduced row-echelon form of Λ. Then each C_i^* is the fundamental circuit of N with respect to B_0 and b_i^*. We note that C_i^* is a circuit of a symmetric matroid $N' := ([n] \cup [n]^*, \mathcal{B}(N) \cap \mathcal{T}_n)$ if C_i^* is a subtransversal. However, it is possible that none of C_i^* is a subtransversal, and hence the base B of N may not have any fundamental circuits; see Example 4.2.15.

Example 4.2.15. Let $M = ([n] \cup [n]^*, \mathcal{T}_n)$, which is a symmetric matroid and an antisymmetric matroid, simultaneously. It is representable over all fields, witnessed by $\Lambda = \begin{bmatrix} I_n \,|\, I_n \end{bmatrix}$. As a symmetric matroid, M has no circuits. Thus, for any base B and an element $e \in B^*$, there is no circuit contained in $B + e$. In contrast, as an antisymmetric matroid, M has the fundamental circuit $\{e, e^*\}$ with respect to each B and e, which coincide with the n supports of the rows of Λ.

The following lemma generalizes the orthogonality of matroids.

Lemma 4.2.16. *If C_1 and C_2 are circuits of an antisymmetric matroid, then $|C_1 \cap C_2^*| \neq 1$.*

Proof. By Lemma 4.2.12, there are transversal bases B_1 and B_2 such that $|C_i \setminus B_i| = 1$. For each $i \in \{1, 2\}$, let $S_i = B_i \cup C_i$ and let q_i be the skew pair contained in S_i. For each $e \in S_1 \setminus (S_2 - q_2)$, by Lemma 4.2.13, $e \in C_1$ if and only if $S_1 - e \in \mathcal{B}$. Similarly, $e^* \in C_2$ if and only if $S_2 - e^* \in \mathcal{B}$, and the latter condition is equivalent to $S_2 - q_2 + e \in \mathcal{B}$ by (Sym). By (Exch′) applied to S_1 and $S_2 - q_2$, we deduce that $|C_1 \cap C_2^*| \neq 1$. □

Lemma 4.2.17. *Let C be a circuit of an antisymmetric matroid M and let $e, f \in C$ be distinct elements such that $C - e$ is a subtransversal. Then there is a circuit D such that $C \cap D^* = \{e, f\}$.*

Proof. By Lemma 4.2.12, M has a transversal base B such that $C \setminus B = \{e\}$. Let D be the fundamental circuit with respect to B and f^*. Then $f \in C \cap D^* \subseteq \{e, f\}$ and thus by Lemma 4.2.16, $C \cap D^* = \{e, f\}$. □

Lemma 4.2.18. *Let $\mathcal{C}$ be a family of subsets C of E such that C contains at most one skew pair. If $\mathcal{C}$ satisfies (△Orth) and (ΔMax), then it satisfies the following:*

(ΔElim′) *For distinct $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$, if $(C_1 \cup C_2) - e$ contains at most one skew pair, then there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.*

Proof. We say a subtransversal $I \subseteq E$ is $\mathcal{C}$-*independent* if I contains no element in $\mathcal{C}$. If a subtransversal I is $\mathcal{C}$-independent and a skew pair $\{e, e^*\}$ does not intersect with I, then $I+e$ or $I+e^*$ is $\mathcal{C}$-independent by (△Orth).

Let C_1, C_2 be distinct elements in $\mathcal{C}$ and let $e \in C_1 \cap C_2$ such that $J := (C_1 \cup C_2) \setminus \{e\}$ contains at most one skew pair. Suppose to contrary that J does not contain any element in $\mathcal{C}$.

We first assume that J is a subtransversal. Then J is $\mathcal{C}$-independent and thus there is a $\mathcal{C}$-independent transversal J' containing J. Then $e^* \in J'$. As C_1 and C_2 are distinct, there is $f \in C_2 \setminus C_1$. By (ΔMax), there is $D \in \mathcal{C}$ such that $f^* \in D \subseteq J' + f^*$. By (△Orth) applied to D and C_2, we have $e^* \in D$. Then $D \cap C_1^* = \{e^*\}$ contradicting (△Orth).

Now we may assume that J has a skew pair, say $\{f, f^*\}$. By (△Orth), $\{e, e^*\}$ is in $C_1 \cup C_2$. By symmetry, we can assume that $\{e^*, f\} \subseteq C_1$. Then by (△Orth), $\{f, f^*\} \subseteq C_2$. In short, $\{e, e^*, f\} \subseteq C_1$ and $\{e, f, f^*\} \subseteq C_2$. Let $K := J - f^*$. Then K is a $\mathcal{C}$-independent subtransversal and thus there is a $\mathcal{C}$-independent transversal K' containing K. By (ΔMax), we have an element $D \in \mathcal{C}$ such that $f^* \in D \subseteq K' + f^*$. By the assumption that J contains no element in $\mathcal{C}$, there is $g \in D \setminus J$. By (ΔMax), $\mathcal{C}$ has an element D' such that $g^* \in D' \subseteq K' + g^*$. As $g \in D \cap (D')^* \subseteq \{g, f^*\}$, we deduce that $f^* \in (D')^*$ by (△Orth). Then $f^* \in C_2 \cap (D')^* \subseteq \{f^*, e\}$ and by (△Orth), $e \in (D')^*$. Then $C_1 \cap (D')^* = \{e\}$ that contradicts (△Orth). □

We remark that (ΔElim′) extends the circuit elimination axiom for matroids.

Proof of Theorem 4.2.9. The forward direction is done by Lemmas 4.2.13 and 4.2.16.

Now to show the converse we assume that $\mathcal{C}$ satisfies the four clauses (C1), (C2), (△Orth), and (ΔMax). By Lemma 4.2.18, $\mathcal{C}$ also satisfies (ΔElim′). Let $\mathcal{B}$ be the set of transversals and almost-transversals that do not contain any $C \in \mathcal{C}$. It is enough to prove that $\mathcal{B} \neq \emptyset$ and $\mathcal{B}$ satisfies (Sym) and (Exch′).

We first show that $\mathcal{B} \neq \emptyset$. Let $C \in \mathcal{C}$. By (C1), $C \neq \emptyset$. We choose an element $e \in C$, and we additionally assume that $\{e, e^*\} \subseteq C$ if C contains a skew pair. Let $I_0 = C - e + e^*$. Then by (△Orth), there is no $D \in \mathcal{C}$ contained in I_0. By (△Orth), if I is a subtransversal containing no set in $\mathcal{C}$, then for each $\{f, f^*\} \subseteq E - I$, at least one of two sets $I + f$ and $I + f^*$ contains no set in $\mathcal{C}$. Hence we can obtain $B \in \mathcal{B} \cap \mathcal{T}_n$ such that $B \supseteq I_0$.

Second we claim (Sym). Let T be a transversal and p, q be distinct skew pairs such that $T+p-q \in \mathcal{B}$. Suppose to the contrary that there is $C \in \mathcal{C}$ such that $C \subseteq T - p + q$. Then $C \cap q \neq \emptyset$ and let $x \in C \cap q$. Replacing T with $T \triangle q$ if necessary, we can assume that $x \in T$. There is $D \in \mathcal{C}$ such that $D \subseteq (T \triangle q) + p$ by (ΔMax). Then $x^* \in D$ because otherwise $D \subseteq T + p - q$. Then $C \cap D^* = \{x\}$, contradicting (△Orth).

Finally, we show (Exch′). Let T, T' be transversals and e, f be elements in $T' \setminus T$. Let $S := T + e$, $S' := T' - f$, and $q := \{f, f^*\}$. We claim that there are no or at least two elements $g \in S \setminus S'$ such that both $S - g$ and $S' + g$ are in $\mathcal{B}$. We may assume that S' does not contain any set in $\mathcal{C}$, since otherwise $S' + x \notin \mathcal{B}$ for any $x \in E$. Then by (ΔElim′) and (△Orth), $\mathcal{C}$ has a unique element D such that $D \subseteq S' + q$. By (ΔMax), there is $C \in \mathcal{C}$ such that $C \subseteq S$. If there is another $C_2 \in \mathcal{C}$ such that $C_2 \subseteq S$, then by (ΔElim′), we deduce that $S - x \notin \mathcal{B}$ for every $x \in S$. Hence we may assume that C is the unique element in $\mathcal{C}$ such that $C \subseteq S$. Then

$$C = \{x \in S : S - x \in \mathcal{B}\} \text{ and } D = \{x \in S' + q : S' + q - x \in \mathcal{B}\}.$$

For each $x \in S \setminus S'$, we have that $S' + q - x^* \in \mathcal{B}$ if and only if $S' + x \in \mathcal{B}$ by (Sym). Therefore, for each $x \in S \setminus S' = S \cap (S' + q)^*$, $x \in C \cap D^*$ if and only if $S - x \in \mathcal{B}$ and $S' + x \in \mathcal{B}$. Then the claim follows (△Orth). □

4.2.2 Representability

Recall that for an r-dimensional linear space V in $\mathbb{F}^n$, the family of minimal supports of vectors in $V \setminus \{\mathbf{0}\}$ is the set of cocircuits of a rank-r matroid M on $[n]$. In addition, the set of bases of M is equal to the set of r-element subsets B of $[n]$ such that $p(V)_B \neq 0$, where p is the Grassmann-Plücker embedding. We show an analogous result for Lagrangian subspaces and antisymmetric matroids. We defined in Theorem 1.1.21 the parameterization Φ of the Lagrangian Grassmannian $\mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ into the projective space of dimension $2^{n-2}(4 + \binom{n}{2}) - 1$.

Proposition 4.2.19. *Let W be a Lagrangian subspace in $\mathbb{F}^E$. Let $\mathcal{B} := \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \Phi(W)_B \neq 0\}$ and let $\mathcal{C}$ be the set of minimal supports C of vectors in $W \setminus \{\mathbf{0}\}$ such that C contains at most one skew pair. Then $\mathcal{B}$ is the set of bases of an antisymmetric matroid, and $\mathcal{C}$ is the set of circuits of an antisymmetric matroid $(E, \mathcal{B}^*)$, where $\mathcal{B}^* := \{B^* : B \in \mathcal{B}\}$.*

Proof. It is trivial that $\mathcal{B}$ is nonempty and satisfies (Sym). It is straightforward to deduce (Exch$'$) from the restricted Grassmann-Plücker relations (rGP). Thus, $M_1 = (E, \mathcal{B})$ is an antisymmetric matroid.

We now prove that $\mathcal{C}$ is the set of circuits of an antisymmetric matroid. By definition, (C1) and (C2) hold. As W is isotropic, $\mathcal{C}$ satisfies ($\triangle$Orth). Let $T \in \mathcal{T}_n$ and $e \in T^*$. Suppose that there is no $C \in \mathcal{C}$ such that $C \subseteq T + e$. Then $\dim W \leq |E - (T + e)| = n - 1$, a contradiction. Thus, (ΔMax) holds. Then by Theorem 4.2.9, $\mathcal{C}$ is the set of circuits of an antisymmetric matroid, say M_2, on E.

Finally, we show that $M_2 = (E, \mathcal{B}^*)$. Let Λ be an $n \times E$ matrix such that its row-space is W. Then $\Phi(W)_B = \det(\Lambda[n, B])$ for each $B \in \mathcal{T}_n \cup \mathcal{A}_n$ by definition.

Claim 1. *For each $B \in \mathcal{B}$, there is no $C \in \mathcal{C}$ such that $C \subseteq B^*$.*

Proof. We denote by $B = \{b_1, \ldots, b_n\}$ and $E - B = \{a_1, \ldots, a_n\}$. Since $B \in \mathcal{B}$, $\Phi(W)_B = \det(\Lambda[n, B])$ is nonzero. Hence Λ is row-equivalent to a matrix Λ' such that $\Lambda'[n, B]$ is an identity matrix. So W has n independent vectors $X_1, \ldots, X_n$ such that $\mathrm{supp}(X_i) \cap B = \{b_i\}$.

Suppose that B is a transversal. As $X_1, \ldots, X_n$ span W, every element $C \in \mathcal{C}$ intersects with B. Thus, $\mathcal{C}$ has no element contained in $B^* = E - B$.

Hence we can assume that B is an almost-transversal. By relabelling, we may assume that $b_2 = b_1^*$ and $a_2 = a_1^*$. Then $B \cap \{a_1, a_1^*\} = \emptyset = (E - B) \cap \{b_1, b_1^*\}$. Hence $0 = \omega(X_1, X_2) = \sum_{e \in \{b_1, a_1, a_2\}} (-1)^{\chi(e)} X_1(e) X_2(e^*)$. Since $X_1(b_1) = 1 = X_2(b_1^*)$, we deduce that $(X_1(a_1), X_1(a_2))$ and $(X_2(a_1), X_2(a_2))$ are not a scalar multiple of each other. Then the support of each nonzero linear combination of $X_1, \ldots, X_n$ intersects with $E - B^* = \{a_1, a_2\} \cup \{b_3, \ldots, b_n\}$. Therefore, $\mathcal{C}$ has no element contained in B^*. ■

Claim 2. *For each pair (C, e) such that $e \in C \in \mathcal{C}$, there is $B \in \mathcal{B}$ such that $C - e \subseteq B^*$.*

Proof. First, suppose that $C - e$ has no skew pair. Then M_2 has a transversal base $B = \{b_1, \ldots, b_n\}$ such that $C - e \subseteq B$ by Lemma 4.2.12. Let $X_1, \ldots, X_n$ be vectors in W such that each $\mathrm{supp}(X_i)$ is the fundamental circuit of M_2 with respect to B and b_i^*. Because $b_i^* \in \mathrm{supp}(X_i) \subseteq B + b_i^*$, n vectors $X_1, \ldots, X_n$ are independent. Hence Λ is row-equivalent to a matrix Λ' consisting of $X_1, \ldots, X_n$ as rows. A submatrix $\Lambda'[n, B^*]$ only has nonzero entries for (i, b_i^*) with $1 \leq i \leq n$. Thus, $\Lambda[n, B^*]$ is nonsigular and $B^* \in \mathcal{B}$.

Now we assume that $C - e$ has a skew pair, say $\{x, x^*\}$. Then M_2 has a transversal base B such that $C - x \subseteq B$ by Lemma 4.2.12. Then $B_2 := B + x - e$ is also a base of M_2 by Lemma 4.2.13. Note that $B_2 \supseteq C - e$. We denote by $B = \{b_1, \ldots, b_n\}$ such that $b_1 = x^*$ and $b_2 = e$. Let $X_1, \ldots, X_n$ be vectors in W such that each $\mathrm{supp}(X_i)$ is the fundamental circuit of M_2 with respect to B and b_i^*. Then

$\mathrm{supp}(X_1) = C \ni e = b_2$. Then by the orthogonality (△Orth), $x^* = b_1 \in \mathrm{supp}(X_2)$. Let $Y_2 := X_2$ and for $i \in [n] \setminus \{2\}$, let $Y_i := X_i - \frac{X_i(b_1)}{X_2(b_1)}$. Then an $n \times E$ matrix Λ' consisting of $Y_1, \dots, Y_n$ is row-equivalent to Λ and its square submatrix $\Lambda'[n, \{b_1^*, b_1\} \cup \{b_3^*, \dots, b_n^*\}] = \Lambda'[n, B_2^*]$ is a nonsingular diagonal matrix. Thus, $\Phi(B_2^*) = \det(\Lambda[n, B_2^*]) \neq 0$ and $B_2^* \in \mathcal{B}$. As $C - e \subseteq B_2$, the claim follows. ■

By the above two claims, $\mathcal{C}$ is the set of circuits of $(E, \mathcal{B}^*)$. □

Example 4.2.20. Let $\Lambda = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ be a $2 \times ([2] \cup [2]^*)$ matrix over a field $\mathbb{F}$. Then its row-space, say W, is in $\mathrm{SpGr}_{\mathbb{F}}(2, 4)$. Then $\mathcal{C} := \{12, 11^*2^*, 21^*2^*\}$ is the set of minimal supports C of nonzero vectors in W such that C contains at most one skew pair. Let $\mathcal{B} := \{B \in \mathcal{T}_2 \cup \mathcal{A}_2 : \det(\Lambda[2, B]) \neq 0\} = \{12, 11^*, 12^*, 21^*, 22^*\}$. Then the set of circuits of an antisymmetric matroid $(E, \mathcal{B})$ is $\{1^*2^*, 121^*, 122^*\} = \mathcal{C}^*$.

Definition 4.2.21. An antisymmetric matroid M on $E = [n] \cup [n]^*$ is *representable over a field* $\mathbb{F}$ or $\mathbb{F}$-*representable* if there is an $n \times E$ matrix Λ over $\mathbb{F}$ such that its row-space is Lagrangian in $\mathbb{F}^E$ and $\mathcal{B}(M) = \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \det(\Lambda[n, B]) \neq 0\}$.

For a Lagrangian subspace W in $\mathbb{F}^E$, let $M(W)$ be an antisymmetric matroid on E such that $\mathcal{C}(M(W))$ is the set of minimal supports C of vectors in $W \setminus \{\mathbf{0}\}$ such that C contains at most one skew pair. Then an antisymmetric M is representable over k if and only if $M = M(W)$ for some Lagrangian subspace W.

In Section 4.2.1, we observed that a linear matroid produces a representable antisymmetric matroid. More strongly, we show that the representability of antisymmetric matroids extends that of matroids.

Proposition 4.2.22. *A matroid M on $[n]$ is representable over a field $\mathbb{F}$ if and only if an antisymmetric matroid* $\mathrm{ant}(M)$ *is representable over* $\mathbb{F}$.

Proof. Recall that, in Section 4.2.1, $\mathrm{ant}(M)$ is defined as an antisymmetric matroid whose set of circuits is $\mathcal{C}(M) \oplus \mathcal{C}(M^\perp) = \mathcal{C}(M) \cup \{C^* : C \in \mathcal{C}(M^\perp)\}$. We denote the rank of M by r.

Suppose that M is representable over $\mathbb{F}$, and let V be an r-dimensional linear subspace in $\mathbb{F}^n$ representing M so that the set $\mathcal{C}(M^\perp)$ of cocircuits of M is exactly the set of minimal supports of nonzero vectors of V. Then $\mathcal{C}(\mathrm{ant}(M)) = \mathcal{C}(M) \oplus \mathcal{C}(M^\perp)$ equals the set of minimal supports of nonzero vectors in $V^\perp \oplus V$. Thus, $\mathrm{ant}(M) = M(V^\perp \oplus V)$.

Suppose that $\mathrm{ant}(M)$ is representable over $\mathbb{F}$. Then there is a Lagrangian subspace W in $\mathbb{F}^{[n] \cup [n]^*}$ such that $\mathcal{C}(\mathrm{ant}(M))^*$ equals the set of minimal supports C of nonzero vectors of W such that C contains at most one skew pair. Let V be the projection of W into the space $\mathbb{F}^{[n]}$ regarding the first n coordinates. Since $\mathcal{C}(\mathrm{ant}(M))^* = \mathcal{C}(M^\perp) \oplus \mathcal{C}(M)$, the projection V is a linear space representing M. □

4.2.3 Even antisymmetric matroids

An antisymmetric matroid $M = ([n] \cup [n]^*, \mathcal{B})$ is *even* if the induced symmetric matroid $([n] \cup [n]^*, \mathcal{B} \cap \mathcal{T}_n)$ is even. The following theorem together with Proposition 4.2.7 shows the one-to-one correspondence between even antisymmetric matroids and even symmetric matroids.

Theorem 4.2.23. *Let $M = ([n] \cup [n]^*, \mathcal{B})$ be an even symmetric matroid. There is a unique $\mathcal{B}' \subseteq \mathcal{A}_n$ such that $M' = ([n] \cup [n]^*, \mathcal{B} \cup \mathcal{B}')$ is an antisymmetric matroid.*

Theorem 4.2.23 can be restated as follows: Every orthogonal $\mathbb{K}$-matroid is an antisymmetric $\mathbb{K}$-matroid. Recall that $\mathbb{K}$ is the Kransner hyperfield that is the terminal object in the category of tracts. See Chapters 5 and 6 for precise definitions of orthogonal $\mathbb{K}$-matroids and antisymmetric $\mathbb{K}$-matroids. We will extend this observation to tracts with $1 = -1$ in Section 6.2.2.

Theorem 4.2.23 is easily deduced from the circuit axiom for even symmetric matroids (Corollary 4.1.51) and that for antisymmetric matroids (Theorem 4.2.9).

Proof. Let $\mathcal{B}'$ be the set of almost-transversals $A \in \mathcal{A}_n$ such that for some $x, y \in E$ with $\{x, x^*\} \subseteq A$ and $\{y, y^*\} \cap A = \emptyset$, both $A - x + y$ and $A - x^* + y^*$ are in $\mathcal{B}$. Let $\mathcal{C}$ be the set of circuits of the even symmetric matroid M. By Corollary 4.1.51 and Theorem 4.2.9, $\mathcal{C}$ is the set of circuits of an antisymmetric matroid, say M', on E.

We claim that $\mathcal{B}(M') = \mathcal{B} \cup \mathcal{B}'$. It is equivalent to show that for each $B \in \mathcal{T}_n \cup \mathcal{A}_n$, the set B is in $\mathcal{B} \cup \mathcal{B}'$ if and only if B does not contain any $C \in \mathcal{C}$.

Suppose that $B \in \mathcal{B} \cup \mathcal{B}'$. If $B \in \mathcal{B} \subseteq \mathcal{T}_n$, then B is not contained in any $C \in \mathcal{C}$ because it is a base of the even symmetric matroid M. If $B \in \mathcal{B}' \subseteq \mathcal{A}_n$, then B is not contained in any $C \in \mathcal{C}$ because B contains a skew pair and C is a subtransversal. Thus, B is not a superset of any elements of $\mathcal{C}$.

Conversely, suppose that B does not contain any $C \in \mathcal{C}$. If $B \in \mathcal{T}_n$, then B is a base of the even symmetric matroid M and equivalently $B \in \mathcal{B}$. Thus, we may assume that $B \in \mathcal{A}_n$. Let $\{x, x^*\}$ and $\{y, y^*\}$ be the skew pairs such that $\{x, x^*\} \subseteq B$ and $\{y, y^*\} \cap B = \emptyset$. By (ΔMax), there is $C \in \mathcal{C}$ such that $C \subseteq B + y$. Then $y \in C$ since $C \not\subseteq B$. As C is a subtransversal, by interchanging x and x^* if necessary, we may assume that $x^* \notin C$. Then there is no $D \in \mathcal{C}$ such that $D \subseteq B - x^* + y^*$, because otherwise $C \cap D^* = \{y\}$ contradicting (△Orth). Hence $B - x^* + y^* \in \mathcal{B}$. Assume that $x \notin C$. Then simlarly there is no $D \in \mathcal{C}$ such that $D \subseteq B - x + y^*$, and hence $B - x + y^* \in \mathcal{B}$, which violates that M is even. Therefore, $x \in C$. Assume that there is $D \in \mathcal{C}$ such that $D \subseteq B - x + y$. If $x^* \in D$, then $C \cap D^* = \{x\}$, contradicting (△Orth). Thus, $x^* \notin D$. Hence $D \subseteq B - x - x^* + y$ and $D \neq C$. Also, $y \in D$ because $D \not\subseteq B$. By (ΔElim′), there is $D' \in \mathcal{C}$ such that $D' \subseteq C + D - y \subseteq B$, a contradiction. Therefore, no such D exists and $B - x + y \in \mathcal{B}$. Thus, we conclude that $B \in \mathcal{B}'$ because $B - x^* + y^*, B - x + y \in \mathcal{B}$. Therefore, the claim is proved.

It remains to check that if there is an antisymmetric matroid M'' such that $\mathcal{B}(M'') \cap \mathcal{T}_n = \mathcal{B}$, then $\mathcal{B}(M'') \cap \mathcal{A}_n = \mathcal{B}'$. It is straightforward from Lemma 4.2.5. □

Proposition 4.2.22 can be generalized by replacing matroids with even delta-matroids, whenever $\mathbb{F}$ has characteristic two. It is due to Theorem 4.2.23.

Corollary 4.2.24. *Let $\mathbb{F}$ be a field of characteristic two. Then an even delta-matroid M on $[n]$ admits an $[n]$-by-$[n]$ symmetric matrix A over $\mathbb{F}$ such that $\mathcal{B}(M) = \{X \subseteq [n] : \det(A[X, X]) \neq 0\}$ if and only if the antisymmetric matroid associated with* lift(M) *is representable over $\mathbb{F}$.* □

Proposition 4.2.22 cannot be further extended to delta-matroids because of Example 4.2.8.

4.2.4 Minors

We define minors of an antisymmetric matroid and show that the representability is closed under taking minors. For an antisymmetric matroid M on E and $i \in E$, Let $\mathcal{B}(M)|i := \{B - i \subseteq E \setminus \{i, i^*\} : B \in \mathcal{B}(M),\ B \cap \{i, i^*\} = \{i\}\}$ if M has a base containing i, and $\mathcal{B}(M)|i := \{B - i^* \subseteq E \setminus \{i, i^*\} : B \in \mathcal{B}(M)\}$ otherwise. Let $\mathcal{C}(M)|i := \mathrm{Min}\{C - i : i^* \notin C \in \mathcal{C}(M) \text{ and } C \neq \{i\}\}$.

Proposition 4.2.25. *Let M be an antisymmetric matroid on E and let $i \in E$. Then*

(i) *$\mathcal{B}(M)|i$ is the set of bases of an antisymmetric matroid on $E \setminus \{i, i^*\}$, and*

(ii) *$\mathcal{C}(M)|i$ is the set of circuits of the same antisymmetric matroid.*

We denote the resulting antisymmetric matroid by $M|i$ and call it an *elementary minor* of M. An antisymmetric matroid N is a *minor* of another antisymmetric matroid M if $N = M|i_1|i_2 \cdots |i_k$ for some $i_1, \dots, i_k$. For a matroid N on $[n]$ and $i \in [n]$, we note that $\mathcal{C}(\mathrm{ant}(N))|i = \mathcal{C}(N/i) \oplus \mathcal{C}(N^\perp \setminus i)$. Equivalently, $\mathrm{ant}(N)|i = \mathrm{ant}(N/i)$. We similarly have that $\mathcal{C}(\mathrm{ant}(N))|i^* = \mathcal{C}(N \setminus i) \oplus \mathcal{C}(N^\perp/i)$ and $\mathrm{ant}(N)|i^* = \mathrm{ant}(N \setminus i)$.

Example 4.2.26. Let $N = U_{3,4}$ be the uniform matroid on [4] of rank 3. Then $\mathcal{C}(\mathrm{ant}(N)) = \{1234\} \cup \{i^*j^* : ij \in \binom{[4]}{2}\}$. Note that $N/4 = U_{2,3}$ and $N \setminus 4 = U_{3,3}$. Thus, $\mathcal{C}(\mathrm{ant}(N))|4 = \{123\} \cup \{i^*j^* : ij \in \binom{[3]}{2}\} = \mathcal{C}(N/4) \oplus \mathcal{C}(N^\perp \setminus 4)$ and $\mathcal{C}(\mathrm{ant}(N))|4^* = \{1^*, 2^*, 3^*\} = \mathcal{C}(N \setminus 4) \oplus \mathcal{C}(N^\perp/4)$.

Proof of Proposition 4.2.25. By definition, $\mathcal{B}(M)|i$ trivially satisfies (Sym) and (Exch). Now we check that $\mathcal{B}(M)|i \neq \emptyset$. Suppose that M has a base B containing i. If $B \cap \{i, i^*\} = \{i\}$, then $\mathcal{B}(M)|i \neq \emptyset$. Therefore, we may assume that B is an almost-transversal containing $\{i, i^*\}$. Let $\{j, j^*\}$ be a skew pair non-intersecting with B. Then by Lemma 4.2.5, $B - i^* + j$ or $B - i^* + j^*$ is a base of M. Thus, $\mathcal{B}(M)|i \neq \emptyset$. Hence we can assume that M has no base containing i. If M has a base which does not include i^*, then by (Sym), M has a base containing $\{i, i^*\}$, a contradiction. Thus, every base of M contains i^*, implying that $\mathcal{B}(M)|i \neq \emptyset$. Therefore, $\mathcal{B}(M)|i$ is the set of bases of an antisymmetric matroid on $E \setminus \{i, i^*\}$.

By definition, $\mathcal{C}(M)|i$ satisfies (C1) and (C2). Let $C_1, C_2 \in \mathcal{C}(M)|i$ and let $D_1, D_2 \in \mathcal{C}(M)$ such that $C_a = D_a - i$ with $a \in \{1, 2\}$. Then $|C_1 \cap C_2^*| = |D_1 \cap D_2^*| \neq 1$, so $\mathcal{C}(M)|i$ satisfies ($\triangle$Orth). Let T be a transversal in $E \setminus \{i, i^*\}$ and let $j \in T^*$. Suppose that $\{i\}$ is not a circuit of M. By (ΔMax), M has a circuit D contained in $(T + i) + j$. Then $D \neq \{i\}$. Hence there is $C \in \mathcal{C}(M)|i$ such that $C \subseteq D - i \subseteq T + j$. So we may assume that $\{i\}$ is a circuit of M. Then every circuit of M does not contain i^* by ($\triangle$Orth). By (ΔMax), M has a circuit D' contained in $(T + i^*) + j$. Then $D' \subseteq T + j$. Thus, $\mathcal{C}(M)|i$ satisfies (ΔMax). By Theorem 4.2.9, $\mathcal{C}(M)|i$ is the set of circuits of an antisymmetric matroid on $E \setminus \{i, i^*\}$.

It remains to show that $\mathcal{B}(M)|i$ and $\mathcal{C}(M)|i$ are associated with the same antisymmetric matroid. We first assume that M has a base containing i. Then $\{i\}$ is not a circuit of M. Let $\mathcal{A}$ be the set of subsets C of $E \setminus \{i, i^*\}$ such that C contains at most one skew pair. We note that if $D \in \mathcal{A}$ and $D + i$ is a subset of some almost-transversal base B of M such that $B \supseteq \{i, i^*\}$, then by Lemma 4.2.5, $D + i$ is a subset of a transversal base of M. Then

$$
\begin{aligned}
\mathcal{C}(M|i) &= \mathrm{Min}\{C - i : i^* \notin C \in \mathcal{C}(M)\} \\
&= \mathrm{Min}\{D \in \mathcal{A} : D \text{ or } D + i \in \mathcal{C}(M)\} \\
&= \mathrm{Min}\{D \in \mathcal{A} : D + i \not\subseteq B \text{ for all } B \in \mathcal{B}(M) \text{ with } i \in B\} \\
&= \mathrm{Min}\{D \in \mathcal{A} : D + i \not\subseteq B \text{ for all } B \in \mathcal{B}(M) \text{ with } B \cap \{i, i^*\} = \{i\}\} \\
&= \mathrm{Min}\{D \in \mathcal{A} : D \not\subseteq B' \text{ for all } B' \in \mathcal{B}(M)|i\}.
\end{aligned}
$$

Thus, $C(M|i)$ is the set of circuits of the antisymmetric matroid $(E \setminus \{i, i^*\}, \mathcal{B}(M)|i)$.

Next, we suppose that M has no base containing i. Then every base of M contains i^*. Hence $\{i\}$ is a circuit of M, and every circuit of M does not contain i^* by ($\triangle$Orth). Thus, $\mathcal{C}(M|i) = \{C : \{i\} \neq C \in \mathcal{C}(M)\}$ and it is equal to the set of circuits of the antisymmetric matroid with the base set $\mathcal{B}(M)|i = \{B - i^* : B \in \mathcal{B}(M)\}$. $\square$

The class of antisymmetric matroids representable over a given field is closed under taking minors by the following proposition.

Proposition 4.2.27. *Let $W \in \mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ and let $\pi : \mathbb{F}^{[n]\cup[n]^*} \to \mathbb{F}^{[n]\cup[n]^*\setminus\{i,i^*\}}$ be the natural projection. Then $M(\pi(W \cap \mathbf{e}_i^{\perp})) = M(W)|i$.*

Lemma 4.2.28. *Let W be in $\mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ and v be a nonzero vector in $\mathbb{F}^{[n]\cup[n]^*}$ such that $\mathrm{supp}(v) \subseteq \{n, n^*\}$. Then the natural projection of $W \cap v^{\perp}$ into $\mathbb{F}^{[n-1]\cup[n-1]^*}$ is in $\mathrm{SpGr}_{\mathbb{F}}(n-1, 2n-2)$.*

Proof. We can assume that $v \notin W$. Let $w_1, \dots, w_n$ be a base of W. By relabelling, we may assume that $w_n \notin v^{\perp}$. Then for each $1 \le i \le n-1$, there is $c_i \in k$ such that $u_i := w_i - c_i w_n \in v^{\perp}$. We denote by π the natural projection from $\mathbb{F}^{[n]\cup[n]^*}$ to $\mathbb{F}^{[n-1]\cup[n-1]^*}$. Then $\pi(u_1), \dots, \pi(u_n)$ is a base of $\pi(W \cap v^{\perp})$ and $\omega(\pi(u_i), \pi(u_j)) = 0$ for all distinct $i, j \in [n-1]$. Therefore, $\pi(W \cap v^{\perp}) \in \mathrm{SpGr}_{\mathbb{F}}(n-1)(2n-2)$. □

Proof of Proposition 4.2.27. Straightforwardly, the set of circuits of $M(\pi(W \cap \mathbf{e}_i^{\perp}))$ equals $\mathrm{Min}\{C \setminus \{i\} : i^* \notin C \in \mathcal{C}(M(W))$ and $C \neq \{i\}\}$. □

4.2.5 Homotopy theorem

A *transversal base graph* $\mathcal{G}_M$ of an antisymmetric matroid M on $[n] \cup [n]^*$ is a graph such that

- its vertex set is $\mathcal{B}(M) \cap \mathcal{T}_n$, and
- two vertices B and B' are adjacent if and only if (i) $|B \setminus B'| = 1$ or (ii) $|B \setminus B'| = 2$ and there is $A \in \mathcal{B}(M) \cap \mathcal{A}_n$ such that $|B \setminus A| = |B' \setminus A| = 1$.

The *weight* of an edge BB' is $\eta(BB') := |B \setminus B'|$. It is obvious that every cycle C in $\mathcal{G}_M$ have even weights, i.e., $\eta(C) := \sum_{e\in E(C)} \eta(e)$ is even.

Theorem 4.2.29 (Homotopy Theorem for antisymmetric matroids)**.** *Let G be the transversal base graph of an antisymmetric matroid with the weights $\eta(e)$ on edges. Then the homology group of G is generated by the cycles in G of weight 6 or 8.*

Section 6.3 will provide the proof. The readers may read Section 6.3 at once since the proof does not demand any knowledge of the rest of Chapter 6.

It is easy to deduce Maurer's and Wenzel's Homotopy Theorems 3.1.13 and 4.1.36 from Theorem 4.2.29. Let N be an even delta-matroid and let M be the antisymmetric matroid such that the set of bases of $\mathrm{lift}(N)$ is exactly the set of transversal bases of M, which always exists by Theorem 4.2.23. Then the base graph G_N of N is identified with the transversal base graph $\mathcal{G}_M$ of M, and each cycle of length k in G_N corresponds to a cycle of weight $2k$ in $\mathcal{G}_M$. Therefore, we deduce Wenzel's Homotopy Theorem for even delta-matroids from Theorem 4.2.29. Similarly, Maurer's Homotopy Theorem for matroids follows as well.

We define a convex polytope $P_M \subseteq \mathbb{R}^n$ associated with an antisymmetric matroid $M = ([n]\cup[n]^*, \mathcal{B})$, by taking P_M as the convex hull of $\chi_B := \frac{1}{2}\mathbf{e}_{[n]} + \frac{1}{2}\mathbf{e}_{B\cap[n]} - \frac{1}{2}\mathbf{e}_{B^*\cap[n]}$ with $B \in \mathcal{B}$. Note that P_M is congruent to a $\frac{1}{2}$-scaling of the convex hull of $\mathbf{e}_{B\cap[n]} - \mathbf{e}_{B^*\cap[n]}$. The following lemma shows that P_M is the same with the base polytope of a delta-matroid $([n], \{B \subseteq [n] : B + ([n] \setminus B)^* \in \mathcal{B}(M)\})$. Figure 4.4 illustrates an example of P_M.

Corollary 4.2.30. *Let $M = ([n] \cup [n]^*, \mathcal{B})$ be an antisymmetric matroid. Then P_M is equal to the base polytope of a delta-matroid $([n], \{B \subseteq [n] : B + ([n] \setminus B)^* \in \mathcal{B}(M)\})$. In particular, for each base B of M, the vector χ_B is a vertex of P_M if and only if B is a transversal.*

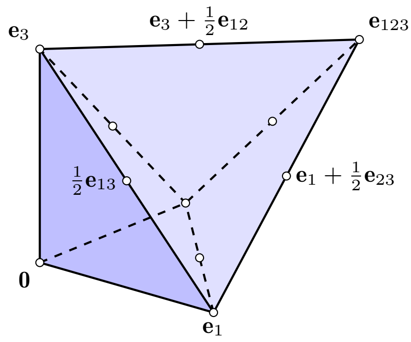


Figure 4.4: The base polytope of an antisymmetric matroid M on $[3] \cup [3]^*$ such that $\mathcal{B}(M) = \{1^*2^*3^*, 12^*3^*, 1^*23^*, 1^*2^*3, 123\} \cup \mathcal{A}_3$. The six mid-points $\frac{1}{2}\mathbf{e}_{ij}$ and $\mathbf{e}_k + \frac{1}{2}\mathbf{e}_{ij}$ with $ijk = [3]$ on 1-dimensional faces correspond to the almost-transversals of M.

Proof. Let A be an almost-transversal base, let p and q be skew pairs such that $p \subseteq A$ and $q \cap A = \emptyset$, and let $S := A - p$. By Lemma 4.2.5, M has two trasnversal bases T_1 and T_2 such that $T_1 = S + ij$ and $T_2 = S + i^*j^*$ for some elements $i \in p$ and $j \in q$. Then $\chi_A = \chi_S$ is lying on the line segment between χ_{T_1} and χ_{T_2}. Therefore, P_M is indeed equal to the convex hull of the vectors $\chi_B = \mathbf{e}_{B\cap[n]}$ with transversal bases $B \in \mathcal{B} \cap \mathcal{T}_n$, i.e., the base polytope associated with a delta-matroid $([n], \{B \subseteq [n] : B + ([n] \setminus B)^* \in \mathcal{B}(M)\})$. Then by Theorem 4.1.33, the vertices of P_M are exactly the vectors χ_B with transversal bases B of M. $\square$

4.2.6 Relation to gaussoids

A gaussoid is a combinatorial structure introduced by Lněnička and Matúš [82] to understand which almost-principal submatrices of a positive definite symmetric matrix can be simultaneously singular. We review an equivalent definition presented by [13, Theorem 1].

A subset $\mathcal{G}$ of $\mathcal{A}_n$ is *allowable* if an almost-transversal A is in $\mathcal{G}$ if and only if $A - p + q$ is in $\mathcal{G}$, where p and q are skew pairs with $p \subseteq A$ and $q \cap A = \emptyset$. An allowable subset $\mathcal{G}$ of $\mathcal{A}_n$ is *incompatible with* a restricted G-P relation $f := \sum_{e \in S_1 \setminus S_2} (-1)^{|S_1 < e| + |S_2 < e|} x_{S_1 - e} x_{S_2 + e} = 0$ if there is precisely one term $x_{S_1 - e} x_{S_2 + e}$ in the polynomial f such that neither $S_1 - e$ nor $S_2 + e$ is in $\mathcal{G}$. Otherwise $\mathcal{G}$ is *compatible with* $f = 0$.

Recall that an edge relation is

$$(-1)^{|L<a|} x_{Sabc} x_{Sbb^*c^*} + (-1)^{|L<b^*|} x_{Sabc^*} x_{Sbb^*c} + (-1)^{|L<c^*|} x_{Sabb^*} x_{Sbcc^*} = 0,$$

where $Sabc = S + \{a, b, c\}$ is a transversal with $|S| = n - 3$ and $L = \{a, c, b^*, c^*\}$. In the edge relation, $Sabc$ and $Sabc^*$ are transversals, and Sbb^*c^*, Sbb^*c^*, $Sabb^*$, and $Sbcc^*$ are almost-transversals.

Definition 4.2.31 ([82]; see [13, Theorem 1])**.** A subset $\mathcal{G}$ of $\mathcal{A}_n$ is a *gaussoid* if it is allowable and is compatible with all edge relations.

Example 4.2.32. Let Σ be a positive definite symmetric matrix over the real field $\mathbb{R}$, and let $\Lambda := [I \mid \Sigma]$. Then $\mathcal{G} := \{B \in \mathcal{A}_n : \det(\Lambda[n, B]) = 0\}$ is a gaussoid, and $\mathcal{B} := \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \det(\Lambda[n, B]) \neq 0\}$ is the family of bases of an antisymmetric matroid. Note that $\mathcal{T}_n \subseteq \mathcal{B}$ because Σ is positive definite, and $\mathcal{G} = \mathcal{A}_n \setminus \mathcal{B}$.

In general, we can obtain a gaussoid from each antisymmetric matroid M such that $\mathcal{B}(M) \supseteq \mathcal{T}_n$.

Proposition 4.2.33. *Let $M = ([n] \cup [n]^*, \mathcal{B})$ be an antisymmetric matroid. If $\mathcal{T}_n \subseteq \mathcal{B}$, then $\mathcal{A}_n \setminus \mathcal{B}$ is a gaussoid.*

Proof. Let $\mathcal{G} := \mathcal{A}_n \setminus \mathcal{B}$. By (Sym), $\mathcal{G}$ is allowable. By (Exch) and the assumption $\mathcal{T}_n \subseteq \mathcal{B}$, the set $\mathcal{G}$ is compatible with all restricted G-P relations and thus it is a gaussoid. □

It is open whether for each gaussoid $\mathcal{G}$, a set family $\mathcal{T}_n \cup (\mathcal{A}_n \setminus \mathcal{G})$ is the family of bases of an antisymmetric matroid. We remark that the previous statement holds if Conjecture 1 in [13] is true.

Chapter 5. Orthogonal matroids with coefficients

We define orthogonal matroids with coefficients in tract by extending idea of Wenzel [116, 119] who introduced even delta-matroids with coefficients in fuzzy rings by making use of Wick relations. Furthermore, we present several equivalent definitions of orthogonal matroids with coefficients. Such equivalence extends the cryptomorphism of matroids with coefficients (Theorem 3.2.12) shown by Baker and Bowler [4] and Anderson [2].

Theorem 5.1.18. *For a tract F, there are natural bijections between any pair of the following:*

(i) *Orthogonal F-matroids.*

(ii) *F-circuit sets of orthogonal matroids.*

(iii) *Orthogonal F-signatures.*

(iv) *Orthogonal F-vector sets.*

We also define weaker notions of orthogonal matroids with coefficients and show their equivalence.

Theorem 5.1.19. *There is a natural bijection between:*

(i) *Moderately weak orthogonal F-matroids.*

(ii) *Weak orthogonal F-signatures.*

Theorem 5.1.20. *There is a natural bijection between:*

(i) *Weak orthogonal F-matroids.*

(ii) *Weak F-circuit sets of orthogonal matroid.*

Every (strong) orthogonal F-matroid is a moderately weak orthogonal F-matroid, and every moderately weak orthogonal F-matroid is a weak orthogonal F-matroid. These three notions of orthogonal F-matroids are not the same in general by Examples 5.2.21 and 5.2.22, but they coincide if F is a partial field [6], the tropical hyperfield $\mathbb{T}$ [102], or the Krasner hyperfield $\mathbb{K}$.

If $F = \mathbb{F}$ is a field, then Theorem 5.1.18 implies Theorem 1.1.19. Recall that Theorem 1.1.19 shows a parameterization of the Lagrangian orthogonal Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n, 2n)$ into the projective space $\mathbb{P}(\mathbb{F}^{2^n})$ and also presents the Wick relations (Wick). Furthermore, orthogonal F-matroids generalize various concepts:

(1) Matroids with coefficients in tracts by Baker and Bowler [4] if underlying orthogonal matroids of orthogonal F-matroids are lifts of matroids.

(2) Ordinary orthogonal matroids if $F = \mathbb{K}$ is the Krasner hyperfield.

(3) Principally unimodular skew-symmetric matrices if $F = \mathbb{U}_0$ is the regular partial field.

(4) Valuated even delta-matroids by Dress and Wenzel [52, 116, 119] and tropical Wick vectors by Rincón [102] if $F = \mathbb{T}$ is the tropical hyperfield.

(5) Oriented even delta-matroid by Wenzel [116, 119], see also [14], if $F = \mathbb{S}$ the sign hyperfield.

We present two applications in Section 5.3. First, we give systematic proofs of old and new theorems on the representability of orthogonal matroids. Second, we show a generalization of Farkas' Lemma for oriented orthogonal matroids.

Structure of this chapter. In Section 5.1 we define three notions of orthogonal matroids over tracts, namely the weak, moderately weak, and strong orthogonal matroids over tracts in terms of Wick functions. We also present their equivalent definitions in terms of orthogonal F-signatures, F-circuit sets of orthogonal matroids, and orthogonal F-vector sets. In Section 5.2, we prove the cryptomorphisms: Theorems 5.1.18, 5.1.19, and 5.1.20. We examine applications in Section 5.3.

5.1 Orthogonal matroids over tracts

Let $E = [n] \cup [n]^*$ be a finite set and let F be a tract endowed with an involution $x \mapsto \overline{x}$. In Subsection 5.1.1, we define strong, moderately weak, and weak orthogonal matroids on E over F in terms of Wick functions. We then establish other cryptomorphic definitions, including orthogonal F-signatures and F-circuit sets of orthogonal matroids in Subsection 5.1.2, and orthogonal F-vector sets in Subsection 5.1.3. We summarize equivalences and implications of various notions in Subsection 5.1.4. In Subsections 5.1.5–5.1.7, we introduce functoriality, duality, and minors, and in Subsection 5.1.8, we explain how orthogonal F-matroids generalize historical works on orthogonal matroids by specifying F.

5.1.1 Wick functions

We describe the first cryptomorphic characterization of strong, moderately weak, and weak orthogonal matroids over tracts in terms of Wick functions.

Definition 5.1.1. A (*strong*) *Wick function on E with coefficients in F* is a map $\varphi : \mathcal{T}_n \to F$ such that:

(W1) φ is not identically zero.

(W2) For all T_1, $T_2 \in \mathcal{T}_n$, we have

$$\sum_{k=1}^{m} (-1)^k \varphi(T_1 \triangle \{x_k, x_k^*\}) \varphi(T_2 \triangle \{x_k, x_k^*\}) \in N_F, \qquad \text{(Wick*)}$$

where $\{x_1, \ldots, x_n\} = (T_1 \triangle T_2) \cap [n]$ with $x_1 < \cdots < x_m$.

Two Wick functions φ and φ' with coefficients in F are *equivalent* if $\varphi = c \cdot \varphi'$ for some $c \in F^\times$, and we call an equivalence class $M_\varphi = [\varphi]$ of (strong) Wick functions a (*strong*) *orthogonal matroid with coefficients in F* or a *orthogonal F-matroid*.

Proposition 5.1.2. *The support* $\operatorname{supp}(\varphi) := \{T \in \mathcal{T}_n : \varphi(T) \neq 0\}$ *of a strong Wick function φ is the set of bases of an ordinary orthogonal matroid.*

Proof. Clearly $\operatorname{supp}(\varphi) \neq \emptyset$ by (W1). Let B_1, B_2 be in $\operatorname{supp}(\varphi)$ with $\{x, x^*\} \subseteq B_1 \triangle B_2$. Let $T_1 = B_1 \triangle \{x, x^*\}$ and $T_2 = B_2 \triangle \{x, x^*\}$, and we write $(B_1 \triangle B_2) \cap [n] = (T_1 \triangle T_2) \cap [n] = \{x_1 < \cdots < x_m\}$. Take $i \in [m]$ such that $\{x_i, x_i^*\} = \{x, x^*\}$. Then we have $\varphi(T_1 \triangle \{x_i, x_i^*\}) \varphi(T_2 \triangle \{x_i, x_i^*\}) = \varphi(B_1)\varphi(B_2) \neq 0$. By (W2), there exists $y \in \{x_1, \ldots, x_m\} \setminus \{x_i\}$ such that $\varphi(T_1 \triangle \{y, y^*\}) \varphi(T_2 \triangle \{y, y^*\}) \neq 0$, implying that $B_j \triangle \{x, x^*\} \triangle \{y, y^*\} = T_j \triangle \{y, y^*\} \in \operatorname{supp}(\varphi)$ for both $j \in \{1, 2\}$. □

We denote by $\underline{M_\varphi}$ the *underlying orthogonal matroid* of an orthogonal F-matroid M_φ, whose family of bases is $\operatorname{supp}(\varphi)$.

Definition 5.1.3. Let $\varphi : \mathcal{T}_n \to F$ be a map such that the support of φ is the set of bases of an orthogonal matroid. We say that φ is a *moderately weak Wick function on E with coefficients in F* if φ satisfies (W1) and the following weakened version of (W2):

(W2′) For all T_1, $T_2 \in \mathcal{T}_n$, if at most four of $\varphi(T_1 \triangle \{x, x^*\})\varphi(T_2 \triangle \{x, x^*\})$'s with $x \in (T_1 \cap T_2) \cap [n]$ are nonzero, then we have (Wick*).

We say that φ is a *weak Wick function on E with coefficients in F* if φ satisfies (W1) and the following:

(W2″) For all T_1, $T_2 \in \mathcal{T}_n$, if $|(T_1 \triangle T_2) \cap [n]| = 4$, then we have (Wick*).

We define (*moderately*) *weak orthogonal F-matroid* as an equivalenc class of (moderately) weak Wick functions.

It is trivial that every moderately weak orthogonal F-matroid is weak. Proposition 5.1.2 shows that every strong orthogonal F-matroid is a moderately weak orthogonal F-matroid. Three notions of orthogonal F-matroids are the same when F is a partial field [6], the tropical hyperfield $\mathbb{T}$ [102], or the Krasner hyperfield $\mathbb{K}$.

Every matroid can be identified with its lift that is an orthogonal matroid. Also, we have a canonical map $\mathrm{Gr}_{\mathbb{F}}(r, n) \to \mathrm{OGr}_{\mathbb{F}}(n, 2n)$ such that $V \mapsto V \oplus V^\perp$ where $V^\perp$ is the orthogonal complement of V with respect to the usual inner product. The following proposition generalizes these observations for F-matroids and orthogonal F-matroids.

Proposition 5.1.4. *There is a natural bijection between*

(i) *the set of all strong F-matroids on $[n]$ and*

(ii) *the set of all strong orthogonal F-matroids M_ψ on $[n] \cup [n]^*$ such that the intersections of bases of $\underline{M_\psi}$ and $[n]$ have the same cardinality.*

Proof. Let $\varphi : [n]^r \to F$ be a strong Grassmann-Plücker function. Define $\psi : \mathcal{T}_n \to F$ to be $\psi(T) := \varphi(a_1, \dots, a_r)$ if $T = B \cup ([n] \setminus B)^*$ for $B = \{a_1 < \cdots < a_r\}$, and $\psi(T) = 0$ otherwise. It is obvious that ψ is not identically zero, and we claim that ψ satisfies (W2). To prove (W2), we take $T_1, T_2 \in \mathcal{T}_n$ with $(T_1 \triangle T_2) \cap [n] = \{x_1 < \cdots < x_m\}$. Suppose without loss of generality that $T_1 \cap [n] = \{b_1 < \cdots < b_{r+1}\}$ and $T_2 \cap [n] = \{c_1 < \cdots < c_{r-1}\}$. If $x_k \in (T_2 \setminus T_1) \cap [n]$, then $\psi(T_1 \triangle \{x_k, x_k^*\}) = \psi(T_2 \triangle \{x_k, x_k^*\}) = 0$. If $x_k = b_j \in (T_1 \setminus T_2) \cap [n]$, then since $|T_1 \cap [x_k]| = j$ and $|T_2 \cap [x_k]| = k - j + 2|T_1 \cap T_2 \cap [x_k]|$, we have

$$\psi(T_1 \triangle \{x_k, x_k^*\}) = \varphi(b_1, \dots, \hat{b_j}, \dots, b_{r+1}) \text{ and } \psi(T_2 \triangle \{x_k, x_k^*\}) = (-1)^{k-j}\varphi(b_j, c_1, \dots, c_{r-1}).$$

It follows that

$$\sum_{k=1}^{m} (-1)^k \psi(T_1 \triangle \{x_k, x_k^*\})\psi(T_2 \triangle \{x_k, x_k^*\}) = \sum_{j=1}^{r+1} (-1)^j \varphi(b_1, \dots, \hat{b_j}, \dots, b_{r+1})\varphi(b_j, c_1, \dots, c_{r-1}) \in N_F.$$

Therefore, ψ is a strong Wick function whose support forms the bases of lift$\underline{M_\varphi}$.

Conversely, let ψ be a strong Wick function on $E = [n] \cup [n]^*$ such that all elements of $\{B \cap [n] : B \in \operatorname{supp}(\psi)\}$ have the same cardinality r. Let $\varphi : [n]^r \to F$ be the (unique) function satisfying (GP1) and (GP2) defined by $\varphi(a_1, \dots, a_r) := \psi(T)$ where $T = \{a_1, \dots, a_r\} \cup ([n] \setminus \{a_1, \dots, a_r\})^*$ for all $\{a_1 <$

$\cdots < a_r\} \subseteq [n]$. Take $J_1 = \{b_1 < \cdots < b_{r+1}\}$, $J_2 = \{c_1, \ldots, c_{r-1}\} \subseteq [n]$, and write $J_1' = J_1 \cup ([n] \setminus J_1)^*$ and $J_2' = J_2 \cup ([n] \setminus J_2)^*$. Then

$$\sum_{j=1}^{r+1}(-1)^j \varphi(b_1, \ldots, \hat{b_j}, \ldots, b_{r+1})\varphi(b_j, c_1, \ldots, c_{r-1}) = \sum_{j=1}^{r+1}(-1)^j \cdot \psi(J_1' \triangle \{b_j, b_j^*\}) \cdot (-1)^{m_k}\psi(J_2' \triangle \{b_j, b_j^*\}),$$

where m_j is the number of elements in J_2 that are less than b_j. Write $(J_1' \triangle J_2') \cap [n] = \{x_1, \ldots, x_m\}$. If $e \in J_2$, then since all elements of $\{B \cap [n] : B \in \operatorname{supp}(\psi)\}$ have cardinality r, we have $\psi(J_2' \triangle \{e, e^*\}) = 0$. Therefore, we have

$$\sum_{j=1}^{r+1}(-1)^j \varphi(b_1, \ldots, \hat{b_j}, \ldots, b_{r+1})\varphi(b_j, c_1, \ldots, c_{r-1}) = \sum_{k=1}^{m}(-1)^k \psi(J_1' \triangle \{x_k, x_k^*\})\psi(J_2' \triangle \{x_k, x_k^*\}) \in N_F.$$

It's not hard to see that the two constructions are inverses of each other. □

Remark 5.1.5. The variant of Proposition 5.1.4 for weak F-matroids and weak orthogonal F-matroids holds and the proof is similar.

5.1.2 Orthogonal signatures and circuit sets

Let $\underline{M}$ be an ordinary orthogonal matroid on E. If $X \in F^E$, we write $X^* \in F^E$ for the function defined by $X^*(i) := X(i^*)$. Notice that this induces an obvious involution $*$ on the subsets of F^E.

Definition 5.1.6. A subset $\mathcal{C} \subseteq F^E$ is an *F-signature of $\underline{M}$* if the following hold:

(i) The support $\underline{\mathcal{C}} = \{\underline{X} : X \in \mathcal{C}\}$ of $\mathcal{C}$ is the set of circuits of $\underline{M}$.

(ii) If $X \in \mathcal{C}$ and $\alpha \in F^\times$, then $\alpha X \in \mathcal{C}$.

We call $\underline{M}_{\mathcal{C}} := \underline{M}$ the underlying orthogonal matroid of $\mathcal{C}$ and call each element of $\mathcal{C}$ an *F-circuit*.

The *inner product* $\langle \cdot, \cdot \rangle$ on F^E with respect to the involution $x \mapsto \overline{x}$ is defined to be

$$\langle X, Y \rangle = \sum_{i \in [n]} (X(i)\overline{Y(i)} + \overline{X(i^*)}Y(i^*)).$$

Note that $\langle Y, X \rangle = \overline{\langle X, Y \rangle}$. Let $\tilde{\cdot} : F^E \to F^E$ be such that $\tilde{X}(i) = X(i)$ if $i \in [n]$ and $\tilde{X}(i) = \overline{X(i)}$ otherwise. Then $\langle X, Y^* \rangle = \sum_{i \in E} \tilde{X}(i)\tilde{Y}(i^*) = \eta(\tilde{X}, \tilde{Y})$.

We say that an F-signature $\mathcal{C}$ of $\underline{M}$ satisfies the 2-*term orthogonality* if the following condition holds:

(O_2) $\langle X, Y^* \rangle \in N_F$ for all $X, Y \in \mathcal{C}$ with $|\underline{X} \cap \underline{Y}^*| = 2$,

Lemma 5.1.7. *Let $\mathcal{C}$ be an F-signature of $\underline{M}$ satisfying the 2-term orthogonality (O_2). If $X, X' \in \mathcal{C}$ and $\underline{X} = \underline{X'}$, then $X = \alpha X'$ for some $\alpha \in F^\times$.*

Proof. Consider two F-circuits X and X' in $\mathcal{C}$ with $\underline{X} = \underline{X'} = C$. Suppose for contradiction that there exist distinct elements $e, f \in C$ with $X(e)/X(f) \neq X'(e)/X'(f)$. Let B be a base of M containing $C \triangle \{e, e^*\}$, and let D be the fundamental circuit $C(B, f^*)$. Then $C \cap D^* = \{e, f\}$. Let Y be an F-circuit in $\mathcal{C}$ such that $\underline{Y} = D$. Then $\langle X, Y^* \rangle = \tilde{X}(e)\tilde{Y}(e^*) + \tilde{X}(f)\tilde{Y}(f^*) \in N_F$ by (O_2) and thus $\tilde{X}(e)/\tilde{X}(f) = \tilde{Y}(f^*)/\tilde{Y}(e^*)$. We also have the same result for X', a contradiction. □

Definition 5.1.8. We call an F-signature $\mathcal{C}$ of $\underline{M}$ a *strong orthogonal F-signature of $\underline{M}$* if

(O) $\langle X, Y^* \rangle \in N_F$ for all $X, Y \in \mathcal{C}$.

We call an F-signature $\mathcal{C}$ of $\underline{M}$ a *weak orthogonal F-signature of $\underline{M}$* if

(O_4) $\langle X, Y^* \rangle \in N_F$ for all $X, Y \in \mathcal{C}$ with $|\underline{X} \cap \underline{Y}^*| \leq 4$.

Remark 5.1.9. Let $(\mathcal{C}, \mathcal{D})$ be a dual pair of F-signatures of a matroid $\underline{N}$ on $[n]$, and let $\mathcal{C}_1$ and $\mathcal{D}_1$ be the obvious embeddings of $\mathcal{C}$ and $\mathcal{D}$ in $F^E = F^{[n] \cup [n]^*}$. By Proposition 4.1.44, $\mathcal{C}_1 \cup \mathcal{D}_1^*$ is an F-signature of lift($\underline{N}$). It is readily seen from definitions that $(\mathcal{C}, \mathcal{D})$ is a strong dual pair of F-signatures of $\underline{N}$ if and only if $\mathcal{C}_1 \cup \mathcal{D}_1^*$ is a strong orthogonal F-signature of lift($\underline{N}$). In addition, $(\mathcal{C}, \mathcal{D})$ is a weak dual pair of F-signatures of $\underline{N}$ if and only if $\mathcal{C}_1 \cup \mathcal{D}_1^*$ is an F-signature of lift($\underline{N}$) which satisfies the following:

(O_3) $\langle X, Y^* \rangle \in N_F$ for all $X, Y \in \mathcal{C}$ with $|\underline{X} \cap \underline{Y}^*| \leq 3$.

We will show later in Example 5.1.12 that for some field K, there exists a K-signature of an orthogonal matroid which satisfies (O_3) but not (O_4).

Definition 5.1.10. A *strong F-circuit set* of $\underline{M}$ is an F-signature $\mathcal{C}$ of $\underline{M}$ satisfying (O_2) and the following property:

(L) For every F-circuit $X \in \mathcal{C}$ and a base B of $\underline{M}$, the vector $\tilde{X}$ is in the linear span of $\{\tilde{X}_e\}_{e \in B^*}$, where X_e is an F-circuit in $\mathcal{C}$ with support $C(B, e)$.

A *weak F-circuit set* of $\underline{M}$ is an F-signature $\mathcal{C}$ of $\underline{M}$ satisfying (O_2) and the next weakened version of (L):

(L′-i) Let B be an arbitrary base of $\underline{M}$, and let $e_1, e_2 \in B^*$ be distinct. Let $X_i \in \mathcal{C}$ be an F-circuit with support $\underline{X_i} = C(B, e_i)$ for $i = 1, 2$. If $\underline{X_1} \cup \underline{X_2}$ is a transversal and if $f \in \underline{X_1} \cap \underline{X_2}$, then there exists an F-circuit $Y \in \mathcal{C}$ such that $Y(f) = 0$ and $\tilde{Y}$ is in the linear span of $\tilde{X_1}$ and $\tilde{X_2}$.

(L′-ii) Let B be an arbitrary base of $\underline{M}$, and let $e_1, e_2, e_3 \in B^*$ be distinct. Let $X_i \in \mathcal{C}$ be an F-circuit with support $\underline{X_i} = C(B, e_i)$ for $i = 1, 2, 3$. If none of $\underline{X_i} \cup \underline{X_j}$ with $1 \leq i < j \leq 3$ is a transversal, then there exists an F-circuit $Y \in \mathcal{C}$ such that $Y(e_1^*) = Y(e_2^*) = Y(e_3^*) = 0$ and $\tilde{Y}$ is in the linear span of $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{X}_3$.

Remark 5.1.11. Let $\mathcal{C}$ be a weak F-circuit set of a matroid $\underline{N}$ on $[n]$. By [4], its dual $\mathcal{D}$ is the F-signature of the dual matroid $\underline{N}^*$ such that $X \perp Y$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ with $|\underline{X} \cap \underline{Y}| = 2$. Let $\mathcal{C}_1$ and $\mathcal{D}_1$ be natural embeddings of $\mathcal{C}$ and $\mathcal{D}$ into $F^{[n] \cup [n]^*}$. Then $\mathcal{C}_1 \cup \mathcal{D}_1^*$ is an F-signature of lift$\underline{N}$ that satisfies (O_2) and (L′-i) by definition, and $\mathcal{C}_1 \cup \mathcal{D}_1^*$ vacuously satisfies (L′-ii). Therefore, $\mathcal{C}_1 \cup \mathcal{D}_1^*$ is a weak F-circuit set of lift($\underline{N}$). If $\mathcal{C}$ is a strong F-circuit set of $\underline{N}$, then $\mathcal{C}_1 \cup \mathcal{D}_1^*$ is a strong F-circuit set of lift($\underline{N}$).

Denote by $\pi : F^{[n] \cup [n]^*} \to F^{[n]}$ the canonical projection map. Then an F-signature $\mathcal{C}$ of lift$\underline{N}$ is a weak (resp. strong) F-circuit set if and only if $\{\pi(X) : X \in \mathcal{C} \text{ with } \underline{X} \subseteq [n]\}$ and $\{\pi(X^*) : X \in \mathcal{C} \text{ with } \underline{X}^* \subseteq [n]\}$ are weak (resp. strong) F-circuit sets of $\underline{N}$ and $\underline{N}^*$ respectively, and those two F-circuit sets are the dual of each other.

Example 5.1.12. Let $\mathbb{F}$ be a field with $|\mathbb{F}^\times| > 1$ and $\text{char}(\mathbb{F}) \neq 3$ and let $x \in \mathbb{F} \setminus \{0, -3\}$. We assume the trivial involution on $\mathbb{F}$. Let $\mathcal{C}$ be a subset of $\mathbb{F}^{[4] \cup [4]^*}$ consisting of the following eight vectors and their scalar multiples by nonzero elements:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & 0 & 0 \\ -1 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -x \\ 1 & -1 & 1 & 0 \end{pmatrix},$$

where $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$ means $X \in \mathbb{F}^{[4]\cup[4]^*}$ such that $X(i) = a_i$ and $X(i^*) = b_i$ with $i \in [4]$. Then $\mathcal{C}$ is a $\mathbb{F}$-signature of the orthogonal matroid whose set of bases is $\{[4], [4]^*\} \cup \{ijk^*l^* : ijkl = [4]\}$. Notice that $\mathcal{C}$ satisfies (O_3) and (L′-i), but neither (O_4) nor (L′-ii).

We prove the following results in Section 5.2.

Theorem 5.1.13. *An F-signature of an orthogonal matroid is a strong orthogonal F-signature if and only if it is a strong F-circuit set.*

Theorem 5.1.14. *Every weak F-circuit set of an orthogonal matroid is a weak orthogonal F-signature.*

The converse of Theorem 5.1.14 is not true; see Example 5.2.21.

5.1.3 Orthogonal F-vector sets

Let $\mathcal{V}$ be a subset of F^E. A vector $X \in \mathcal{V}$ is *elementary* in $\mathcal{V}$ if (i) it is nonzero, and (ii) it has a minimal support in $\mathcal{V} \setminus \{\mathbf{0}\}$, and (iii) its support $\underline{X}$ is a transversal. A transversal $T \in \mathcal{T}_n$ is a *support base* of $\mathcal{V}$ if there is no $X \in \mathcal{V} \setminus \{\mathbf{0}\}$ such that $\underline{X} \subseteq T$. A *fundamental circuit form* for $\mathcal{V}$ with respect to a support base B is $\{X^{\mathcal{V}}_{B,e} : e \in B^*\}$ where $X^{\mathcal{V}}_{B,e} \in \mathcal{V}$ is such that $\mathrm{supp}(X^{\mathcal{V}}_{B,e}) \subseteq B \triangle \{e, e^*\}$ and $X^{\mathcal{V}}_{B,e}(e) = 1$. We simply write $X^{\mathcal{V}}_{B,e}$ as X_e if it is clear from the context.

Definition 5.1.15. We call $\mathcal{V} \subseteq F^E$ an *orthogonal F-vector set* if the following hold:

(V1) For all elementary vectors $X, Y \in \mathcal{V}$, if $|\underline{X} \cap \underline{Y}^*| \leq 2$, then $\langle X, Y^* \rangle \in N_F$.

(V2) Support bases exist, and for each support base B, there exists a corresponding fundamental circuit form.

(V3) $\mathcal{V}$ is exactly the set of vectors $X \in F^E$ such that for every support base B of $\mathcal{V}$, $\tilde{X}$ belongs to the linear span of $\{\tilde{X}_e : e \in B^*\}$.

The axiom (V3) implies the uniqueness of the fundamental circuit form for each support base of an orthogonal F-vector set $\mathcal{V}$, and that every fundamental circuit form of an orthogonal F-vector set $\mathcal{V}$ consists of elementary vectors of $\mathcal{V}$. When $F = \mathbb{F}$ is a field, a subset $\mathcal{W} \subseteq \mathbb{F}^{[n]}$ is an $\mathbb{F}$-vector set if and only if it is a linear subspace [2]. We give an analog of this for orthogonal $\mathbb{F}$-vector sets.

Theorem 5.1.16. *Let $\mathbb{F}$ be a field and $\mathcal{V}$ be a subset of $\mathbb{F}^E$.*

(i) *If $\mathcal{V}$ is an orthogonal $\mathbb{F}$-vector set, then it is a Lagrangian subspace with respect to the inner product $\langle \cdot, (\cdot)^* \rangle$.*

(ii) *Whenever* $\mathrm{char}(\mathbb{F}) \neq 2$*, the converse of* (i) *holds.*

We delay the proof of Theorem 5.1.16(i) to Section 5.2.5. Theorem 5.1.16(ii) can be deduced from the results of Oum [95, Propositions 4.2 and 4.3(i)]. The condition that $\mathrm{char}(\mathbb{F}) \neq 2$ in (ii) is crucial, since otherwise $\mathcal{V} = \{(x, x) : x \in \mathbb{F}\}$ is a Lagrangian subspace of $\mathbb{F}^{[1]\cup[1]^*}$ but not an orthogonal F-vector set.

Lemma 5.1.17 ([95])**.** *Let $\mathbb{F}$ be a field and let $\mathcal{V} \subseteq \mathbb{F}^E$ be a Lagrangian subspace with respect to $\langle \cdot, (\cdot)^* \rangle$.*

(i) *There is a support base of $\mathcal{V}$.*

(ii) *If* $\mathrm{char}(\mathbb{F}) \neq 2$*, then for each support base* B *of* $\mathcal{V}$ *and* $x \in B^*$*, there exists a unique vector* $X \in \mathcal{V}$ *such that* $\underline{X} \subseteq B \triangle \{x, x^*\}$ *and* $X(x) = 1$*.*

Proof of Theorem 5.1.16(ii). Since $\mathcal{V}$ is isotropic, it satisfies (V1). By Lemma 5.1.17, (V2) holds. Since the n vectors in each fundamental circuit form are independent, $\mathcal{V}$ satisfies (V3). Therefore, $\mathcal{V}$ is an orthogonal F-vector set. □

5.1.4 Main theorems

We prove the equivalence of four notions of strong orthogonal matroids over tracts.

Theorem 5.1.18. *Let* $E = [n] \cup [n]^*$ *and let* F *be a tract endowed with an involution* $x \mapsto \overline{x}$*. Then there are natural bijections between any pair of the following:*

(1) *Strong orthogonal* F*-matroids on* E*.*

(2) *Strong orthogonal* F*-signatures on* E*.*

(3) *Strong* F*-circuit sets of orthogonal matroids on* E*.*

(4) *Orthogonal* F*-vector sets on* E*.*

Similarly, we have the next two equivalences for weaker notions.

Theorem 5.1.19. *Let* $E = [n] \cup [n]^*$ *and let* F *be a tract endowed with an involution* $x \mapsto \overline{x}$*. Then there is a natural bijection between:*

(1) *Moderately weak orthogonal* F*-matroids on* E*.*

(2) *Weak orthogonal* F*-signatures on* E*.*

Theorem 5.1.20. *Let* $E = [n] \cup [n]^*$ *and let* F *be a tract endowed with an involution* $x \mapsto \overline{x}$*. Then there is a natural bijection between:*

(1) *Weak orthogonal* F*-matroids on* E*.*

(2) *Weak* F*-circuit sets of orthogonal matroids on* E*.*

We will provide proofs of Theorems 5.1.18, 5.1.19, and 5.1.20 in Section 5.2. Since the notions of weak and strong orthogonal F-matroid coincide if F is a partial field [6], the tropical hyperfield $\mathbb{T}$ [102], or the Krasner hyperfield $\mathbb{K}$, it follows that the three notions of orthogonal F-matroids are equivalent when F is any of these specific tracts.

We summarize our results in Figure 5.1. Additionally, we remark that in Figure 5.1, each inclusion is strict for certain tracts; see Examples 5.2.21 and 5.2.22.

5.1.5 Functoriality.

Now we discuss the behavior of strong and weak orthogonal matroids over tracts with respect to tract homomorphisms. The following propositions are all straightforward from the definitions.

Proposition 5.1.21. *Let* $f : F_1 \to F_2$ *be a tract homomorphism, and let* φ *be a strong Wick function with coefficients in* F_1*. Then the composition* $f \circ \varphi$ *is a strong Wick function with coefficients in* F_2*. The same results hold for weak and moderately weak Wick functions.* □

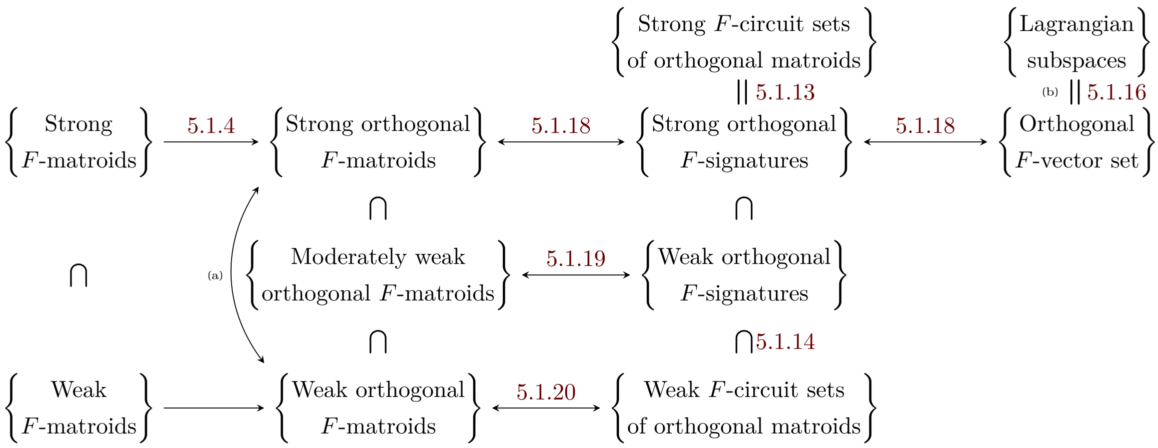


Figure 5.1: Summary of results in Section 5.1.1–5.1.4. In (a), we assume that $F \in \{\mathbb{T}, \mathbb{K}\}$ or F is a partial field [6, 102]. In (b), we assume that F is a field with $\mathrm{char}(F) \neq 2$.

By the above proposition, we define the *pushforward* operator f_* taking orthogonal F_1-matroids to orthogonal F_2-matroids by $f_*([\varphi]) := [f \circ \varphi]$. Notice that the pushforwards are functorial: if $F_1 \xrightarrow{f} F_2 \xrightarrow{g} F_3$ are tract homomorphisms, then $(g \circ f)_* = g_* \circ f_*$. If $F_2 = \mathbb{K}$, the terminal object of the category of tracts, then the orthogonal $\mathbb{K}$-matroid $[f \circ \varphi]$ is the same thing as $\underline{M}_\varphi$.

Proposition 5.1.22. *Let F_1 and F_2 be tracts equipped with involutions ι_1 and ι_2, respectively, and let $f : F_1 \to F_2$ be a tract homomorphism that respects the involutions, i.e. $f \circ \iota_1 = \iota_2 \circ f$. If $\mathcal{C}$ is a strong* (*resp. weak or moderately weak*) *orthogonal F_1-signature of an ordinary orthogonal matroid $\underline{M}$, then $f_*(\mathcal{C}) := \{cf(X) : c \in F_2^\times, X \in \mathcal{C}\}$ is a strong* (*resp. weak or moderately weak*) *orthogonal F_2-signature of $\underline{M}$. The same results hold for strong and weak circuit sets over tracts of an orthogonal matroid.* □

Therefore, we also have the *pushforward* operator f_* taking orthogonal F_1-signatures (resp. F_1-circuit set) to orthogonal F_2-signatures (resp. F_2-circuit set). If $F_2 = \mathbb{K}$, then $f_*(\mathcal{C})$ is the same thing as the set of circuits of $\underline{M}_{\mathcal{C}}$.

However, the simple pushforwards of orthogonal F-vector sets are not defined properly. In fact, there are a tract homomorphism $f : F_1 \to F_2$ respecting the involutions and an orthogonal F_1-vector set $\mathcal{V}$ such that the set $f_*(\mathcal{V}) = \{cf(X) : c \in F_2^\times, X \in \mathcal{V}\}$ is not an orthogonal F_2-vector set; see Example 5.2.25.

Proposition 5.1.23. *Let F_1, F_2 be tracts, and let φ_1, φ_2 be strong Wick functions with coefficients in F_1, F_2, respectively, with the same underlying orthogonal matroid $\underline{M}$. Then $\varphi_1 \times \varphi_2 : \mathcal{T}_n \to F_1 \times F_2$ defined as $(\varphi_1 \times \varphi_2)(T) = (\varphi_1(T), \varphi_2(T))$ is a strong Wick function with coefficients in the product $F_1 \times F_2$. The same results hold for weak and moderately weak Wick functions.* □

5.1.6 Duality.

Let $\varphi : \mathcal{T}_n \to F$ be a strong Wick function over F. Its *dual strong Wick function* $\varphi^* : \mathcal{T}_n \to F$ is defined as

$$\varphi^*(T) := \varphi(T^*)$$

for all $T \in \mathcal{T}_n$. It is indeed a strong Wick function with underlying orthogonal matroid $(\underline{M}_\varphi)^*$ from definitions. We define the duals of weak and moderately weak Wick F-functions in the same way.

Given a strong (resp. weak) orthogonal F-signature $\mathcal{C}$, we can define its *dual strong (resp. weak) orthogonal F-signature* $\mathcal{C}^*$ by setting

$$\mathcal{C}^* := \{X^* : X \in \mathcal{C}\},$$

and the underlying orthogonal matroid of $\mathcal{C}^*$ is $(\underline{M}_{\mathcal{C}})^*$. The duals of strong and weak F-circuit sets of orthogonal matroids are defined in the same way.

5.1.7 Minors.

Let φ be a strong or (moderately) weak Wick function on E with coefficients in F and take $e \in E$. Then we define $\varphi|e$ to be the function from the set of transversals of $E \setminus \{e, e^*\}$ to F as

$$(\varphi|e)(T) := \begin{cases} \varphi(T \cup \{e\}) & \text{if } e \text{ is nonsingular in } \underline{M}_\varphi, \\ \varphi(T \cup \{e^*\}) & \text{otherwise.} \end{cases}$$

Proposition 5.1.24. *Let φ be a strong Wick F-function φ on E and $e \in E$. Then $\varphi|e$ is a strong Wick F-function and $\underline{M}_{\varphi|e} = \underline{M}_\varphi|e$. The same holds for weak and moderately weak Wick F-functions.* □

We define minors of strong or weak orthogonal signatures as follows. Let $\mathcal{C}$ be a strong or weak orthogonal F-signature of an orthogonal matroid $\underline{M}$ on E. For $e \in E$, let $\mathcal{C}|e$ be the set of functions in

$$\{\pi(X) \in F^{E\setminus\{e,e^*\}} : X \in \mathcal{C} \text{ with } X(e^*) = 0 \text{ and } \underline{X} \neq \{e\}\}$$

that have minimal supports, where $\pi : F^E \to F^{E\setminus\{e,e^*\}}$ is the obvious projection.

The next proposition is a direct consequence of Proposition 4.1.52.

Proposition 5.1.25. *Let $\underline{M}$ be an orthogonal matroid on E and let $e \in E$. If $\mathcal{C}$ is a strong orthogonal F-signature of $\underline{M}$, then $\mathcal{C}|e$ is a strong orthogonal F-signature of $\underline{M}|e$. If $\mathcal{C}$ is a weak orthogonal F-signature of $\underline{M}$, then $\mathcal{C}|e$ is a weak orthogonal F-signature of $\underline{M}|e$.* □

Minors of a strong or weak F-circuit set of an orthogonal matroid are defined in the same way as minors of an orthogonal F-signature, and an analogue of Proposition 5.1.25 holds.

One possible candidate of a minor of an orthogonal F-vector set $\mathcal{V} \subseteq F^E$ with respect to $e \in E$ is

$$\mathcal{V}|e := \{\pi(X) \in F^{E\setminus\{e,e^*\}} : X \in \mathcal{V} \text{ with } X(e^*) = 0\},$$

which coincides with the deletion and the contraction of an F-vector set of a matroid in [2, Section 4.2]. However, $\mathcal{V}|e$ is not necessarily an orthogonal F-signature in general, even if F is a partial field and the underlying orthogonal matroid of $\mathcal{V}$ is the lift of a matroid; see Example 5.2.24. We remark that if F is a field, then $\mathcal{V}|e$ is an orthogonal F-vector set by [95, Proposition 3.8].

5.1.8 Other related works

We briefly indicate how our notions of strong and weak orthogonal F-matroids generalize various flavors of orthogonal matroids in the literature.

Example 5.1.26. If the support of a strong or weak orthogonal matroid on E over F is the lift of an ordinary matroid on $[n]$, then an orthogonal matroid on E over F is the same thing as a strong or weak matroid on $[n]$ over F in the sense of [4]. This follows from Proposition 5.1.4.

Example 5.1.27. A strong or weak orthogonal matroid over the Krasner hyperfield $\mathbb{K}$ is the same thing as an ordinary orthogonal matroid.

Example 5.1.28. When $F = \mathbb{F}$ is a field, a strong or weak Wick $\mathbb{F}$-matroid is the same thing as a projective solution to Wick equations in $\mathbb{P}(\mathbb{F}^{2^n})$. In addition, when $\mathrm{char}(\mathbb{F}) \neq 2$, a strong or weak orthogonal $\mathbb{F}$-matroid is the same thing as a maximal isotropic subspace of $\mathbb{F}^{2n}$ in the usual sense. Indeed, a weak Wick function with coefficients in the field $\mathbb{F}$ automatically satisfies (W2). This follows from [6, Theorem 1.6].

Example 5.1.29. A strong or weak orthogonal matroid over the regular partial field $\mathbb{U}_0$ is the same thing as a regular orthogonal matroid in the sense of [61]. This follows from the discussion on [61, Page 33] and [6, Theorem 1.6].

Example 5.1.30. A strong or weak orthogonal matroid over the tropical hyperfield $\mathbb{T}$ is the same thing as a valuated orthogonal matroid in the sense of [52] or a tropical Wick vector in the sense of [102]. This follows from [102, Theorem 5.1].

Example 5.1.31. A strong orthogonal matroid over the sign hyperfield $\mathbb{S}$ is the same thing as an oriented orthogonal matroid in the sense of [115, 118]. This follows from the discussion at the top of page 241 of [118].

5.2 Cryptomorphisms for orthogonal matroids over tracts

In this section, we give proofs of the main theorems of Chapter 5 and confirm Figure 5.1. Our plan is as follows. We first construct strong or weak orthogonal signatures from strong or moderately weak Wick functions in Subsection 5.2.1, and show the converse in Subsection 5.2.2. In Subsection 5.2.3 we show the equivalence between weak orthogonal F-matroids and weak F-circuit sets using the constructions in Sections 5.2.1 and 5.2.2. In Subsection 5.2.4 we prove Theorem 5.1.13 that orthogonal F-signatures and F-circuit sets coincide for the strong case. In Subsection 5.2.5 we show the equivalence between strong orthogonal signatures and orthogonal vector sets, as well as Theorem 5.1.16(i). We sum up all main theorems in Subsection 5.2.6. Subsection 5.2.7 provides several pathological examples.

Recall that $\mathcal{T}_n$ denotes the family of all transversals of $E = [n] \cup [n]^*$. For every $i \in E$, let $\bar{i}$ be the element in $[n]$ such that $\{i, i^*\} \cap [n] = \{\bar{i}\}$. For $i, j \in [n]$, let $(i, j]$ be the subset $\{k \in [n] : i < k \leq j\}$ if $i \leq j$, and $(j, i]$ otherwise. For $T \in \mathcal{T}_n$ and $i, j \in E$, let $m_{i,j}^T$ denote $|T \cap (\bar{i}, \bar{j}]|$. We often omit the superscription T in $m_{i,j}^T$ if it is clear from the context. If $\alpha, \beta \in F^\times$, we write $\frac{\beta}{\alpha}$ for $\alpha^{-1}\beta$. We often denote a finite set $S = \{a_1, a_2, \ldots, a_m\}$ by enumerating its elements, such as $a_1 a_2 \ldots a_m$.

5.2.1 From Wick functions to orthogonal signatures

Let φ be a weak Wick function on E with coefficients in a tract F. We denote by $\underline{M} = \underline{M}_\varphi$ the underlying orthogonal matroid of $[\varphi]$. We first suggest a candidate for the orthogonal signature induced from the given Wick function φ.

Recall that the set of bases of $\underline{M}$ is $\mathrm{supp}(\varphi) = \{B \in \mathcal{T}_n : \varphi(B) \neq 0\}$. For each circuit C of $\underline{M}$, we define a function $X \in F^E$ such that $\underline{X} = C$ as follows. Let $T \supseteq C$ be a transversal such that $T \triangle \{x, x^*\} \in \mathrm{supp}(\varphi)$ for all $x \in C$, which exists by Lemma 4.1.46. Then for every $e, f \in C$, we set

$$\frac{\tilde{X}(e)}{\tilde{X}(f)} = (-1)^{m_{e,f}^T} \frac{\varphi(T \triangle \{e, e^*\})}{\varphi(T \triangle \{f, f^*\})}. \tag{5.1}$$

We call X an *F-circuit of φ with support C*.

Lemma 5.2.1. *The ratio $\frac{\tilde{X}(e)}{\tilde{X}(f)}$ is independent of the choice of T. Explicitly, let T_1, T_2 be distinct transversals containing C such that both $T_1 \triangle \{x, x^*\}$ and $T_2 \triangle \{x, x^*\}$ are bases of $\underline{M}$ for all $x \in C$. Then*

$$(-1)^{m_1} \frac{\varphi(T_1 \triangle \{e, e^*\})}{\varphi(T_1 \triangle \{f, f^*\})} = (-1)^{m_2} \frac{\varphi(T_2 \triangle \{e, e^*\})}{\varphi(T_2 \triangle \{f, f^*\})}.$$

where $m_i = |T_i \cap (\overline{e}, \overline{f}]|$ for each $i = 1, 2$.

Proof. We proceed by induction on $|T_1 \setminus T_2|$. Since $T_1 \triangle \{x, x^*\}$ and $T_2 \triangle \{x, x^*\}$ with $x \in C \neq \emptyset$ are distinct bases of $\underline{M}$, we know that $|T_1 \setminus T_2| = |B_1 \setminus B_2|$ is even and at least 2.

Suppose that $|T_1 \setminus T_2| = 2$. Write $T_1 \setminus T_2 = \{a, b\}$ so that $T_1 \triangle T_2 = \{a, a^*, b, b^*\}$. Then neither $T_1 \triangle \{a, a^*\}$ nor $T_1 \triangle \{b, b^*\}$ is a base since they contain C. Thus, $\varphi(T_1 \triangle \{a, a^*\}) = \varphi(T_1 \triangle \{b, b^*\}) = 0$. Denote $m = |\{\overline{a}, \overline{b}\} \cap (\overline{e}, \overline{f}]|$. Note that $m_1 + m \equiv m_2 \pmod 2$. By the axiom (W2) applied to $T_1 \triangle \{e, e^*, f, f^*\}$ and $T_1 \triangle \{a, a^*, b, b^*\} = T_2$, we have

$$\varphi(T_1 \triangle \{f, f^*\}) \varphi(T_2 \triangle \{e, e^*\}) + (-1)^{m+1} \varphi(T_1 \triangle \{e, e^*\}) \varphi(T_2 \triangle \{f, f^*\}) \in N_F,$$

which implies the desired equality.

Now we assume that $|T_1 \setminus T_2| > 2$. Fix $x \in C$ and let $B_i := T_i \triangle \{x, x^*\}$ with $i \in \{1, 2\}$. Then B_1 and B_2 are bases of $\underline{M}$. Take $y \in T_1 \setminus T_2$. By the symmetric exchange axiom, there is $z \in (T_1 \setminus T_2) \setminus \{y\}$ such that $B_1 \triangle \{y, y^*, z, z^*\}$ is a base of $\underline{M}$. Let $T_0 := T_1 \triangle \{y, y^*, z, z^*\}$. We claim that for every $w \in C$, the transversal $T_0 \triangle \{w, w^*\}$ is a base of $\underline{M}$. Indeed, since $T_0 \triangle \{x, x^*\} = B_1 \triangle \{y, y^*, z, z^*\}$, we may assume that $w \neq x$. Then by Lemma 4.1.45, $T_1 \triangle \{w, w^*\} = B_1 \triangle \{x, x^*, w, w^*\}$ is a base of $\underline{M}$. Both $B_1 \triangle \{x, x^*, y, y^*\} = T_1 \triangle \{y, y^*\}$ and $B_1 \triangle \{x, x^*, z, z^*\} = T_1 \triangle \{z, z^*\}$ contain C and hence neither of them is a base. Then the symmetric exchange forces that $T_0 \triangle \{w, w^*\} = B_1 \triangle \{x, x^*, y, y^*, z, z^*, w, w^*\}$ is a base. Notice that $|T_0 \triangle T_1| = 4$ and $|T_0 \triangle T_2| < |T_1 \triangle T_2|$. Therefore, we conclude the desired equality by the claim and the induction hypothesis. □

Proposition 5.2.2. *Every circuit C of $\underline{M}$ corresponds to an F-circuit $X \in F^E$ of φ with support C such that X is well-defined and unique up to multiplication by an element in $F^\times$.*

Proof. Lemma 5.2.1 shows the uniqueness. We assert furthermore that X is well-defined. This can be proved directly. Let T be a transversal such that $C \subseteq T$ and $T \triangle \{x, x^*\} \in \mathrm{supp}(\varphi)$ for all $x \in C$. Take $e, f, g \in C$. Since $m_{e,f} + m_{f,g} + m_{e,g} \equiv 0 \pmod 2$, we have

$$\begin{aligned}\frac{\tilde{X}(e)}{\tilde{X}(f)} \frac{\tilde{X}(f)}{\tilde{X}(g)} &= (-1)^{m_{e,f}} \frac{\varphi(T \triangle \{e, e^*\})}{\varphi(T \triangle \{f, f^*\})} \cdot (-1)^{m_{f,g}} \frac{\varphi(T \triangle \{f, f^*\})}{\varphi(T \triangle \{g, g^*\})} \\ &= (-1)^{m_{e,g}} \frac{\varphi(T \triangle \{e, e^*\})}{\varphi(T \triangle \{g, g^*\})} = \frac{\tilde{X}(e)}{\tilde{X}(g)}.\end{aligned}$$ □

By Proposition 5.2.2, the set $\mathcal{C}_\varphi$ of all projective F-circuits induced from φ is an F-signature of $\underline{M}$.

Theorem 5.2.3. *Let F be a tract. If φ is a strong Wick function over F, then $\mathcal{C}_\varphi$ is a strong orthogonal F-signature.*

Proof. Let $X_1, X_2 \in \mathcal{C}_\varphi$. We may assume that $\underline{X_1} \cap \underline{X_2}^* \neq \emptyset$. By Lemma 4.1.46, there is a transversal T_i containing $\underline{X_i}$ such that $T_i \triangle \{e, e^*\}$ is a base for every $e \in \underline{X_i}$ and $i = 1, 2$. Note that

$$\varphi(T_1 \triangle \{e, e^*\}) \varphi(T_2 \triangle \{e, e^*\}) = 0$$

for all $e \in (T_1 \triangle T_2) \setminus (\underline{X_1} \triangle \underline{X_2})$. Write $T_1 \cap T_2^* = \{e_1, e_2, \ldots, e_a\}$ with $\overline{e_1} < \overline{e_2} < \cdots < \overline{e_a}$ and write $\underline{X_1} \cap \underline{X_2}^* = \{e_{\alpha_1}, \ldots, e_{\alpha_b}\}$ with $\alpha_1 < \cdots < \alpha_b$. Then $(T_1 \triangle T_2) \cap [n] = \{\overline{e_1}, \ldots, \overline{e_a}\}$. Let $m_j := |T_1 \cap (\overline{e_{\alpha_1}}, \overline{e_{\alpha_j}}]|$ and $n_j := |T_2 \cap (\overline{e_{\alpha_1}}, \overline{e_{\alpha_j}}]|$ for each $j \in [b]$. Since $(T_1 \triangle T_2) \cap (\overline{e_{\alpha_1}}, \overline{e_{\alpha_j}}] = \{\overline{e_k} : \alpha_1 < k \le \alpha_j\}$, we have $m_j + n_j \equiv \alpha_j - \alpha_1 \pmod 2$. By (W2) applied to T_1 and T_2, we have

$$N_F \ni \sum_{i=1}^{a} (-1)^i \varphi(T_1 \triangle \{e_i, e_i^*\}) \varphi(T_2 \triangle \{e_i, e_i^*\}) = \sum_{i=1}^{b} (-1)^{\alpha_i} \varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}) \varphi(T_2 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}).$$

Therefore,

$$\begin{aligned}
\langle X_1, X_2^* \rangle &= \sum_{i=1}^{b} \tilde{X}_1(e_{\alpha_i}) \tilde{X}_2(e_{\alpha_i}^*) \\
&= \tilde{X}_1(e_{\alpha_1}) \tilde{X}_2(e_{\alpha_1}^*) \sum_{i=1}^{b} \frac{\tilde{X}_1(e_{\alpha_i})}{\tilde{X}_1(e_{\alpha_1})} \frac{\tilde{X}_2(e_{\alpha_i}^*)}{\tilde{X}_2(e_{\alpha_1}^*)} \\
&= \tilde{X}_1(e_{\alpha_1}) \tilde{X}_2(e_{\alpha_1}^*) \sum_{i=1}^{b} (-1)^{m_i} \frac{\varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})}{\varphi(T_1 \triangle \{e_{\alpha_1}, e_{\alpha_1}^*\})} (-1)^{n_i} \frac{\varphi(T_2 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})}{\varphi(T_2 \triangle \{e_{\alpha_1}, e_{\alpha_1}^*\})} \\
&= (-1)^{\alpha_1} \frac{\tilde{X}_1(e_{\alpha_1}) \tilde{X}_2(e_{\alpha_1}^*)}{\varphi(T_1 \triangle \{e_{\alpha_1}, e_{\alpha_1}^*\}) \varphi(T_2 \triangle \{e_{\alpha_1}, e_{\alpha_1}^*\})} \cdot \sum_{i=1}^{b} (-1)^{\alpha_i} \varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}) \varphi(T_2 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}) \in N_F.
\end{aligned}$$

□

Theorem 5.2.4. *Let F be a tract. If φ is a moderately weak Wick function over F, then $\mathcal{C}_\varphi$ is a weak orthogonal F-signature.*

Proof. It is not hard to see that $\mathcal{C}_\varphi$ satisfies (O_4) if we replace (W2) with (W2′) in the proof of Theorem 5.2.3. □

5.2.2 From orthogonal signatures to Wick functions.

Throughout this subsection, $\mathcal{C} \subseteq F^E$ is an F-signature of an ordinary orthogonal matroid $\underline{M}$ satisfying the 2-term orthogonality:

(O_2) $\langle X, Y^* \rangle \in N_F$ for all $X, Y \in \mathcal{C}$ with $|\underline{X} \cap \underline{Y}^*| = 2$.

Recall that by Lemma 5.1.7, for each circuit C of $\underline{M}$, the F-circuit $X \in \mathcal{C}$ (and equivalently, $\tilde{X}$) with $\underline{X} = C$ is unique up to multiplication by an element in $F^\times$.

We first set $\gamma(B, B) = 1$ for every base B of $\underline{M}$. Let B_1, B_2 be two bases of $\underline{M}$ with $|B_1 \triangle B_2| = 4$. We can write $B_1 = T \triangle \{f, f^*\}$ and $B_2 = T \triangle \{e, e^*\}$ for some transversal T containing e and f. Let $X \in \mathcal{C}$ be the F-circuit whose support $\underline{X}$ is the fundamental circuit $C(B_1, f)$. Then $\underline{X} = C(B_2, e) \subseteq T$, and in particular, $e, f \in \underline{X}$. We define

$$\gamma(B_1, B_2) := (-1)^{m_{e,f}^T} \frac{\tilde{X}(e)}{\tilde{X}(f)}.$$

Proposition 5.2.5. *$\gamma(B_1, B_2)$ is well-defined.*

Proof. By Lemma 5.1.7, $\gamma(B_1, B_2)$ is independent of the choice of X for fixed T. Let $T_1 = T \triangle \{e, e^*\} \triangle \{f, f^*\} \ni e^*, f^*$. Let $X_1 \in \mathcal{C}$ be such that $\underline{X_1} = C(B_1, e^*) = C(B_2, f^*) \ni e^*, f^*$. It suffices to show that

$$(-1)^{m_{e,f}^T} \frac{\tilde{X}(e)}{\tilde{X}(f)} = (-1)^{m_{e,f}^{T_1}} \frac{\tilde{X}_1(f^*)}{\tilde{X}_1(e^*)}.$$

Since $(T \triangle T_1) \cap (\overline{e}, \overline{f}] = \max\{\overline{e}, \overline{f}\}$, we have $|m_{e,f}^T - m_{e,f}^{T_1}| = 1$. By (O_2), $\tilde{X}(e)\tilde{X}_1(e^*) + \tilde{X}(f)\tilde{X}_1(f^*) = \langle X, X_1^* \rangle \in N_F$ and therefore we obtain the desired equality. □

The next lemma is obvious from the definition.

Lemma 5.2.6. *If B_1, B_2 are bases of $\underline{M}$ with $|B_1 \triangle B_2| = 4$, then we have $\gamma(B_1, B_2) = \gamma(B_2, B_1)^{-1}$.* □

Now we define a candidate for a Wick function on E with coefficients in F whose underlying matroid is exactly $\underline{M}$. Fix a base B_0 of $\underline{M}$, and let $\varphi_{\mathcal{C}} : \mathcal{T}_n \to F$ be such that:

(i) $\varphi_{\mathcal{C}}(B_0) = 1$ ($\notin N_F$).

(ii) For each base B of $\underline{M}$ other than B_0, we set

$$\varphi_{\mathcal{C}}(B) := \gamma(B', B)\varphi_{\mathcal{C}}(B'),$$

where B' is a base of $\underline{M}$ such that $|B \setminus B'| = 2$ and $|B \setminus B_0| = |B' \setminus B_0| + 2$.

(iii) For each non-base transversal T, we set $\varphi(T) = 0$.

We show the well-definedness of $\varphi_{\mathcal{C}}$ using Maurer's Homotopy Theorem [118]. We recall the statement below. The *base graph* $G_{\underline{M}}$ of an $\underline{M}$ is a graph whose vertex set is $\mathcal{B}(\underline{M})$ and two vertices B_1, B_2 are adjacent if $|B_1 \setminus B_2| = 2$.

Theorem 4.1.36. *Let $\underline{M}$ be an orthogonal matroid. Then the homology group of $G_{\underline{M}}$ is generated by the cycles of length at most four.*

Lemma 5.2.7. *The following hold for the base graph $G_{\underline{M}}$ of an orthogonal matroid $\underline{M}$ with an F-signature $\mathcal{C}$ satisfying (O_2).*

(i) *If B_1, B_2, B_3, B_1 is a directed cycle of length 3 in $G_{\underline{M}}$, then*

$$\gamma(B_1, B_2)\gamma(B_2, B_3)\gamma(B_3, B_1) = 1.$$

(ii) *If B_1, B_2, B_3, B_4, B_1 is a directed cycle of length 4 in $G_{\underline{M}}$, then*

$$\gamma(B_1, B_2)\gamma(B_2, B_3)\gamma(B_3, B_4)\gamma(B_4, B_1) = 1.$$

Proof. (i) If B_1, B_2, B_3 are bases of $\underline{M}$ with $|B_i \setminus B_j| = 2$ for all distinct $i, j \in [3]$, then there exist a transversal T and distinct elements $e_1^*, e_2^*, e_3^* \in T$ such that $B_i = T \triangle \{e_i, e_i^*\} \ni e_i$ for each $i \in [3]$. For distinct $i, j \in [3]$, let $T_{ij} := T \triangle \{e_i, e_i^*\} \triangle \{e_j, e_j^*\}$ and $X_{ij} \in \mathcal{C}$ be the F-circuit with $\underline{X_{ij}} = C(B_i, e_j) = C(B_j, e_i) \subseteq T_{ij}$. Then

$$\gamma(B_i, B_j) = (-1)^{m_{ij}} \frac{\tilde{X}_{ij}(e_i)}{\tilde{X}_{ij}(e_j)},$$

where $m_{ij} := |T_{ij} \cap (\overline{e_i}, \overline{e_j}]|$. Let Y be the F-circuit with $\underline{Y} = C(B_1, e_1^*) \subseteq T$. Then $\underline{Y} = C(B_2, e_2^*) = C(B_3, e_3^*)$ and thus $\{e_1^*, e_2^*, e_3^*\} \subseteq \underline{Y}$. By ($O_2$), if $i \neq j \in [3]$, we have that $\tilde{X}_{ij}(e_i)\tilde{Y}(e_i^*) + \tilde{X}_{ij}(e_j)\tilde{Y}(e_j^*) = \langle X_{ij}, Y^* \rangle \in N_F$. Thus,

$$\frac{\tilde{X}_{12}(e_1)}{\tilde{X}_{12}(e_2)} \frac{\tilde{X}_{23}(e_2)}{\tilde{X}_{23}(e_3)} \frac{\tilde{X}_{31}(e_3)}{\tilde{X}_{31}(e_1)} = \left(-\frac{\tilde{Y}(e_2^*)}{\tilde{Y}(e_1^*)}\right)\left(-\frac{\tilde{Y}(e_3^*)}{\tilde{Y}(e_2^*)}\right)\left(-\frac{\tilde{Y}(e_1^*)}{\tilde{Y}(e_3^*)}\right) = -1.$$

By relabelling, we may assume that $\overline{e_1} < \overline{e_2} < \overline{e_3}$. Then $(T_{12} \cap (\overline{e_1}, \overline{e_2}]) \triangle (T_{23} \cap (\overline{e_2}, \overline{e_3}]) \triangle (T_{13} \cap (\overline{e_1}, \overline{e_3}]) = \{\overline{e_2}\}$. Hence $m_{12} + m_{23} + m_{13}$ is odd and therefore $\gamma(B_1, B_2)\gamma(B_2, B_3)\gamma(B_3, B_1) = 1$.

(ii) By (i) and Lemma 5.2.6, we may assume that the directed cycle B_1, B_2, B_3, B_4, B_1 is not generated by directed cycles of length 3. Then $|B_i \setminus B_{i+1}| = 2$ and $|B_i \setminus B_{i+2}| = 4$ for all $i \in [4]$, where all subscripts are read modulo 4. Thus, there exist a transversal T and distinct elements $e_1, e_2, e_3, e_4 \in T$ such that $B_1 = T_{12}$, $B_2 = T_{13}$, $B_3 = T_{34}$, and $B_4 = T_{24}$, where $T_I = T \triangle \bigcup_{i\in I}\{e_i, e_i^*\}$ for all $I \subseteq [4]$. In addition, none of T, T_{14}, T_{23}, and T_{1234} is a base.

Let $X_1, X_3, Y_3, Y_1 \in \mathcal{C}$ be F-circuits such that $\underline{X_i} \subseteq T_i$ and $\underline{Y_i} \subseteq T_{2j4}$ for each $\{i, j\} = \{1, 3\}$. Then

$$\begin{aligned}
\gamma(T_{12}, T_{13}) &= (-1)^{m_1} \frac{\tilde{X}_1(e_3)}{\tilde{X}_1(e_2)},\\
\gamma(T_{13}, T_{34}) &= (-1)^{m_3} \frac{\tilde{X}_3(e_4)}{\tilde{X}_3(e_1)},\\
\gamma(T_{12}, T_{24}) &= (-1)^{n_3} \frac{\tilde{Y}_3(e_1^*)}{\tilde{Y}_3(e_4^*)},\\
\gamma(T_{24}, T_{34}) &= (-1)^{n_1} \frac{\tilde{Y}_1(e_2^*)}{\tilde{Y}_1(e_3^*)},
\end{aligned}$$

where $m_1 := |T_1 \cap (\overline{e_2}, \overline{e_3}]|$, $m_3 := |T_3 \cap (\overline{e_1}, \overline{e_4}]|$, $n_3 := |T_{124} \cap (\overline{e_1}, \overline{e_4}]|$, and $n_1 := |T_{234} \cap (\overline{e_2}, \overline{e_3}]|$. Note that $m_1 + m_3 + n_3 + n_1$ is even, since $(T_1 \cap (\overline{e_2}, \overline{e_3}]) \triangle (T_{234} \cap (\overline{e_2}, \overline{e_3}]) = \{\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\} \cap (\overline{e_2}, \overline{e_3}]$, and $(T_3 \cap (\overline{e_1}, \overline{e_4}]) \triangle (T_{124} \cap (\overline{e_1}, \overline{e_4}]) = \{\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\} \cap (\overline{e_1}, \overline{e_4}]$.

Suppose for contradiction that $X_1(e_1^*) \neq 0$, i.e., $e_1^* \in \underline{X_1}$. Notice that $\underline{X_1} = C(B_1, e_2)$ is the unique circuit of $\underline{M}$ contained in T_1, and the subtransversal $T_1 \setminus \{e_1^*\}$ is independent. Since T_1 is not a base, $T = (T_1 \setminus \{e_1^*\}) \cup \{e_1\}$ is a base, a contradiction. Thus, $X_1(e_1^*) = 0$. Similarly, one can check that all of $X_1(e_4)$, $X_3(e_2)$, $X_3(e_3^*)$, $Y_3(e_2^*)$, $Y_3(e_3)$, $Y_1(e_1)$, and $Y_1(e_4^*)$ are zero, because none of T, T_{14}, T_{23}, and T_{1234} is a base of $\underline{M}$. Therefore, by (O$_2$), we have

$$\tilde{X}_1(e_2)\tilde{Y}_1(e_2^*) + \tilde{X}_1(e_3)\tilde{Y}_1(e_3^*) = \langle X_1, Y_1^* \rangle \in N_F,$$

and

$$\tilde{X}_3(e_1)\tilde{X}_3(e_1^*) + \tilde{X}_3(e_4)\tilde{Y}_3(e_4^*) = \langle X_3, Y_3^* \rangle \in N_F.$$

Therefore,

$$\gamma(T_{12}, T_{13})\gamma(T_{13}, T_{34}) = (-1)^{m_1+m_3} \frac{\tilde{X}_1(e_3)}{\tilde{X}_1(e_2)} \frac{\tilde{X}_3(e_4)}{\tilde{X}_3(e_1)} = (-1)^{n_1+n_3} \frac{\tilde{Y}_1(e_2^*)}{\tilde{Y}_1(e_3^*)} \frac{\tilde{Y}_3(e_1^*)}{\tilde{Y}_3(e_4^*)} = \gamma(T_{24}, T_{34})\gamma(T_{12}, T_{24}).$$

By Lemma 5.2.6, we obtain that

$$\begin{aligned}
&\gamma(B_1, B_2)\gamma(B_2, B_3)\gamma(B_3, B_4)\gamma(B_4, B_1)\\
&= \gamma(T_{12}, T_{13})\gamma(T_{13}, T_{34})\gamma(T_{24}, T_{34})^{-1}\gamma(T_{12}, T_{24})^{-1} = 1.
\end{aligned}$$ □

Corollary 5.2.8. $\varphi_{\mathcal{C}}$ *is well-defined.*

Proof. It suffices to show that for arbitrary paths $P = B_0 B_1 \dots B_k$ and $P' = B'_0 B'_1 \dots B'_\ell$ in $G_{\underline{M}}$, if $B_0 = B'_0$ and $B_k = B'_\ell$, then

$$\prod_{i=0}^{k-1} \gamma(B_i, B_{i+1}) = \prod_{j=0}^{\ell-1} \gamma(B_j, B_{j+1}).$$

This is straightforward from Lemmas 5.2.6, 5.2.7, and Theorem 4.1.36. □

Theorem 5.2.9. *If $\mathcal{C}$ satisfies the orthogonality (O), then $\varphi_{\mathcal{C}}$ is a strong Wick function on E with coefficients in F.*

Proof. We only need to prove (W2). Take $T_1, T_2 \in \mathcal{T}_n$ with $T_1 \cap T_2^* = \{e_1, \dots, e_a\}$, where $\overline{e_1} < \dots < \overline{e_a}$. If T_1 is a base of $\underline{M}$, then $\varphi(T_1 \triangle \{i, i^*\}) = 0$ for all $i \in [n]$ and thus $\sum_{i=1}^{a}(-1)^i \varphi(T_1 \triangle \{e_i, e_i^*\})\varphi(T_2 \triangle \{e_i, e_i^*\}) \in N_F$. Therefore, we may assume that T_1 is not a base, and similarly we may assume that T_2 is not a base. Then there exist $X_1, X_2 \in \mathcal{C}$ such that $\underline{X_i} \subseteq T_i$ for $i = 1, 2$. Write $\underline{X_1} \cap \underline{X_2}^* = \{e_{\alpha_1}, \dots, e_{\alpha_b}\}$ with $\alpha_1 < \dots < \alpha_b$. For $i \in [a] \setminus \{\alpha_1, \dots, \alpha_b\}$, at least one of $T_1 \triangle \{e_i, e_i^*\}$ and $T_2 \triangle \{e_i, e_i^*\}$ is not a base. Hence

$$\sum_{i=1}^{a}(-1)^i \varphi(T_1 \triangle \{e_i, e_i^*\})\varphi(T_2 \triangle \{e_i, e_i^*\}) = \sum_{i=1}^{b}(-1)^{\alpha_i} \varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})\varphi(T_2 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}).$$

Therefore, we may assume that $b \geq 1$. We can also assume that there exists $c \in [b]$ such that both $B_1 := T_1 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\}$ and $B_2 := T_2 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\}$ are bases. Then $\underline{X_1} = C(T_1 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\}, e_{\alpha_c})$ and $\underline{X_2} = C(T_2 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\}, e_{\alpha_c}^*)$, and therefore $T_j \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}$ is a base for each $i \in [b]$ and $j = 1, 2$.

For each $i \in [b]$, let $m_i := |T_1 \cap (\overline{e_{\alpha_c}}, \overline{e_{\alpha_i}}]|$ and $n_i := |T_2 \cap (\overline{e_{\alpha_c}}, \overline{e_{\alpha_i}}]|$. By the definition of $\varphi_{\mathcal{C}}$, we have

$$\frac{\tilde{X}_1(e_{\alpha_i})}{\tilde{X}_1(e_{\alpha_c})} = (-1)^{m_i} \frac{\varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})}{\varphi(T_1 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\})} \quad \text{and} \quad \frac{\tilde{X}_2(e_{\alpha_i}^*)}{\tilde{X}_2(e_{\alpha_c}^*)} = (-1)^{n_i} \frac{\varphi(T_2 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})}{\varphi(T_2 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\})}.$$

Since $(T_1 \triangle T_2) \cap (\overline{e_{\alpha_c}}, \overline{e_{\alpha_i}}]$ equals $\{\overline{e_k} : \alpha_c < k \leq \alpha_i\}$ if $c < i$ and $\{\overline{e_k} : \alpha_i < k \leq \alpha_c\}$ otherwise, we have $m_i + n_i \equiv \alpha_i - \alpha_c \pmod 2$. By the orthogonality relation (O), we have

$$\sum_{i=1}^{b} \tilde{X}_1(e_{\alpha_i})\tilde{X}_2(e_{\alpha_i}^*) = \langle X_1, X_2^* \rangle \in N_F.$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{b}(-1)^{\alpha_i} \varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})\varphi(T_2 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}) \\
&= (-1)^{\alpha_c} \sum_{i=1}^{b}(-1)^{m_i+n_i} \varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\})\varphi(T_1 \triangle \{e_{\alpha_i}, e_{\alpha_i}^*\}) \\
&= (-1)^{\alpha_c} \frac{\varphi(T_1 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\})}{\tilde{X}_1(e_{\alpha_c})} \frac{\varphi(T_2 \triangle \{e_{\alpha_c}, e_{\alpha_c}^*\})}{\tilde{X}_2(e_{\alpha_c}^*)} \sum_{i=1}^{b} \tilde{X}_1(e_{\alpha_i})\tilde{X}_2(e_{\alpha_i}^*) \in N_F.
\end{aligned}$$

□

Theorem 5.2.10. *If $\mathcal{C}$ satisfies (O_4), then $\varphi_{\mathcal{C}}$ is a moderately weak Wick function on E with coefficients in F.*

Proof. It yields if we replace (O) with (O_4) in the proof of Theorem 5.2.9. □

5.2.3 Weak Wick functions and weak circuit sets

In this section, we prove the equivalence between the weak Wick functions and the weak circuit sets, using the constructions in Sections 5.2.1 and 5.2.2.

Theorem 5.2.11. *Let $\mathcal{C}$ be a weak F-circuit set of an orthogonal matroid. Then $\varphi_{\mathcal{C}}$ is a weak Wick F-function.*

Proof. Denote $\varphi := \varphi_{\mathcal{C}}$. Let T_1 be a transversal, and let $e_1, e_2, e_3, e_4 \in T_1$ be such that $\overline{e_1} < \overline{e_2} < \overline{e_3} < \overline{e_4}$. Let T_2 be another transversal such that $T_2 \setminus T_1 = \{e_1^*, e_2^*, e_3^*, e_4^*\}$. We may assume that neither T_1 nor T_2 is a base, and there is $k \in [4]$ such that both $T_1 \triangle \{e_k, e_k^*\}$ and $T_2 \triangle \{e_k, e_k^*\}$ are bases of $\underline{M}$. Let X and Y be F-circuits in $\mathcal{C}$ such that $\underline{X} \subseteq T_1$ and $\underline{Y} \subseteq T_2$. Then for each $i \in [4]$, we have

$$\frac{\varphi(T_1 \triangle \{e_i, e_i^*\})}{\varphi(T_1 \triangle \{e_k, e_k^*\})} = (-1)^{m_i} \frac{\tilde{X}(e_i)}{\tilde{X}(e_k)} \quad \text{and} \quad \frac{\varphi(T_2 \triangle \{e_i, e_i^*\})}{\varphi(T_2 \triangle \{e_k, e_k^*\})} = (-1)^{n_i} \frac{\tilde{Y}(e_i^*)}{\tilde{Y}(e_k^*)},$$

where $m_i := m_{e_i, e_k}^{T_1} = |T_1 \cap (\overline{e_i}, \overline{e_k}]|$ and $n_i := m_{e_i, e_k}^{T_2} = |T_2 \cap (\overline{e_i}, \overline{e_k}]|$. Note that $m_i + n_i \equiv k - i \pmod 2$. Hence, we have

$$\sum_{i \in [4]} (-1)^{i+k} \frac{\varphi(T_1 \triangle \{e_i, e_i^*\}) \varphi(T_2 \triangle \{e_i, e_i^*\})}{\varphi(T_1 \triangle \{e_k, e_k^*\}) \varphi(T_2 \triangle \{e_k, e_k^*\})} = 1 + \sum_{i \in [4] \setminus \{k\}} \frac{\tilde{X}(e_i) \tilde{Y}(e_i^*)}{\tilde{X}(e_k) \tilde{Y}(e_k^*)}. \tag{$*$}$$

By the strong symmetric exchange axiom, at least one of $\theta_i := \varphi(T_1 \triangle \{e_i, e_i^*\}) \varphi(T_2 \triangle \{e_i, e_i^*\})$ with $i \in [4] \setminus \{k\}$ is nonzero. If exactly one of θ_i is nonzero, then ($*$) is $1 + \frac{\tilde{X}(e_i)\tilde{Y}(e_i^*)}{\tilde{X}(e_k)\tilde{Y}(e_k^*)} \in N_F$ by (O$_2$). Therefore we may assume that at least two of θ_i are nonzero. We denote by a, b, c the distinct elements of $[4] \setminus \{k\}$.

Suppose that θ_a and θ_b are nonzero but $\theta_c = 0$. Then $\{e_a, e_b, e_k\} \subseteq \underline{X}$ and $\{e_a^*, e_b^*, e_k^*\} \subseteq \underline{Y}$. By interchanging roles of T_1 and T_2 if necessary, we may assume that $T_2 \triangle \{e_c, e_c^*\}$ is not a base of $\underline{M}$. Then $e_c^* \notin \underline{Y}$. Because $\{e_a, e_b, e_k\} \subseteq \underline{X} \subseteq T_1$, neither $T_1 \triangle \{e_a, e_a^*, e_k, e_k^*\}$ nor $T_1 \triangle \{e_b, e_b^*, e_k, e_k^*\}$ is a base. Hence $\mathcal{C}$ has F-circuits Z_a and Z_b such that $\underline{Z_i} \subseteq T_1 \triangle \{e_i, e_i^*, e_k, e_k^*\}$ for each $i \in \{a, b\}$. Because $T_1 \triangle \{e_a, e_a^*\}$, $T_1 \triangle \{e_k, e_k^*\}$, and $T_1 \triangle \{e_b, e_b^*\}$ are bases, we have $\{e_a^*, e_c, e_k^*\} \subseteq \underline{Z_a}$. Because $T_2 \triangle \{e_c, e_c^*\}$ is not a base, $e_b \notin \underline{Z_a}$. Similarly, $\{e_b^*, e_c, e_k^*\} \subseteq \underline{Z_b}$ and $e_a \notin \underline{Z_b}$. Then $\underline{Z_a} \cup \underline{Z_b}$ is a transversal, and by the circuit elimination axiom ($\triangle$Elim), $\underline{M}$ has a circuit C contained in $(\underline{Z_a} \cup \underline{Z_b}) \setminus \{e_c\} \subseteq T_2 \setminus \{e_c\}$. Then $\underline{Y} = C$ and hence $\tilde{Y}$ is in the linear span of $\tilde{Z}_a$ and $\tilde{Z}_b$ by (L$'$-i). Rescaling Z_a and Z_b if necessary, we may assume that $Z_a(e_a^*) = Y(e_a^*)$ and $Z_b(e_b^*) = Y(e_b^*)$. Then $\tilde{Y}(e_k^*) - \tilde{Z}_a(e_k^*) - \tilde{Z}_b(e_k^*) \in N_F$. By (O$_2$), $\frac{\tilde{Z}_i(e_k^*)}{\tilde{Z}_i(e_i^*)} = -\frac{\tilde{X}(e_i)}{\tilde{X}(e_k)}$ for $i \in \{a, b\}$ and thus ($*$) is equal to $1 - \frac{\tilde{Z}_a(e_k^*)}{\tilde{Y}(e_k^*)} - \frac{\tilde{Z}_b(e_k^*)}{\tilde{Y}(e_k^*)} \in N_F$.

Now we consider the case where θ_a, θ_b, θ_c are all nonzero. Then there are F-circuits Z_a, Z_b, Z_c in $\mathcal{C}$ such that $\underline{Z_i} \subseteq T_1 \triangle \{e_i, e_i^*, e_k, e_k^*\}$ with $i \in \{a, b, c\}$. It can be easily checked that $\{e_a, e_b, e_c, e_k^*\} \triangle \{e_i, e_i^*\} \subseteq \underline{Z_i}$ for every $i \in \{a, b, c\}$. Then by (L$'$-ii), $\tilde{Y}$ is in the linear span of $\tilde{Z}_1$, $\tilde{Z}_2$, and $\tilde{Z}_3$. Rescaling Z_i if necessary, we may assume that $Z_i(e_i^*) = Y(e_i^*)$ for each $i \in \{a, b, c\}$. Then $\tilde{Y}(e_k^*) - \tilde{Z}_a(e_k^*) - \tilde{Z}_b(e_k^*) - \tilde{Z}_c(e_k^*) \in N_F$. By (O$_2$), $\frac{\tilde{Z}_i(e_k^*)}{\tilde{Z}_i(e_i^*)} = -\frac{\tilde{X}(e_i)}{\tilde{X}(e_k)}$ with $i = a, b, c$ and therefore ($*$) is equal to $1 - \frac{\tilde{Z}_a(e_k^*)}{\tilde{Y}(e_k^*)} - \frac{\tilde{Z}_b(e_k^*)}{\tilde{Y}(e_k^*)} - \frac{\tilde{Z}_c(e_k^*)}{\tilde{Y}(e_k^*)} \in N_F$. □

To prove the converse of Theorem 5.2.11, we consider the following weaker replacement of orthogonality (O$_2$):

(O$'_2$) Let $X, Y \in \mathcal{C}$ be such that $\underline{X}$ and $\underline{Y}$ are fundamental circuits with respect to the same base of $\underline{M}_{\mathcal{C}}$, then $\langle X, Y^* \rangle \in N_F$.

Lemma 5.2.12. *Let $\mathcal{C}$ be an F-signature of an orthogonal matroid. If $\mathcal{C}$ satisfies* (O$'_2$) *and* (L$'$-i), *then it satisfies* (O$_2$).

Proof. Suppose for contradiction that (O$_2$) does not hold. Let X and Y be F-circuits in $\mathcal{C}$ such that $|\underline{X} \cup \underline{Y}|$ is minimized subject to $|\underline{X} \cap \underline{Y}^*| = 2$ and $\langle X, Y^* \rangle \notin N_F$. Write $\underline{X} \cap \underline{Y}^* = \{e, f\}$. Then $J := (\underline{X} \cup \underline{Y}) \setminus \{e^*, f\}$ is dependent in $\underline{M}_{\mathcal{C}}$, because otherwise there is a base $B \supseteq J$ such that X and Y are fundamental circuits with respect to B and thus $\langle X, Y^* \rangle \in N_F$ by (O$'_2$), a contradiction. Let C be a circuit contained in J which minimizes $|\underline{X} \cup C|$. Note that $C \cap \{e, f^*\} = \emptyset$ by ($\triangle$Orth), and there are $x \in C \cap (\underline{X} \setminus \underline{Y})$ and $y \in C \cap (\underline{Y} \setminus \underline{X})$ by (C2). Because of the minimality of $|\underline{X} \cup C|$, we deduce that $J_2 := (\underline{X}\triangle\{f, f^*\}) \cup (C \setminus \{y\})$ is independent. Let B_2 be a base containing J_2. Then $\underline{X}$ and C are fundamental circuits with respect to B_2. Let Z be an F-circuit whose support is C. By (L′-i), there is an F-circuit X_2 such that $X_2(x) = 0$ and $\tilde{X}_2$ is in the linear span of $\tilde{X}$ and $\tilde{Z}$. Then $\underline{X_2} \cup \underline{Y} \subsetneq \underline{X} \cup \underline{Y}$, and for some $\alpha \in F^\times$, we have $X_2(e) = \alpha X(e)$ and $X_2(f^*) = \alpha X(f^*)$. Therefore, $\alpha\langle X, Y^*\rangle = \langle X_2, Y^*\rangle \in N_F$, a contradiction. □

We note that by Lemma 5.2.12, an F-signature of an orthogonal matroid is a strong F-circuit set if and only if it satisfies (L) and (O_2). In addition, (O_2) in Lemma 5.1.7 can be replaced by (O_2').

Theorem 5.2.13. *Let φ be a weak Wick function. Then $\mathcal{C}_\varphi$ is a weak F-circuit set of $\underline{M}_\varphi$.*

Proof. By Lemma 5.2.12, it suffices to show that $\mathcal{C}_\varphi$ satisfies (O_2'), (L′-i), and (L′-ii).

Let X and Y be F-circuits in $\mathcal{C}_\varphi$ such that $\underline{X} = C(B, f)$ and $\underline{Y} = C(B, e)$ for some base B and distinct elements $e, f \in B^*$. We denote $T_1 := B\triangle\{f, f^*\} \supseteq \underline{X}$ and $T_2 := B\triangle\{e, e^*\} \supseteq \underline{Y}$. Then

$$\frac{\tilde{X}(e)}{\tilde{X}(f)} = (-1)^{m^{T_1}_{e,f}} \frac{\varphi(T_1\triangle\{e, e^*\})}{\varphi(T_1\triangle\{f, f^*\})} = (-1)^{m^{T_2}_{e,f}} \frac{\varphi(T_2\triangle\{f, f^*\})}{\varphi(T_2\triangle\{e, e^*\})} = -\frac{\tilde{Y}(f^*)}{\tilde{Y}(e^*)},$$

and hence $\langle X, Y^*\rangle \in N_F$. Therefore, $\mathcal{C}_\varphi$ satisfies (O_2').

Now we show that (L′-i) holds. Let B be a base of $\underline{M}_\varphi$ and $e_1, e_2 \in B^*$ be distinct elements. Let X_1 and X_2 be F-circuits in $\mathcal{C}$ such that $\underline{X_i} = C(B, e_i)$ for $i = 1, 2$. Suppose that $X_1(e_2^*) = X_2(e_1^*) = 0$ and there is an element $f \in \underline{X_1} \cap \underline{X_2}$. Let Y be an F-circuit whose support is a subset of $(\underline{X_1} \cup \underline{X_2}) \setminus \{f\}$. We claim that $\tilde{Y}$ belongs to the linear span of $\tilde{X}_1$ and $\tilde{X}_2$. We may assume that $\tilde{X}_i(e_i) = \tilde{Y}(e_i)$. Thus it suffices to show that $\tilde{Y}(g) - \tilde{X}_1(g) - \tilde{X}_2(g) \in N_F$ for all $g \in (\underline{X_1} \cup \underline{X_2}) \setminus \{e_1, e_2\}$.

Let $Z \in \mathcal{C}$ be such that $\underline{Z} = C(B, f^*)$. By ($O_2'$), $\tilde{Z}(e_i^*)\tilde{X}_i(e_i) + \tilde{Z}(f^*)\tilde{X}_i(f) \in N_F$ with $i = 1, 2$. Again by (O_2'), $-Z(f^*)(X_1(f) + X_2(f)) = \tilde{Z}(e_1^*)\tilde{Y}(e_1) + \tilde{Z}(e_2^*)\tilde{Y}(e_2) \in N_F$. Hence $X_1(f) + X_2(f) \in N_F$. So we may assume that $g \neq f$. By symmetry, we may assume that $g \in \underline{X_1}$, implying that $B\triangle\{e_1, e_1^*, g, g^*\}$ is a base. Let $W \in \mathcal{C}$ be such that $\underline{W} = C(B, g^*)$. Let $T_1 := B\triangle\{g, g^*\}$ and $T_2 := B\triangle\{e_1, e_1^*, e_2, e_2^*, f, f^*\}$. Then $\underline{W} \subseteq T_1$ and $\underline{Y} \subseteq T_2$. Since $B\triangle\{e_2, e_2^*, f, f^*\}$ and $B\triangle\{e_1, e_1^*, g, g^*\}$ are bases of $\underline{M}_\varphi$, both $Y(e_1)$ and $W(e_1^*)$ are nonzero. We rewrite $\{e_1, e_2, f^*, g\} \subseteq T_2$ by $\{x_1, x_2, x_3, x_4\}$ with $\overline{x_1} < \overline{x_2} < \overline{x_3} < \overline{x_4}$, and let $k \in [4]$ be such that $x_k = e_1$. Since $m^{T_1}_{x_i,x_k} + m^{T_2}_{x_i,x_k} \equiv k - i \pmod 2$, by (W2″), we have that

$$\begin{aligned}\sum_{i=1}^{4} \tilde{W}(x_i^*)\tilde{Y}(x_i) &= \tilde{W}(x_k^*)\tilde{Y}(x_k) \sum_{i=1}^{4} \frac{\tilde{W}(x_i^*)\tilde{Y}(x_i)}{\tilde{W}(x_k^*)\tilde{Y}(x_k)} \\ &= \tilde{W}(x_k^*)\tilde{Y}(x_k) \sum_{i=1}^{4} (-1)^{k-i} \frac{\varphi(T_1\triangle\{x_i, x_i^*\})\varphi(T_2\triangle\{x_i, x_i^*\})}{\varphi(T_1\triangle\{x_k, x_k^*\})\varphi(T_2\triangle\{x_k, x_k^*\})} \in N_F.\end{aligned}$$

By (O_2'), $\tilde{W}(e_i^*)\tilde{Y}(e_i) = \tilde{W}(e_i^*)\tilde{Z}_i(e_i) = -\tilde{W}(g^*)\tilde{Z}_i(g)$ for each i. Since $W(g^*) \neq 0$ and $Y(f^*) = 0$, we conclude that $Y(g) - Z_1(g) - Z_2(g) \in N_F$. Therefore (L′-i) holds.

Finally, we show that (L′-ii) holds. Let B be a base of $\underline{M}_\varphi$ and let X_1, X_2, X_3 be F-circuits in $\mathcal{C}$ such that their supports are $C(B, e_1)$, $C(B, e_2)$, $C(B, e_3)$ for some distinct $e_1, e_2, e_3 \in B^*$, and $X_i(e_j^*) \neq 0$ for all $i \neq j$. Then $B\triangle\{e_1, e_1^*, e_2, e_2^*\}$, $B\triangle\{e_1, e_1^*, e_3, e_3^*\}$, and $B\triangle\{e_2, e_2^*, e_3, e_3^*\}$ are all bases. Let Y be an F-circuit in $\mathcal{C}$ whose support is $C(B\triangle\{e_1, e_1^*, e_2, e_2^*\}, e_3)$. Then $\{e_1, e_2, e_3\} \subseteq \underline{Y} \subseteq B\triangle\{e_1, e_1^*, e_2, e_2^*, e_3, e_3^*\}$. We claim that $\tilde{Y}$ belongs to the linear span of $\tilde{X}_i$ with $i = 1, 2, 3$. We may assume that $X_i(e_i) = Y(e_i)$ for each i. Hence it suffices to show that $\tilde{Y}(f) - \tilde{X}_1(f) - \tilde{X}_2(f) - \tilde{X}_3(f) \in N_F$ for all $f \in B$. Denote $\alpha := \tilde{X}_1(e_2^*)\tilde{Y}(e_2) \in F^\times$. Then $\tilde{X}_1(e_3^*)\tilde{Y}(e_3) = -\alpha$ by (O_2') applied to X_1 and Y. Applying again (O_2') to X_1 and X_2, we have $\tilde{X}_2(e_1^*)\tilde{Y}(e_1) = -\alpha$. Similarly, we deduce that $\tilde{X}_2(e_3^*)\tilde{Y}(e_3) = \tilde{X}_3(e_1^*)\tilde{Y}(e_1) =$

$-\tilde{X}_3(e_2^*)\tilde{Y}(e_2) = \alpha$. Then $X_{i+1}(e_i^*) + X_{i+2}(e_i^*) \in N_F$ for each i, where the subscripts are read modulo 3. Thus we may assume that $f \neq e_1^*, e_2^*, e_3^*$.

Let $T_1 := B\triangle\{f, f^*\}$ and $T_2 := B\triangle\{e_1, e_1^*, e_2, e_2^*, e_3, e_3^*\} \supseteq \underline{Y}$. Let Z be an F-circuit in $\mathcal{C}$ such that $\underline{Z} = C(B, f^*) \subseteq T_1$. Note that $X_i(f) \neq 0$ if and only if $T_1\triangle\{e_i, e_i^*\}$ is a base. Hence if $X_i(f) = 0$ for all i, then by (W2″), $\varphi(T_2\triangle\{f, f^*\}) = 0$ so $Y(f) = 0$. Therefore we may assume that at least one of $X_i(f)$ is nonzero. By relabelling, we may assume that $X_1(f) \neq 0$ and hence $T_1\triangle\{e_1, e_1^*\}$ is a base. Then $Z(e_1) \neq 0$.

We rewrite $\{e_1, e_2, e_3, f^*\} \subseteq T_2$ by $\{x_1, x_2, x_3, x_4\}$ with $\overline{x_1} < \overline{x_2} < \overline{x_3} < \overline{x_4}$, and let $k \in [4]$ be such that $x_k = e_1$. Note that $m^{T_1}_{x_i,x_k} + m^{T_2}_{x_i,x_k} \equiv k - i \pmod 2$. Then by (W2″),

$$\sum_{i=1}^{4} \tilde{Z}(x_i^*)\tilde{Y}(x_i) = \tilde{Z}(x_k^*)\tilde{Y}(x_k) \sum_{i=1}^{4} (-1)^{k-i} \frac{\varphi(T_1\triangle\{x_i, x_i^*\})\varphi(T_2\triangle\{x_i, x_i^*\})}{\varphi(T_1\triangle\{x_k, x_k^*\})\varphi(T_2\triangle\{x_k, x_k^*\})} \in N_F.$$

By (O$_2'$), $\tilde{Z}(e_i^*)\tilde{Y}(e_i) = \tilde{Z}(e_i^*)\tilde{X}_i(e_i) = -\tilde{Z}(f^*)\tilde{X}_i(f)$ for each i. Because $f^* \in \underline{Z}$, we deduce that $\tilde{Y}(f) - \sum_{i=1}^{3} \tilde{X}_i(f) = \tilde{Y}(f) + \tilde{Z}(f^*)^{-1} \sum_{i=1}^{3} \tilde{Z}(e_i^*)\tilde{Y}(e_i) \in N_F$. □

5.2.4 Strong orthogonal signatures and strong circuit sets

Let $\mathcal{C}$ be an F-signature of an orthogonal matroid $\underline{M}$ on E satisfying (O$_2$). We say that $X \in F^E$ is *consistent with* $\mathcal{C}$ if for each base B of $\underline{M}$, the vector $\tilde{X}$ belongs to the linear span of $\{\tilde{X}_e : e \in B^*\}$, where X_e is the unique F-circuit in $\mathcal{C}$ such that $\underline{X_e} = C(B, e)$ and $X_e(e) = 1$. Hence (L) is equivalent to that every F-circuit in $\mathcal{C}$ is consistent with $\mathcal{C}$.

The *orthogonal complement* of $\mathcal{W} \subseteq F^E$ is $\mathcal{W}^\perp := \{X \in F^E : \langle X, Y^*\rangle \in N_F \text{ for all } Y \in \mathcal{W}\}$. Therefore, the orthogonality (O) is equivalent to that $\mathcal{C} \subseteq \mathcal{C}^\perp$.

Lemma 5.2.14. *Let $\mathcal{C}$ be an F-signature of an orthogonal matroid on E satisfying (O$_2$). If $X \in F^E$ is consistent with $\mathcal{C}$, then $X \in \mathcal{C}^\perp$.*

Proof. We claim that $\langle X, Y^*\rangle \in N_F$ for all $Y \in \mathcal{C}$. We may assume that $\underline{X} \cap \underline{Y}^* \neq \emptyset$. Write $\underline{X} \cap \underline{Y}^* = \{e_0^*, e_1, \ldots, e_\ell\}$, and let B be a base of the underlying orthogonal matroid $\underline{M}_{\mathcal{C}}$ such that $\underline{Y}\triangle\{e_0, e_0^*\} \subseteq B$. Then $\{e_0^*, \ldots, e_\ell^*\} \subseteq B$. We denote by $m := |\underline{X} \cap B^*|$, and if $\underline{X} \cap (B \setminus \underline{Y})^*$ is nonempty, then we enumerate its elements as $e_{\ell+1}, e_{\ell+2}, \ldots, e_m$. Then $\underline{X} \cap B^* = \{e_1, \ldots, e_m\}$. For $0 \leq i \leq m$, let X_i be the F-circuit in $\mathcal{C}$ such that $\underline{X_i} = C(B, e_i)$ and $X_i(e_i) = 1$. Then $\tilde{X} - \sum_{i=1}^{m} \tilde{X}(e_i)\tilde{X}_i \in (N_F)^E$ since X is consistent with $\mathcal{C}$. Note that $\underline{Y} = C(B, e_0) = \underline{X_0}$. By multiplying Y with $Y(e_0)^{-1} \in F^\times$, we can assume that $Y(e_0) = 1$. For each $1 \leq i \leq m$, $\tilde{X}_0(e_i^*) + \tilde{X}_i(e_0^*) = \langle X_0, X_i^*\rangle \in N_F$ by (O$_2$) and so $\tilde{Y}(e_i^*) = -\tilde{X}_i(e_0^*)$. Therefore,

$$\langle X, Y^*\rangle = \tilde{X}(e_0^*) + \sum_{i=1}^{m} \tilde{X}(e_i)\tilde{Y}(e_i^*) = \tilde{X}(e_0^*) - \sum_{i=1}^{m} \tilde{X}(e_i)\tilde{X}_i(e_0^*) \in N_F.$$ □

Lemma 5.2.15. *Let $\mathcal{C}$ be an orthogonal F-signature of an orthogonal matroid on E. If $X \in \mathcal{C}^\perp$, then X is consistent with $\mathcal{C}$.*

Proof. Let B be a base of $\underline{M}_{\mathcal{C}}$. Write $\underline{X} \cap B^* = \{e_1, \ldots, e_m\}$, and let X_i be the F-circuit in $\mathcal{C}$ such that $\underline{X_i} = C(B, e_i)$ and $X_i(e_i) = 1$. We claim that $\tilde{X}(f) - \sum_i \tilde{X}(e_i)\tilde{X}_i(f) \in N_F$ for all $f \in E$. We may assume that $f \in B$. Let $Y \in \mathcal{C}$ be such that $\underline{Y} = C(B, f^*)$ and $Y(f^*) = 1$. If $f^* = e_i$, then $X_i(f) = X_i(e_i^*) = 0$ and $Y(e_i^*) = Y(f) = 0$. Otherwise, we have $\tilde{X}_i(f) + \tilde{Y}(e_i^*) = \langle X_i, Y^*\rangle \in N_F$ and hence $-\tilde{X}_i(f) = \tilde{Y}(e_i^*)$. Therefore, by the orthogonality (O),

$$\tilde{X}(f) - \sum_i \tilde{X}(e_i)\tilde{X}_i(f) = \tilde{X}(f) + \sum_i \tilde{X}(e_i)\tilde{Y}(e_i^*) = \langle X, Y^*\rangle \in N_F.$$ □

We now prove Theorem 5.1.13 using the previous lemmas.

Proof of Theorem 5.1.13. Let $\mathcal{C}$ be an F-signature of an orthogonal matroid. Suppose that $\mathcal{C}$ is orthogonal. Then $\mathcal{C} \subseteq \mathcal{C}^\perp$ and $\mathcal{C}$ satisfies (O$_2$). By Lemma 5.2.15, $\mathcal{C}$ satisfies (L). Conversely, suppose $\mathcal{C}$ is a strong F-circuit set, then by Lemma 5.2.14, we deduce that $\mathcal{C} \subseteq \mathcal{C}^\perp$, or equivalently, $\mathcal{C}$ is orthogonal. □

5.2.5 Orthogonal signatures and orthogonal vector sets

In [2], Anderson showed the equivalence between strong F-matroids and F-vector sets for matroids. The orthogonal complement of an F-cocircuit set of an ordinary matroid $\underline{M}$ (i.e., an F-circuit set of the dual matroid $\underline{M}^*$) is an F-vector set of $\underline{M}$, and nonzero vectors having minimal supports in an F-vector set of $\underline{M}$ form an F-cocircuit set of $\underline{M}$. We prove that the strong orthogonal F-signatures and the orthogonal F-vector sets can be derived from each other in a similar sense.

Lemma 5.2.16. *Let $\mathcal{V}$ be an orthogonal F-vector set. Then there exists a unique ordinary orthogonal matroid $\underline{M}$ whose set of bases equals the set of support bases of $\mathcal{V}$. Furthermore, the set of supports of elementary vectors in $\mathcal{V}$ equals the set of circuits of $\underline{M}$.*

Proof. Let $\mathcal{B}$ be the set of support bases of $\mathcal{V}$. It suffices to check that $\mathcal{B} \neq \emptyset$ and $\mathcal{B}$ satisfies the symmetric exchange axiom.

We first show that $\mathcal{B} \neq \emptyset$. We may assume that $\mathcal{V}$ has an elementary vector X, since otherwise every transversal is a support base. Let $I_0 = \underline{X} \setminus \{e, e^*\}$ for an arbitrary $e \in \underline{X}$. We say that a subtransversal is $\mathcal{V}$-*independent* if it does not contains any $\underline{Y}$ where $Y \in \mathcal{V} \setminus \{\mathbf{0}\}$. Then I_0 is $\mathcal{V}$-independent.

We claim that if a subtransversal I is $\mathcal{V}$-independent and $f \in [n] \setminus \overline{I}$, then $I \cup \{f\}$ or $I \cup \{f^*\}$ is $\mathcal{V}$-independent. Suppose for contradiction that neither $I \cup \{f\}$ nor $I \cup \{f^*\}$ is $\mathcal{V}$-independent. Then there are $Y_1, Y_2 \in \mathcal{V} \setminus \{\mathbf{0}\}$ such that $\underline{Y_1} \subseteq I \cup \{f\}$ and $\underline{Y_2} \subseteq I \cup \{f^*\}$. We may assume that Y_1 and Y_2 are elementary. Since I is $\mathcal{V}$-independent, $f \in \underline{Y_1}$ and $f^* \in \underline{Y_2}$. Then $\langle Y_1, Y_2^* \rangle = \tilde{Y}_1(f)\tilde{Y}_2(f^*) \notin N_F$, which contradicts (V1). By the claim, for $i = 0, 1, 2, \ldots$, there is a $\mathcal{V}$-independent set I_{i+1} such that $I_i \subseteq I_{i+1}$ and $|I_{i+1}| = |I_i| + 1$, unless $|I_i| \geq n$. Then for $k := n - |I_0|$, the subtransversal I_k is a $\mathcal{V}$-independent set of size n and hence I_k is a support base of $\mathcal{V}$, implying that $\mathcal{B} \neq \emptyset$.

Next we show that $\mathcal{B}$ satisfies the symmetric exchange axiom. Let $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$. By (V2), there is a fundamental circuit form $\{X_g : g \in B_1^*\}$ of $\mathcal{V}$ with respect to B_1, where $\underline{X_g} \subseteq B_1 \triangle \{g, g^*\}$ and $X_g(g) = 1$. Let $X := X_{e^*}$. Note that X is elementary in $\mathcal{V}$ by (V3). Since B_2 is a support base, $\underline{X} \not\subseteq B_2$. Thus there is $f \in \underline{X} \setminus B_2 \subseteq (B_1 \triangle \{e, e^*\}) \setminus B_2 = (B_1 \setminus B_2) \setminus \{e\}$. It suffices to show that $B_1 \triangle \{e, e^*\} \triangle \{f, f^*\}$ is a support base of $\mathcal{V}$. If not, then there is $Y \in \mathcal{V} \setminus \{\mathbf{0}\}$ with support $\underline{Y} \subseteq B_1 \triangle \{e, e^*\} \triangle \{f, f^*\}$. We may assume that Y is elementary in $\mathcal{V}$. By (V1), $\tilde{X}(f)\tilde{Y}(f^*) = \langle X, Y^* \rangle \in N_F$ and thus $Y(f^*) = 0$. Then $\underline{Y} \subseteq B_1 \triangle \{e, e^*\}$. Since B_1 is a support base, $e^* \in \underline{Y}$. By (V3), $Y = Y(e^*)X$, which contradicts the fact that $Y(f) = 0 \neq X(f)$. Therefore, $B_1 \triangle \{e, e^*, f, f^*\}$ is a support base.

From the definitions of $\mathcal{B}$ and $\underline{M}$, it is straightforward to see that the set of circuits of $\underline{M}$ equals the set of supports of elementary vectors of $\mathcal{V}$. □

In Lemma 5.2.15, if we assume additionally that X is elementary in $\mathcal{C}^\perp$, then X is indeed in $\mathcal{C}$ rather than merely being consistent with $\mathcal{C}$, as the next lemma shows. For $\mathcal{W} \subseteq F^E$, let $\mathrm{Elem}(\mathcal{W})$ be the set of elementary vectors in $\mathcal{W}$.

Lemma 5.2.17. *Let $\mathcal{C}$ be an orthogonal F-signature of an orthogonal matroid on E. Then* $\mathrm{Elem}(\mathcal{C}^\perp) = \mathcal{C}$.

Proof. Denote $\underline{M} := \underline{M}_{\mathcal{C}}$. Note that $\mathcal{C} \subseteq \mathcal{C}^\perp$, since $\mathcal{C}$ is orthogonal.

We first show that $\mathrm{Elem}(\mathcal{C}^\perp) \supseteq \mathcal{C}$. Suppose $X \in \mathcal{C}$ is not elementary in $\mathcal{C}^\perp$. Then there is $X' \in \mathcal{C}^\perp \setminus \{\mathbf{0}\}$ such that $\underline{X'} \subsetneq \underline{X}$. Let $e \in \underline{X} \setminus \underline{X'}$ and let B be a base of $\underline{M}$ containing $\underline{X} \triangle \{e, e^*\}$. Choose $f \in \underline{X'}$ and $Y \in \mathcal{C}$ so that $\underline{Y} = C(B, f^*)$. Then $\langle X', Y^* \rangle = \tilde{X}'(f)\tilde{Y}(f^*) \notin N_F$, a contradiction.

Next, we prove that $\mathrm{Elem}(\mathcal{C}^\perp) \subseteq \mathcal{C}$. Let X be an elementary vector in $\mathcal{C}^\perp$. Suppose for contradiction that $\underline{X}$ is independent in $\underline{M}$. Take an element $e \in \underline{X}$ and a base B of $\underline{M}$ containing $\underline{X}$, and let $Y \in \mathcal{C}$ be such that $\underline{Y} = C(B, e^*)$. Then $\langle X, Y^* \rangle = \tilde{X}(e)\tilde{Y}(e^*) \notin N_F$, a contradiction. Therefore, $\underline{X}$ is dependent in $\underline{M}$. Then there is $X' \in \mathcal{C}$ such that $\underline{X'} \subseteq \underline{X}$. Since $\mathcal{C} \subseteq \mathcal{C}^\perp$ and X is elementary in $\mathcal{C}^\perp$, we have $\underline{X} \subseteq \underline{X'}$. Hence $\underline{X} = \underline{X'}$. Now it suffices to show $X = \alpha X'$ for some $\alpha \in F^\times$. For $e \in \underline{X}$, we may assume that $X(e) = X'(e) = 1$. Suppose that $X \neq X'$. Then $X(f) \neq X'(f)$ for some $f \in \underline{X}$. For a base B of $\underline{M}$ containing $\underline{X} \triangle \{e, e^*\}$, let $Y \in \mathcal{C}$ be such that $\underline{Y} = C(B, f^*)$ and $Y(f^*) = 1$. Because $\tilde{X}(f) + \tilde{Y}(e^*) = \langle X, Y^* \rangle \in N_F$, we have $\tilde{X}(f) = -\tilde{Y}(e^*)$. We similarly deduce that $\tilde{X}'(f) = -\tilde{Y}(e^*)$, which contradicts the fact that $X(f) \neq X'(f)$. Thus, $X = X' \in \mathcal{C}$. □

Theorem 5.2.18. *The following hold:*

(i) *If $\mathcal{C}$ is an orthogonal F-signature, then $\mathcal{C}^\perp$ is an orthogonal F-vector set and $\mathcal{C} = \mathrm{Elem}(\mathcal{C}^\perp)$.*

(ii) *If $\mathcal{V}$ is an orthogonal F-vector set, then $\mathrm{Elem}(\mathcal{V})$ is an orthogonal F-signature and $\mathcal{V} = \mathrm{Elem}(\mathcal{V})^\perp$.*

Proof. (i) By Lemma 5.2.17, $\mathrm{Elem}(\mathcal{C}^\perp) = \mathcal{C}$ and thus $\mathcal{C}^\perp$ satisfies (V1). In addition, the set of support bases of $\mathcal{C}^\perp$ is equal to the set of bases of $\underline{M}_{\mathcal{C}}$. Therefore, by ($\Delta$Max), $\mathcal{C}^\perp$ satisfies (V2). By Lemmas 5.2.14 and 5.2.15, $\mathcal{C}^\perp$ satisfies (V3).

(ii) Let $\mathcal{C} := \mathrm{Elem}(\mathcal{V})$. By (V1), $\mathcal{C}$ satisfies the 2-term orthogonality relation (O_2). By Lemma 5.2.16, the set of support bases of $\mathcal{V}$ coincides with the set of support bases of $\mathcal{C}$. Moreover, it is the set of bases of some ordinary orthogonal matroid $\underline{M}$. Then $\mathcal{C}$ is an F-signature of $\underline{M}$ and every fundamental circuit form of $\mathcal{C}$ is a fundamental circuit form of $\mathcal{V}$. Conversely, by (V3), every fundamental circuit form of $\mathcal{V}$ is a fundamental circuit form of $\mathcal{C}$. Therefore, $X \in F^E$ is in $\mathcal{V}$ if and only if it is consistent with $\mathcal{C}$. The latter condition implies that $X \in \mathcal{C}^\perp$ by Lemma 5.2.14. Then $\mathcal{C} \subseteq \mathcal{V} \subseteq \mathcal{C}^\perp$. Therefore, $\mathcal{C}$ is an orthogonal F-signature of $\underline{M}$. By Lemma 5.2.15, if $X \in \mathcal{C}^\perp$, then X is consistent with $\mathcal{C}$. Hence $\mathcal{C}^\perp \subseteq \mathcal{V}$ and we conclude $\mathcal{C}^\perp = \mathcal{V}$. □

We finish the discussion of orthogonal vector sets with the proof of Theorem 5.1.16(i) that if F is a field, then every orthogonal F-vector set is a Lagrangian subspace.

Proof of Theorem 5.1.16(i). By (V2) and (V3), $\mathcal{V}$ is an n-dimensional linear subspace of $F^{[n]\cup[n]^*}$. Let $\mathcal{C} := \mathrm{Elem}(\mathcal{V})$. By Theorem 5.2.18(ii), $\langle X, Y^* \rangle = 0$ for all $X, Y \in \mathcal{C}$ and $\mathcal{V} = \mathcal{C}^\perp$. By Lemma 5.2.15, $\mathcal{V}$ is the subspace spanned by $\mathcal{C}$ and thus $\langle X, Y^* \rangle = 0$ for all $X, Y \in \mathcal{V}$. Hence $\mathcal{V}$ is isotropic and therefore Lagrangian. □

Example 5.2.19. By [4, Corollary 3.45], if F is a *doubly distributive partial hyperfield* such as a field, $\mathbb{S}$, $\mathbb{T}$, or $\mathbb{K}$, and if M is a strong F-matroid, then every vector (resp. covector) of M is orthogonal to all covectors (resp. vectors) of M. For the proof, it is crucial to show that if F is a doubly distributive partial hyperfield, then every weak F-matroid is automatically a strong F-matroid. In the orthogonal case, if F is a field and $\mathcal{W} \subseteq F^{[n]\cup[n]^*}$ is an orthogonal F-vector set, then $\mathcal{W}^\perp = \mathcal{W}$ by Theorem 5.1.16(i). So one may ask naturally whether this fact can be generalized to doubly distributive partial hyperfields. However, it is false even if we take $F = \mathbb{K}$, the Krasner hyperfield. Let $\underline{N}$ be the orthogonal matroid on $[5] \cup [5]^*$ in which a transversal B is a base of $\underline{N}$ if and only if $|B \cap [5]|$ is even and $B \neq 1^*2345, 12^*345$. A simple computer search shows that $|\mathcal{C}| = 15$, $|\mathcal{V}| = 256$, and $|\mathcal{V}^\perp| = 169$, where $\mathcal{C}$ is the unique $\mathbb{K}$-circuit set of $\underline{N}$ and $\mathcal{V} := \mathcal{C}^\perp$ is the corresponding orthogonal $\mathbb{K}$-vector set.

5.2.6 Natural bijections

Summarizing the results in Subsections 5.2.1–5.2.5, we prove the equivalence between various notions of orthogonal matroids with coefficients in tracts, described in Theorems 5.1.18, 5.1.19, and 5.1.20. As a corollary, we deduce Theorem 5.1.14.

The following lemma is straightforward from definitions.

Lemma 5.2.20. *Let F be a tract. Let $\mathcal{C}$ be an F-signature of an orthogonal matroid satisfying* (O$_2$)*, and let φ be a weak Wick F-function. Then $\mathcal{C}_{\varphi_{\mathcal{C}}} = \mathcal{C}$ and $[\varphi_{\mathcal{C}_\varphi}] = [\varphi]$.*

Proof of Theorem 5.1.18. By Theorems 5.2.3, 5.2.9, and Lemma 5.2.20, there is a natural bijection between (1) and (2). By Theorem 5.1.13, (2) and (3) are identical. By Theorem 5.2.18, there is a natural bijection between (2) and (4). □

Proof of Theorem 5.1.19. It is straightforward from Theorems 5.2.4, 5.2.10, and Lemma 5.2.20. □

Proof of Theorem 5.1.20. It is concluded by Theorem 5.2.11, 5.2.13, and Lemma 5.2.20. □

Proof of Theorem 5.1.14. It is an immediate corollary of Theorems 5.1.19 and 5.1.20. □

5.2.7 More examples.

Strong orthogonal F-matroids generalize strong F-matroids by Proposition 5.1.4, and strong orthogonal F-signatures of orthogonal matroids generalize strong dual pairs of F-signatures of matroids by Remark 5.1.9. Baker and Bowler showed in [4] the equivalence of weak F-matroids and weak dual pairs of F-signatures. By Theorem 5.1.19, moderately weak orthogonal F-matroids and weak orthogonal F-signatures are equivalent. However, in the previous equivalence, moderately weak orthogonal F-matroids cannot be replaced by weak orthogonal F-matroids, as the class of weak orthogonal F-matroids is strictly larger than the class of moderately weak orthogonal F-matroids for some tract F. This is true even if we restrict the classes of weak and moderately weak orthogonal F-matroids to those whose underlying orthogonal matroids are lifts of matroids.

Example 5.2.21. Let F be the tract $(\{1\}, \{1+1, 1+1+1\})$ with the trivial involution and let $\underline{M}$ be the lift of the uniform matroid $U_{3,6}$. The set of bases of $\underline{M}$ is $\{abcd^*e^*f^* : abcdef = [6]\}$. Since $F^\times = \{1\}$, the function $\varphi : \mathcal{T}_6 \to F$ whose support is the set of bases of $\underline{M}$ is uniquely determined. Because $\underline{M}$ is the lift of a matroid, for all transversals T_1 and T_2 with $|(T_1 \triangle T_2) \cap [6]| = 4$, at most three of $\varphi(T_1 \triangle \{i_j, i_j^*\})\varphi(T_2 \triangle \{i_j, i_j^*\})$ with $1 \le j \le 4$ are nonzero, where $(T_1 \triangle T_2) \cap [6] = \{i_1 < i_2 < i_3 < i_4\}$. Therefore, φ is a weak Wick F-function. Consider $T_1' = \{1,2,3,4,5^*,6^*\}$ and $T_2' = (T_1')^*$, we have $\sum_{i=1}^{6} (-1)^i \varphi(T_1' \triangle \{i, i^*\})\varphi(T_2' \triangle \{i, i^*\}) = 1+1+1+1 \notin N_F$. Hence φ is not a moderately weak Wick F-function. Similarly, if we take $\mathcal{C}$ to be the unique F-signature of $\underline{M}$, then it is readily seen that $\mathcal{C}$ is a weak F-circuit set but not a weak orthogonal F-signature.

We also have an instance showing where the class of strong F-matroids is strictly larger than the class of moderately weak F-matroids, i.e., the class of strong orthogonal F-signatures is strictly larger than the class of weak orthogonal F-signatures.

Example 5.2.22. Let F be the tract $(\{1\},\{1+1,1+1+1,1+1+1+1\})$ endowed with the trivial involution and let $\underline{M}$ be the lift of $U_{4,8}$. Let $\mathcal{C}$ be the unique F-signature of $\underline{M}$. Then for $X,Y\in\mathcal{C}$ whose supports are $[5]$ and $[5]^*$, respectively, we have $\langle X,Y^*\rangle=1+1+1+1+1\notin N_F$ and thus (O) does not hold. However, it is obvious that (O_4) holds by our choice of F.

By Theorem 3.2.13, if $\mathcal{C}$ is an F-signature of the lift of a matroid satisfying the 3-term orthogonality (O_3), then $\varphi_{\mathcal{C}}$ is a weak Wick F-function. However, this is false in general for orthogonal matroids, even if F is a field.

Example 5.2.23. Consider the K-signature $\mathcal{C}$ defined in Example 5.1.12, which satisfies (O_3) but not (O_4). Note that $\mathcal{C}_{\varphi_{\mathcal{C}}}=\mathcal{C}$ and thus by Theorem 5.2.4, $\varphi_{\mathcal{C}}$ is not a moderately weak Wick K-function. Since $E(\underline{M}_{\mathcal{C}})=[4]\cup[4]^*$, (W2′) and (W2′′) are equivalent for $\varphi_{\mathcal{C}}$. Thus $\varphi_{\mathcal{C}}$ is not a weak Wick function.

More precisely, we can compute $\varphi_{\mathcal{C}}$ by setting $\varphi_{\mathcal{C}}([4])=1$ and check whether it satisfies (W2′′). By definition, it is easily seen that

$$\varphi_{\mathcal{C}}(B)=\begin{cases}1 & \text{if } B=[4] \text{ or } 1^*23^*4,\\ -1 & \text{if } B\in\{ijk^*\ell^*: ijk\ell=[4]\}\setminus\{1^*23^*4\},\\ -x & \text{if } B=[4]^*,\\ 0 & \text{otherwise.}\end{cases}$$

Then for $T_1=1234^*$ and $T_2=1^*2^*3^*4$,

$$\sum_{i=1}^{4}(-1)^i\varphi(T_1\triangle\{i,i^*\})\varphi(T_2\triangle\{i,i^*\})=-1-1-1-x\neq 0$$

since $x\in K\setminus\{0,-3\}$. Therefore, $\varphi_{\mathcal{C}}$ does not satisfies (W2′′).

In Sections 5.1.5 and 5.1.7, we promised to show that the minors and the pushforward operations of an orthogonal F-vector set are not properly defined. Recall that for $\mathcal{W}\subseteq F^E$ and $e\in E$, $\mathcal{W}|e=\{\pi(X)\in F^{E\setminus\{e,e^*\}}: X\in\mathcal{W}\text{ with }X(e^*)=0\}$, where $\pi: F^E\to F^{E\setminus\{e,e^*\}}$ is the canonical projection. For an orthogonal F-signature $\mathcal{C}$ and the corresponding F-vector set $\mathcal{V}:=\mathcal{C}^\perp$, it is readily seen that $\mathcal{V}|e\subseteq(\mathcal{C}|e)^\perp$. Example 5.2.24 provides an instance where $\mathcal{V}|e\neq(\mathcal{C}|e)^\perp$. If $f:F\to F'$ is a tract homomorphism commuting with involutions of F and F', one can check that $f_*(\mathcal{V})\subseteq(f_*(\mathcal{C}))^\perp$. It might not be an equality, as Example 5.2.25 shows.

Example 5.2.24. Let $\underline{M}$ be the lift of $U_{1,3}$. Then $\mathcal{C}(\underline{M})=\{12,13,23,1^*2^*3^*\}$. Consider the following orthogonal $\mathbb{U}_0$-signature of $\underline{M}$:

$$\mathcal{C}:=\{(1,-1,0,0,0,0),\ (1,0,1,0,0,0),\ (0,1,1,0,0,0),\ (0,0,0,1,1,-1)\},$$

where the coordinates of the vectors are indexed by $1,2,3,1^*,2^*,3^*$ in order. Let $\mathcal{V}:=\mathcal{C}^\perp$ be the orthogonal $\mathbb{U}_0$-vector set. Then $\mathcal{V}|3=\{(1,-1,0,0),\ (1,0,0,0),\ (0,1,0,0)\}$ and $(\mathcal{C}|3)^\perp=\mathcal{V}|3\cup\{(1,1,0,0)\}$, where the coordinates of vectors are indexed by $1,2,1^*,2^*$ in order.

Example 5.2.25. Similarly, let $\mathcal{C}=\{(1,1,0,0,0,0),\ (1,0,1,0,0,0),\ (0,1,1,0,0,0),\ (0,0,0,1,1,1)\}\subseteq(\mathbb{F}_2)^{[3]\cup[3]^*}$ be the orthogonal $\mathbb{F}_2$-signature of the lift of $U_{1,3}$, where the coordinates of each vector are indexed by $1,2,3,1^*,2^*,3^*$ in order. Let $\mathcal{V}:=\mathcal{C}^\perp$. Then it is an orthogonal $\mathbb{F}_2$-vector set by Theorem 5.2.18 and $(1,1,1,0,0,0)\notin\mathcal{V}$. For the tract homomorphism $f:\mathbb{F}_2\to\mathbb{K}$, it is easily checked that $(1,1,1,0,0,0)\in(f_*(\mathcal{C}))^\perp\setminus f_*(\mathcal{V})$ and $\mathrm{Elem}(f_*(\mathcal{V}))=f_*(\mathcal{C})$. Thus, $f_*(\mathcal{V})$ is not an orthogonal $\mathbb{K}$-vector set by Theorem 5.2.18.

5.3 Applications

5.3.1 Representability of orthogonal matroids

An ordinary orthogonal matroid M is *representable* over a tract F if there is a strong orthogonal F-matroid whose underlying orthogonal matroid is M. Respectively, M is *weakly reprsentable* over F if there is a weak orthogonal F-matroid whose underlying orthogonal matroid is M. When $F = \mathbb{F}$ is a field, the representability of orthogonal matroids was introduced using skew-symmetric matrices in [25], and coincides with our definition by [115, Theorem 2.2]. Note that whenever M is the lift of a matroid N, the orthogonal matroid M is representable over a field $\mathbb{F}$ if and only if the matroid N admits a usual matrix representation over $\mathbb{F}$ by [25, (4.4)].

The following theorem by Baker and Jin [6, Theorem 4.3] will be used repeatedly in this subsection, which is an analog of Theorem 3.1.4.

Theorem 5.3.1 ([6]). *Let P be a partial field and let $\varphi : \mathcal{T}_n \to P$ be a function. Then φ is a strong Wick function if and only if it is a weak Wick function. In particular, an orthogonal matroid is representable over P if and only if it is weakly representable over P.*

For a tract F and a nonnegative integer k, let $N_F^{\leq k}$ be the set of elements in $N_F \subseteq \mathbb{N}[F^\times]$ that are formal sums of at most k elements of $F^\times$. To check whether a map $\varphi : \mathcal{T}_n \to F$ is a weak Wick function, we only need the information of $N_F^{\leq 4}$ rather than N_F.

One impressive result in matroid theory is that if a matroid is representable over $\mathbb{F}_2$ and $\mathbb{F}_3$, then it is representable over all fields (Theorem 3.3.5). Geelen [61] extended this result to orthogonal matroids.

Theorem 4.1.13. *Let M be an orthogonal matroid. Then the following are equivalent:*

(i) *M is representable over $\mathbb{F}_2$ and $\mathbb{F}_3$.*

(ii) *M is representable over the regular partial field $\mathbb{U}_0$.*

(iii) *M is representable over all fields.*

The proof in [61] involves technical matrix calculations. However, using the theory of orthogonal matroids over tracts, we are able to give a short and conceptual proof.

Proof. If M is representable over $\mathbb{F}_2$ and $\mathbb{F}_3$ via strong Wick functions φ_1 and φ_2, respectively, then by Proposition 5.1.23, $\varphi_1 \times \varphi_2$ is a strong Wick function over $\mathbb{F}_2 \times \mathbb{F}_3$ with underlying orthogonal matroid M. Let f be the map from the set $\mathbb{F}_2 \times \mathbb{F}_3 = \{0, (1, \pm 1)\}$ to the set $\mathbb{U}_0 = \{0, \pm 1\}$ given by $f(1, \pm 1) = \pm 1$ and $f(0) = 0$, then we have $f(N^{\leq 4}_{\mathbb{F}_2 \times F_3}) = N^{\leq 4}_{\mathbb{U}_0}$. Therefore, $\varphi_0 := f \circ (\varphi_1 \times \varphi_2)$ is a weak Wick function over $\mathbb{U}_0$ and hence a strong Wick function by Theorem 5.3.1, and we have (i) implies (ii). For every field F, since there is a natural tract homomorphism $\mathbb{U}_0 \to F$ induced by the map $\mathbb{Z} \to F$, we have (ii) implies (iii) using Proposition 5.1.21. It is trivial that (iii) implies (i). □

It is worth noting that the map f defined in the above proof is not a tract homomorphism.

We say that an orthogonal matroid is *regular* if it satisfies one of the three equivalent conditions in Theorem 4.1.13. We now give two more characterizations of regular orthognal matroids without a specific minor $\mathrm{lift}(M_4) = ([4] \cup [4]^*, \{T \in \mathcal{T}_n : |T \cap [4]| \text{ is even}\})$.

An *ordered field* is a field together with a strict total order $\prec$ such that for every $x, y, z \in F$, (i) if $x \prec y$, then $x + z \prec y + z$, and (ii) if $0 \prec x$ and $0 \prec y$, then $0 \prec xy$. For instance, the real field $\mathbb{R}$ with the usual order is an ordered field.

Theorem 5.3.2. *Let M be an orthogonal matroid with no minor isomorphic to* $\mathrm{lift}(M_4)$ *and let* $(\mathbb{F}, \prec)$ *be an ordered field. Then the following are equivalent:*

(i) *M is regular.*

(ii) *M is representable over $\mathbb{F}_2$ and $\mathbb{F}$.*

(iii) *M is representable over $\mathbb{F}_2$ and the sign hyperfield $\mathbb{S}$.*

To show Theorem 5.3.2, we need the following lemma on orthogonal matroids with no minor isomorphic to $\mathrm{lift}(M_4)$.

Lemma 5.3.3. *Let F be a tract and φ a weak Wick function over F. If $\underline{M}_\varphi$ has no minor isomorphic to* $\mathrm{lift}(M_4)$*, then for all transversals T_1 and T_2 with $T_1 \setminus T_2 = \{i_1, i_2, i_3, i_4\}$, at least one of $\varphi(T_1 \triangle \{i_j, i_j^*\})\varphi(T_2 \triangle \{i_j, i_j^*\})$ with $j \in [4]$ is zero.*

Proof. Suppose for contradiction that all products are nonzero. Then all of the eight transversals $T_k \triangle \{i_j, i_j^*\}$ with $k \in [2]$ and $j \in [4]$ are bases of $\underline{M}_\varphi$. Let $S := T_1 \setminus \{i_1, i_2, i_3, i_4\}$. Then $M|S$ is isomorphic to $\mathrm{lift}(M_4)$, a contradiction. □

Proof of Theorem 5.3.2. If M is representable over $\mathbb{F}_2$ and $\mathbb{S}$ via Wick functions φ_1 and φ_2, respectively, then by Proposition 5.1.23, $\varphi_1 \times \varphi_2$ is a Wick function over $\mathbb{F}_2 \times \mathbb{S}$ with underlying orthogonal matroid M. Let g be the map from the set $\mathbb{F}_2 \times \mathbb{S} = \{0, (1, \pm 1)\}$ to the set $\mathbb{U}_0 = \{0, \pm 1\}$ given by $g(0) = 0$ and $g(1, \pm 1) = \pm 1$. Then $g(N^{\leq 3}_{\mathbb{F}_2 \times \mathbb{S}}) = N^{\leq 3}_{\mathbb{U}_0}$. Hence by Lemma 5.3.3, $\varphi_0 := g \circ (\varphi_1 \times \varphi_2)$ is a weak Wick function over $\mathbb{U}_0$. By Theorem 5.3.1, φ_0 is a strong Wick function, and we have (iii) implies (i). The direction (i) implies (ii) follows trivially from Theorem 3.3.5. Finally, let $\sigma : K \to \mathbb{S}$ be such that

$$\sigma(x) := \begin{cases} 1 & \text{if } x \succ 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Then σ is a tract homomorphism and thus we have (ii) implies (iii) by Proposition 5.1.21. □

Remark 5.3.4. The condition that an orthogonal matroid M does not have minors isomorphic to $\mathrm{lift}(M_4)$ is sufficient but not necessary for the characterizations of regular orthogonal matroids in Theorem 5.3.2. In fact, $\mathrm{lift}(M_4)$ itself is representable over the regular partial field $\mathbb{U}_0$ by setting $\varphi(T) = 1$ if T is a base, and $\varphi(T) = 0$ otherwise, and hence representable over all fields and the sign hyperfield $\mathbb{S}$. It is still an open question whether Theorem 5.3.2 holds for all orthogonal matroids.

Duchamp [56, Proposition 1.5] proved that an orthogonal matroid M is isomorphic to a twist of the lift of a matroid if and only if M has no minor isomorphic to the orthogonal matroid $\mathrm{lift}(M_3)$ on $[3] \cup [3]^*$ whose set of bases is $\{1^*2^*3^*, 1^*23, 12^*3, 123^*\}$. Note that $\mathrm{lift}(M_3) = \mathrm{lift}(M_4)|4$. Therefore, if M is isomorphic to the lift of a matroid, then it does not have minors isomorphic to $\mathrm{lift}(M_4)$. As a consequence, we have the following result originally proved by Bland and Las Vergnas [12, Proposition 6.1].

Corollary 5.3.5 ([12])**.** *A matroid is regular if and only if it is binary and orientable.* □

Our proof of Theorem 5.3.2 does not require any result on the excluded minors of regular orthogonal matroids, in contrast to that the original proof of Corollary 5.3.5 in [12] relies on Tutte's excluded-minor characterization of regular matroids.

We also extend Whittle's theorem [122, Theorem 1.2] that a matroid is representable over both $\mathbb{F}_3$ and $\mathbb{F}_4$ if and only if it is representable over the sixth-root-of-unity partial field R_6 to orthogonal matroids.

Theorem 5.3.6. *Let M be an orthogonal matroid. Then the following are equivalent:*

(i) *M is representable over the sixth-root-of-unity partial field R_6.*

(ii) *M is representable over $\mathbb{F}_3$ and $\mathbb{F}_4$.*

(iii) *M is representable over $\mathbb{F}_3$, $\mathbb{F}_{p^2}$ for all primes p, and $\mathbb{F}_q$ for all primes q with $q \equiv 1 \pmod 3$.*

To show Theorem 5.3.6, we need the following lemma on R_6. This lemma is a consequence of results in van Zwam's thesis [113, Lemma 2.5.12 and Table 4.1]

Lemma 5.3.7 ([113])**.** *Let p be a prime.*

1. *There is a tract homomorphism $R_6 \to \mathbb{F}_{p^2}$.*

2. *If $p \equiv 1 \pmod 3$, then there is a tract homomorphism $R_6 \to \mathbb{F}_p$.*

3. *There is a tract isomorphism $R_6 \cong \mathbb{F}_3 \times \mathbb{F}_4$.*

Proof of Theorem 5.3.6. The proof is a straightforward application of Propositions 5.1.21, 5.1.23, and Lemma 5.3.7, and is similar to the proof of Theorem 4.1.13. In particular, the only nontrivial part that if M is representable over $\mathbb{F}_3$ and $\mathbb{F}_4$ then M is representable over R_6 is guaranteed by the tract isomorphism $R_6 \cong \mathbb{F}_3 \times \mathbb{F}_4$ and Proposition 5.1.23. □

5.3.2 Farkas' Lemma for oriented orthogonal matroids

Farkas' Lemma is a fundamental result in linear programming, which provides a dichotomy for the solvability of a system of linear inequalities. The following proposition is known as a generalization of Farkas' Lemma for oriented matroids; see [11, Corollary 3.4.6]

Proposition 5.3.8 (Farkas' Lemma for oriented matroids)**.** *Let M be an oriented matroid and let $e \in (\underline{M})$. Then there is an $\mathbb{S}$-circuit C with $C(e) \neq 0$ or an $\mathbb{S}$-cocircuit D with $D(e) \neq 0$, but not both.*

We show the following extension for oriented orthogonal matroids.

Proposition 5.3.9 (Farkas' Lemma for oriented orthogonal matroids)**.** *Let M be an oriented orthogonal matroid and let $\{e, e^*\} \subseteq E(\underline{M})$ be a skew pair. Then there is an $\mathbb{S}$-circuit C of M such that $\underline{C} \cap \{e, e^*\} \neq \emptyset$ and $C(f) \in \{0, +\}$ for all $f \in E(\underline{M})$. Moreover, there are no two $\mathbb{S}$-circuits C_1 and C_2 satisfying the above condtion along with $C_1(e) = + = C_2(e^*)$.*

The previous proposition is a special case of the following result taking $X = E(M) \setminus \{e, e^*\}$.

Proposition 5.3.10. *Let M be an oriented orthogonal matroid. Let $\{e, e^*\}$ be a skew pair, X be a union of skew pairs in $E(M) \setminus \{e, e^*\}$, and Y be a transversal in $E(M) \setminus (X \cup \{e, e^*\})$. Then there is an $\mathbb{S}$-circuit C of M such that*

- *$\underline{C} \cap \{e, e^*\} \neq \emptyset$,*
- *$\underline{C} \subseteq \{e, e^*\} \cup X \cup Y$, and*

- $C(f) \in \{0, +\}$ *for every* $f \in \{e, e^*\} \cup X$.

Moreover, there are no two $\mathbb{S}$*-circuits* C_1 *and* C_2 *satisfying the above condtions along with* $C_1(e) = + = C_2(e^*)$.

Proof. We first prove the existence. We proceed by induction on $|X|/2 \geq 0$. It is obvious for $X = \emptyset$ by the axiom (ΔMax) for orthogonal matroids, and thus we may assume that $|X|/2 \geq 1$. Take an arbitrary skew pair $\{x, x^*\}$ in X. By the induction hypothesis applied to $X \setminus \{x, x^*\}$ and $Y \cup \{x\}$, there is a signed circuit C_1 of M such that $\underline{C_1} \cap \{e, e^*\} \neq \emptyset$, $\underline{C_1} \subseteq \{e, e^*\} \cup X \cup Y \setminus \{x^*\}$, and $C_1(f) \in \{0, +1\}$ for every $f \in \{e, e^*\} \cup X \setminus \{x, x^*\}$. We may assume that $C_1(x) = -1$ since otherwise it suffices to take $C = C_1$. Similarly, applying the induction hypothesis to $X \setminus \{x, x^*\}$ and $Y \cup \{x^*\}$, there is a circuit C_2 of M such that $\underline{C_2} \cap \{e, e^*\} \neq \emptyset$, $\underline{C_2} \subseteq \{e, e^*\} \cup X \cup Y \setminus \{x\}$, and $C_2(f) \in \{0, +1\}$ for every $f \in \{e, e^*\} \cup X \setminus \{x, x^*\}$. We also may assume that $C_2(x^*) = -1$. Then $\langle C_1, C_2 \rangle = C_1(x)C_2(x^*) = +1 \notin N_{\mathbb{S}}$, a contradiction.

We now assume that there are two $\mathbb{S}$-circuits C_1 and C_2 satisfying the above conditions along with $C_1(e) = + = C_2(e^*)$. Then $\langle C_1, C_2^* \rangle = C_1(e)C_2(e^*) + \sum_{f \in X} C_1(f)C_2(f^*) \in \mathbb{N}[\{+\}] \setminus \{0\}$ and thus $\langle C_1, C_2^* \rangle \notin N_{\mathbb{S}}$, contradicting (O). □

Remark 5.3.11. Matt Baker, Changxin Ding, and the author [5] make use of Proposition 5.3.9 to prove that the set of quasi-trees of an orientable ribbon graph $\mathbb{G}$ is a $K(\mathbb{G})$-torsor, i.e., the critical group $K(\mathbb{G})$ acts simply and transitively on the set of spanning quasi-trees, where the critical group of an orientable ribbon graph was introduced in [86]. It gives a bijective proof that the number of spanning quasi-trees in $\mathbb{G}$ equals the order of the critical group $K(\mathbb{G})$. We remark that the critical group $K(G)$ of a connected graph G is the cokernel of a reduced Laplacian matrix of G, and its order equals the number of spanning trees in G, which is an extension of Kirchhoff's Matirx-Tree Theorem; see [10].

Chapter 6. Antisymmetric matroids with coefficients

The Grassmannian $\mathrm{Gr}_{\mathbb{F}}(r,n)$ is the set of r-dimensional linear subspaces in the n-dimensional vector space $\mathbb{F}^n$. It can be parameterized into the projective space of dimension $\binom{n}{r}-1$ by the Grassmann-Plücker embedding p. The image of p is exactly cut out by the Grassmann-Plücker relations, which are homogeneous quadrics. For every linear subspace $V \in \mathrm{Gr}_k(r,n)$, the support of the Plücker vector $p(V)$ forms the bases of a matroid M, and the set of the minimal supports of nonzero vectors in V forms the circuits of the dual matroid $M^\perp$. Therefore, matroids are regarded as the combinatorial essence of linear spaces. Dress and Wenzel [48, 51] founded matroids with coefficients by defining Grassmann-Plücker relations over extensive field-like objects, and this line of research was culminated by Baker and Bowler [4] who presented several equivalent definitions of matroids with coefficients; see Section 3.2. Orthogonal matroids play a similar role for the Lagrangian orthogonal Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n,2n)$, and in Chapter 5, we establish orthogonal matroids with coefficients in tracts in several equivalent ways. This chapter explores a parallel theory concerning antisymmetric matroids and the Lagrangian symplectic Grassmannian.

We define antisymmetric matroids with coefficients in two equivalent ways: (i) antisymmetric F-matroids in Subsection 6.1.1 and (ii) antisymmetric F-circuit sets in Subsection 6.1.2.

Theorem 6.4.1. *For a tract F, there is a natural bijection between antisymmetric F-matroids and antisymmetric F-circuit sets.*

The key lemma is the Homotopy Theorem for antisymmetric matroids (Theorem 4.2.29) that will be proved in Section 6.3. If $F = \mathbb{F}$ is a field, Theorem 6.4.1 implies Theorems 1.1.21 and 4.2.2 stating that the Lagrangian Grassmannian $\mathrm{SpGr}_{\mathbb{F}}(n,2n)$ is parameterized into the projective space of dimension $2^n + \binom{n}{2}2^{n-2} - 1$ and it is set-theoretically cut out by the restricted Grassmann-Plücker relations (rGP). Furthermore, antisymmetric matroids with coefficients in tracts F generalize

(1) matroids with coefficients in tracts by Baker and Bowler [4] (Subsection 6.2.1), and

(2) orthogonal matroids with coefficients in tracts F if $-1 = 1$ in F (Subsection 6.2.2).

They are also compatible with several other concepts such as

(3) the symplectic Dressian and isotropic tropical linear spaces [102, 9] if $F = \mathbb{T}$ is the tropical hyperfield (Subsection 6.2.3), and

(4) oriented gaussoids [13] if $F = \mathbb{S}$ is the sign hyperfield (Subsection 6.2.4).

We show an analogy of Theorem 3.1.4 regarding antisymmetric matroids and the restricted G-P relations.

Theorem 6.5.2. *Let $\mathbb{F}$ be a field. For $X \in \mathbb{P}(\mathbb{F}^{2^n+\binom{n}{2}2^{n-2}})$, the following are equivalent:*

(i) *X satisfies all restricted Grassmann-Plücker relations.*

(ii) *X satisfies all $3/4$-term restricted Grassmann-Plücker relations and the support of X forms the set of bases of an antisymmetric matroid.*

(iii) *There is an* $n \times 2n$ *matrix* $\Lambda = [A_1 \mid A_2]$ *over* $\mathbb{F}$ *such that* $A_1 A_2^t$ *is symmetric and* $X_B = \det(A[B])$ *with* $B \in \mathcal{T}_n \cup \mathcal{A}_n$.

Structure of the chapter. In Section 6.1 we define antisymmetric matroids with coefficients in two ways, and in Section 6.2 we examine its connections to other concepts such as matroids with coefficients [4], symplectic Dressian [9], and oriented gaussoids [13]. In Section 6.3 we prove the Homotopy Theorem for antisymmetric matroids (Theorem 4.2.29). In Section 6.4 we prove Theorem 6.4.1. We show Theorem 6.5.2 in Section 6.5.

6.1 Antisymmetric matroids over tracts

We define antisymmetric matroids with coefficients in two different manners. We will prove their equivalence in Section 6.4.

6.1.1 Antisymmetric F-matroids

We define an antisymmetric matroid with coefficients in a tract by introducing the restricted Grassmann-Plücker relations over tracts.

Definition 6.1.1. A *restricted Grassmann–Plücker function on* $E = [n] \cup [n]^*$ *with coefficients in a tract* F is a function $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ that satisfies the following.

(rGP1) φ is not identically zero.

(rGP2) If $A \in \mathcal{A}_n$ and skew pairs p, q such that $p \subseteq A$ and $q \cap A = \emptyset$, then

$$\varphi(A) = (-1)^{i+j} \varphi(A - p + q)$$

where $i, j \in [n]$ such that $p = \{i, i^*\}$ and $q = \{j, j^*\}$.

(rGP3) For $S \in \binom{E}{n+1}$ and $T \in \binom{E}{n-1}$ such that S contains exactly one skew pair and T has no skew pair,

$$\sum_{x \in S \setminus T} (-1)^{|S<x|+|T<x|} \varphi(S - x)\varphi(T + x) \in N_F. \qquad \text{(rGP*)}$$

Then $\mathcal{B} = \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \varphi(B) \neq 0\}$ is nonempty and satisfies (Sym), and (Exch), and we call a pair $(E, \mathcal{B})$ the *underlying antisymmetric matroid* of φ. Two restricted G-P functions φ and φ' are *equivalent* if $\varphi' = c \cdot \varphi$ for some $c \in F^\times$. An *antisymmetric matroid with coefficient in* F or an *antisymmetric* F*-matroid* is an equivalence class $[\varphi]$ of restricted G-P functions with coefficients in F.

The F*-Lagrangian Grassmannian* $\mathrm{SpGr}_F(n, 2n)$ is the set of antisymmetric F-matroids on $[n] \cup [n]^*$. The antisymmetric $\mathbb{K}$-matroids can be regarded as the antisymmetric matroids. We call an antisymmetric $\mathbb{S}$-matroids *oriented antisymmetric matroids* and call an antisymmetric $\mathbb{T}$-matroids *valuated antisymmetric matroids*. In Section 6.1.3, we show that those encompass oriented matroids and valuated matroids, respectively.

For a restricted G-P function φ with coefficient in F and a tract morphism $f : F \to F'$, the composition $f \circ \varphi$ is a restricted G-P function with coefficients in F'. Therefore, we obtain a pushforward operation f_* such that for each antisymmetric F-matroid M, $f_* M$ is an antisymmetric F'-matroid. In particular, if $F' = \mathbb{K}$, then $f_* M$ is identified with the underlying antisymmetric matroid of M.

We will frequently use the following lemma without explicitly referring to it.

Lemma 6.1.2. *Let S be a subtransversal of size $n-2$ and let i, j be distinct elements in $[n]$ such that $S \cap \{i, i^*\} = \emptyset = S \cap \{j, j^*\}$. Then $i + j \equiv 1 + \sum_{z \in \{i,i^*,j,j^*\}} |(S + \{i, i^*, j, j^*\}) < z| \pmod 2$.* □

6.1.2 Antisymmetric F-circuit sets

We can identify a set $E = [n] \cup [n]^*$ with $\mathbb{K}^E$. Then an antisymmetric matroid M on E is an antisymmetric $\mathbb{K}$-matroid, and the circuits $C \subseteq E$ of M are vectors in $\mathbb{K}^E$. Furthermore, (△Orth) and (ΔMax) in Theorem 4.2.9 can be rephrased as follows. Note that $-1 = 1$ in $\mathbb{K}$.

(△Orth) $\sum_{i=1}^n \big(X(i)Y(i^*) + (-1)X(i^*)Y(i)\big) \in N_{\mathbb{K}}$ for all $X, Y \in \mathcal{C}(M) \subseteq \mathbb{K}^E$.

(ΔMax) For every $S \subseteq E$ such that $|S| = n+1$ and S contains exactly one skew pair, there is $X \in \mathcal{C}(M)$ such that $\mathrm{supp}(X) \subseteq S$.

Replacing the Krasner hyperfield $\mathbb{K}$ with an arbitrary tract F in (ΔMax), we define an antisymmetric F-circuit set which is equivalent to an antisymmetric F-matroid.

Definition 6.1.3. A set $\mathcal{C}$ of vectors in F^E is *prepared* if the following conditions hold:

(i) $\mathbf{0} \notin \mathcal{C}$.

(ii) The support of each vector in $\mathcal{C}$ contains at most one skew pair.

(iii) If $X \in \mathcal{C}$, then $cX \in \mathcal{C}$ for all $c \in F^\times$.

(iv) For $X, Y \in F^E$, if $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$ and $Y \in \mathcal{C}$, then $X = cY$ for some $c \in F^\times$,

Definition 6.1.4. An *antisymmetric F-circuit set* is a prepared set $\mathcal{C}$ of vectors in F^E satisfying the following two properties:

(ΔOrth*) $\omega(X, Y) := \sum_{i=1}^n \big(X(i)Y(i^*) - X(i^*)Y(i)\big)$ for all $X, Y \in \mathcal{C}$.

(ΔMax*) For every $S \subseteq E$ such that $|S| = n+1$ and S contains exactly one skew pair, there is $X \in \mathcal{C}(M)$ such that $\mathrm{supp}(X) \subseteq S$.

Lemma 6.1.5. *If $\mathcal{C}$ is an antisymmetric F-circuit set, then $\underline{\mathcal{C}} := \{\underline{X} : X \in \mathcal{C}\}$ is the set of circuits of an antisymmetric matroid.* □

Remark 6.1.6. Definition 6.1.4 is written differently in Section 1.3. It is straightforward that the definition of antisymmetric F-circuit sets in Section 1.3 implies all conditions of the definition in this section except for Definition 6.1.3(iv). This remaining condition can be derived easily from other conditions. The readers could verify it by mimicking the proof of Lemma 5.1.7, and we omit details.

The set of circuits of an antisymmetric matroid is identified with an antisymmetric $\mathbb{K}$-circuit set. For a field $\mathbb{F}$ and a Lagrangian subspace W of $\mathbb{F}^E$, let $\mathcal{C}$ be the set of nonzero vectors X in W such that $\underline{X}$ is minimal and $\underline{X}$ contains at most one skew pair. Then $\mathcal{C}$ is an antisymmetric $\mathbb{F}$-circuit set. Conversely, if $\mathcal{C}' \subseteq \mathbb{F}^E$ is an antisymmetric $\mathbb{F}$-circuit set, then the span of $\mathcal{C}'$ is isotropic by (ΔOrth*) and has dimension n by (ΔMax*).

6.1.3 Basic examples

We first recall basic examples when $F = \mathbb{K}$ is the Krasner hyperfield or $F = \mathbb{F}$ is a field.

Example 6.1.7. An antisymmetric $\mathbb{K}$-matroid is identified with the set of bases of an antisymmetric matroid. An antisymmetric $\mathbb{K}$-circuit set is identified with the set of circuits of an antisymmetric matroid.

Example 6.1.8. Let $\mathbb{F}$ be a field. Then an antisymmetric $\mathbb{F}$-matroid is equal to a point in the projective space of dimension $2^{n-2}(4 + \binom{n}{2}) - 1$ satisfying the restricted G-P relations (rGP). For a Lagrangian subspace W in $\mathbb{F}^{[n]\cup[n]^*}$, if $\mathcal{C}$ is the set of vectors X in $W \setminus \{\mathbf{0}\}$ such that $\underline{X}$ is minimal and $\underline{X}$ contains at most one skew pair, then $\mathcal{C}$ is an antisymmetric $\mathbb{F}$-circuit set.

We give an explicit explanation of why Theorem 6.4.1 implies Theorems 1.1.21 and 4.2.2. Let $W \in \mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ and let M_1 be an antisymmetric $\mathbb{F}$-circuit set obtained by collecting all nonzero vectors X in W such that $\underline{X}$ is minimal and $\underline{X}$ contains at most one skew pair. Note that W is equal to the span of M_1. Let $M_2 := \Phi(W)$ be an antisymmetric $\mathbb{F}$-matroid. By Proposition 4.2.19, their underlying antisymmetric matroids $\underline{M_1}$ and $\underline{M_2}$ are different, but $\mathcal{B}(\underline{M_1}) = \mathcal{B}(\underline{M_2})^*$. Note that $W^* := \{X^* : X \in W\}$ is also a Lagrangian subspace in k^E, where $X^* \in \mathbb{F}^E$ such that $X^*(i) = X(i^*)$ for each $i \in E$. Therefore, the equivalence of antisymmetric $\mathbb{F}$-circuit sets and antisymmetric $\mathbb{F}$-matroids implies that Φ is a parameterization of $\mathrm{SpGr}_{\mathbb{F}}(n, 2n)$ and its image is set-theoretically cut out by the restricted G-P relations (rGP).

6.2 Connections to other concepts

6.2.1 Matroids with coefficients

We show that every F-matroid naturally induces an antisymmetric F-matroid, extending a map $\mathrm{Gr}_{\mathbb{F}}(r, n) \to \mathrm{SpGr}_{\mathbb{F}}(n, 2n)$; $V \mapsto V \oplus V^\perp$ where $V^\perp$ is the orthogonal complement of V with respect to the usual inner product.

Lemma 6.2.1. *Let ψ be a Grassmann-Plücker function of rank r on $[n]$ with coefficients in a tract F. Then a function $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ such that for each $B \in \mathcal{T}_n \cup \mathcal{A}_n$,*

$$\varphi(B) = \begin{cases} \psi(B \cap [n]) \cdot \psi^\perp(B^* \cap [n]) & \text{if } |B \cap [n]| = r, \\ 0 & \text{otherwise,} \end{cases}$$

is a restricted G-P function.

Proof. We first show that φ satisfies (rGP2). Let $A \in \mathcal{A}_n$ such that $A = (B - i + j) \cup ([n] \setminus B)^*$ for some $B \in \binom{[n]}{r}$, $i \in B$, and $j \in [n] \setminus B$. Note that $\{i, i^*\} \cap A = \emptyset$, $\{j, j^*\} \subseteq A$, and $\mathrm{sign}(B) \cdot \mathrm{sign}(B - i + j) = (-1)^m$ where $m := 1 + \sum_{z \in \{i, i^*, j, j^*\}} |(A + \{i, i^*\}) < z|$. Then

$$\begin{aligned} \varphi((B - i + j) \cup ([n] \setminus B)^*) &= \psi(B - i + j) \cdot \psi^\perp([n] \setminus B) \\ &= (-1)^m \cdot \psi^\perp([n] \setminus (B - i + j)) \cdot \psi(B) \\ &= (-1)^m \cdot \varphi(B \cup ([n] \setminus (B - i + j))^*). \end{aligned}$$

Now we claim (rGP3). Let $S \in \binom{[n]\cup[n]^*}{n+1}$ and $T \in \binom{[n]\cup[n]^*}{n-1}$ such that S contains exactly one skew pair and T has no skew pair. Let $S_1 = S \cap [n]$, $S_2 = S \cap [n]^*$, $T_1 = T \cap [n]$, and $T_2 = T \cap [n]^*$. We can assume that either $|S_2| = |T_2| = n - r$ or $|S_1| = |T_1| = r$. In the former case, $|S_1| = r+1$ and $|T_1| = r-1$, and thus

$$\sum_{x \in S \setminus T} (-1)^{|S<x|+|T<x|} \varphi(S-x)\varphi(T+x)$$
$$= \psi^{\perp}(S_2)\psi^{\perp}(T_2) \sum_{x \in S_1 \setminus T_1} (-1)^{|S_1<x|+|T_1<x|} \psi(S_1-x)\psi(T_1+x) \in N_F.$$

The latter case holds similarly. □

The *F-Grassmannian* $\mathrm{Gr}_F(r,n)$ is the set of F-matroids of rank r on $[n]$. If $F = \mathbb{F}$ is a field, then it is the ordinary Grassmannian over $\mathbb{F}$. Note that $\mathbb{S}$-matroids and $\mathbb{T}$-matroids are identical with oriented matroids [11] and valuated matroids [54], respectively. Hence, the $\mathbb{T}$-Grassmannian $\mathrm{Gr}_{\mathbb{T}}(r,n)$ equals to the *Dressian* $\mathrm{Dr}(r,n)$ that is the set of valuated matroids of rank r on $[n]$.

Theorem 6.2.2. *Let F be a tract. There is an injective map $\mathrm{Gr}_F(r,n) \to \mathrm{SpGr}_F(n,2n)$ such that the following diagram commutes,*

$$\begin{array}{ccc} \mathrm{Gr}_F(r,n) & \longrightarrow & \mathrm{SpGr}_F(n,2n) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{\mathbb{K}}(r,n) & \longrightarrow & \mathrm{SpGr}_{\mathbb{K}}(n,2n) \end{array}$$

where the vertical arrows mean taking underlying matroids or underlying antisymmetric matroids. □

By Theorem 6.2.2, every oriented matroid is an oriented antisymmetric matroid. Also, every valuated matroid is a valuated antisymmetric matroid, equivalently, the Dressian $\mathrm{Dr}(r,n) = \mathrm{Gr}_{\mathbb{T}}(r,n)$ is a subset of the Lagrangian Grassmannian $\mathrm{SpGr}_{\mathbb{T}}(n,2n)$ over the tropical hyperfield.

Remark 6.2.3. In [19, Page 78], it is asked whether one can develop a theory of oriented symplectic matroids. Note that Lagrangian (symplectic) matroids[1] are exactly symmetric matroids. Booth, Borovik, Gelfand, and White [16] introduced oriented Lagrangian (symplectic) matroids, but their notion does not generalize oriented matroids; see [16, Page 640]. In this sense, our theory of antisymmetric F-matroids is interesting as oriented antisymmetric matroids naturally encompass oriented matroids.

Remark 6.2.4. The F-matroids have several cryptomorphic definitions [4, 2] in terms of F-circuits and F-vectors as we reviewed in Section 3.2. It is straightforward to show that a dual pair of F-signature of a matroid M induces an antisymmetric F-circuit set such that the circuit set of the underlying antisymmetric matroid is exactly $\mathcal{C}(\mathrm{ant}(M))$, which extends the diagrams in Section 4.2.1 and in Theorem 6.2.2 more directly.

6.2.2 Even symmetric matroids with coefficients

We define orthogonal matroids with coefficients in tracts in Chapter 5 as a generalization of both matroids with coefficients in tracts and the Lagrangian orthogonal Grassmannian $\mathrm{OGr}_{\mathbb{F}}(n,2n)$. Let M be an orthogonal F-matroid. Recall that by Corollary 4.1.51 and Theorem 4.2.9, the circuit set of an orthogonal matroid is the circuit set of an antisymmetric matroid. Therefore, if we consider M as an orthogonal F-signature of $\underline{M}$ (Definition 5.1.8), then it satisfies all axioms for antisymmetric F-matroids (Definition 6.1.4) except for (ΔOrth*). The only difference between (ΔOrth*) and (O) is that the former

[1] For the Lagrangian case, the word 'symplectic' is often omitted; see [18, Page 81].

contains the negative signs whereas the latter does not. Hence, these two axioms (ΔOrth*) and (O) are the same whenever $1 = -1$ in a tract F. Therefore, we deduce the next proposition, which generalizes the trivial fact that over a field of characteristic two, every skew-symmetric matrix (with zero diagonals) is a symmetric matrix.

Proposition 6.2.5. *Let F be a tract with $-1 = 1$. Then an orthogonal F-matroid is an antisymmetric F-matroid.* □

Example 6.2.6. The tropical hyperfield $\mathbb{T}$ is a tract with $-1 = 1$. Thus, every valuated orthogonal matroid in [71] is a valuated antisymmetric matroid.

6.2.3 Symplectic Dressian and isotropic tropical linear spaces

Balla and Olarte [9] introduced the symplectic Dressian $\mathrm{SpDr}(r, 2n)$ as a tropical counterpart to the symplectic Grassmannian, which is based on the work of De Concini [45] showing that the symplectic Grassmannian is cut out by the Grassmann-Plücker relations together with certain linear relations. We show that each tropical symplectic Plücker vector in $\mathrm{SpDr}(n, 2n)$ naturally induces a valuated antisymmetric matroid by restricting the coordinates of the vector to transversals and almost-transversals. They also investigated isotropic tropical linear spaces, which were first introduced by Rincón [102] to study valuated even delta-matroids. We show that if we collect certain minimal vectors in an isotropic tropical linear space associated with a rank-n valuated matroid on $2n$ elements, then it forms an antisymmetric $\mathbb{T}$-circuit set. It is analogous to a relation between Lagrangian subspaces over a field $\mathbb{F}$ and antisymmetric $\mathbb{F}$-circuit sets shown in Example 6.1.8

Valuated matroids were first introduced by Dress and Wenzel [54], and they are equivalent to $\mathbb{T}$-matroids. For integers $0 \leq r \leq n$, the Dressian $\mathrm{Dr}(r, n)$ is the set of valuated matroids of rank r on $[n]$. A *tropical linear space* is a subset of $\mathbb{T}^n$ defined as

$$L_\mu = \left\{ x \in \mathbb{T}^n : \sum_{i \in S} \mu_{S-i} x_i \in N_{\mathbb{T}} \text{ for every } S \in \binom{[n]}{r+1} \right\}$$

for some valuated matroid $[\mu] \in \mathrm{Dr}(r, n)$. In the remainder of this section, we regard the ground set of valuated matroids in $\mathrm{Dr}(r, 2n)$ as $[n] \cup [n]^*$.

Definition 6.2.7 ([9])**.** A *tropical symplectic Plücker vector* is a $\mathbb{T}$-matroid $[\mu] \in \mathrm{Dr}(r, 2n)$ satisfying the *tropical symplectic relations*:

$$\sum_{i \in [n] \setminus (S \cup S^*)} \mu(S + \{i, i^*\}) \in N_{\mathbb{T}}$$

for every subset S of E with $|S| = r - 2$. The *symplectic Dressian* $\mathrm{SpDr}(r, 2n)$ is the set of tropical symplectic Plücker vectors in $\mathrm{Dr}(r, 2n)$.

Definition 6.2.8 ([102, 9])**.** A tropical linear space $L \subseteq \mathbb{T}^{[n] \cup [n]^*}$ is *isotropic* if for all $X, Y \in L$,

$$\alpha(X, Y) = \sum_{i=1}^{n} (X(i)Y(i^*) + X(i^*)Y(i)) \in N_{\mathbb{T}}.$$

Proposition 6.2.9 ([9])**.** *For $n \leq 3$ and $[\mu] \in \mathrm{Dr}(n, 2n)$, the valuated matroid $[\mu]$ is a symplectic Plücker vector if and only if the tropical linear space L_μ is isotropic.*

Proposition 6.2.9 does not hold for $n \geq 4$ by [9, Examples B and C]. We show that symplectic Plücker vectors and isotropic tropical linear spaces naturally produce valuated antisymmetric matroids and antisymmetric $\mathbb{T}$-circuit sets, respectively. We note that valuated antisymmetric matroids and antisymmetric $\mathbb{T}$-circuit sets are equivalent by Theorem 6.4.1 applied to $F = \mathbb{T}$.

Proposition 6.2.10. *If $[\mu] \in \mathrm{SpDr}(n, 2n)$ is a tropical symplectic Plücker vector, then $[\mu|_{\mathcal{T}_n \cup \mathcal{A}_n}]$ is an antisymmetric $\mathbb{T}$-matroid.*

Proof. Let $\mu' := \mu|_{\mathcal{T}_n \cup \mathcal{A}_n}$, i.e., $\mu' : \mathcal{T}_n \cup \mathcal{A}_n \to \mathbb{T}$ such that $\mu'(B) = \mu(B)$ for all $B \in \mathcal{T}_n \cup \mathcal{A}_n$. Since $[\mu] \in \mathrm{Dr}(n, 2n)$, μ' satisfies (rGP3). Since μ satisfies the tropical symplectic relations applied to subtransversals S of size $n-2$, we have $\mu'(S+p) = \mu'(S+q)$ where p, q are distinct skew pairs in $([n] \cup [n]^*) - S$. Therefore, μ' satisfies (rGP2). Thus, it suffices to show that μ' is nontrivial, i.e., $\mu'(B) \neq 0$ for some $B \in \mathcal{T}_n \cup \mathcal{A}_n$.

Since $[\mu]$ is a valuated matroid, there is an n-element subset X of $[n] \cup [n]^*$ such that $\mu(X) \neq 0$. We choose X minimizing $|X \cap X^*|$. We may assume that $\frac{1}{2}|X \cap X^*| \geq 2$. Let p be a skew pair contained in X and let $S := X - p \in \binom{[n] \cup [n]^*}{n-2}$. By the tropical symplectic relations applied to S, we obtain a skew pair $q \neq p$ such that $\mu(S - p + q) \neq 0$. Let $x \in p$ and $y \in q$, and then applying the tropical 3-term Plücker relation to $S + y$ and $(S - p + q) - y$, we deduce that $\mu(S + y - x) \neq 0$ or $\mu(S + y - x^*) \neq 0$. It contradicts the minimality. □

Proposition 6.2.11. *Let $L_\mu \subseteq \mathbb{T}^{[n] \cup [n]^*}$ be a tropical linear space associated with a rank-n valuated matroid $[\mu] \in \mathrm{Dr}(n, 2n)$. Let $\mathcal{C}$ be the set of vectors $X \in L_\mu \setminus \{\mathbf{0}\}$ such that $\underline{X}$ is minimal and contains at most one skew pair. Then if L_μ is isotropic, then $\mathcal{C}$ is an antisymmetric $\mathbb{T}$-circuit set.*

Proof. Clearly, $\mathcal{C}$ is prepared and satisfies (ΔOrth*). Because $[\mu]$ has rank n, for every $(n+1)$-element subset S of $[n] \cup [n]^*$, there is a vector $X \in L_\mu$ such that $\underline{X} \subseteq S$. Hence $\mathcal{C}$ satisfies (ΔMax*). □

6.2.4 Oriented gaussoids

We show that every oriented gaussoid [13, Section 5] is an oriented antisymmetric matroid. We review the definition of oriented gaussoids first. Let Σ be an n-by-n symmetric matrix that is real and positive definite. For $L \subseteq [n]$ and distinct $i, j, k \in [n] \setminus L$, the following equation holds:

$$\det(\Sigma[L, L]) \det(\Sigma[L + ij, L + ik]) - \det(\Sigma[L + i, L + i]) \det(\Sigma[L + j, L + k]) + \det(\Sigma[L + i, L + j]) \det(\Sigma[L + i, L + k]) = 0.$$

We remark that this equation corresponds to the edge relations for $[I_n \mid \Sigma]$. For $I \subseteq [n]$, let p_I be an unknown representing the $I \times I$ principal minor of Σ. For $K \subseteq [n]$ and distinct $i, j \in [n] \setminus K$, let $a_{ij|K}$ be an unknown representing the $(K+i) \times (K+j)$ almost-principal minor of Σ. Then $a_{ij|K} = a_{ji|K}$. Let $\mathcal{PA}$ be the set of all such unkowns p_I and $a_{ij|K}$. An oriented gaussoid is defined as follows, which can be regarded as a counterpart of a positive definite symmetric matrix over the sign hyperfield $\mathbb{S}$.

Definition 6.2.12 ([13])**.** An *oriented gaussoid* is a map $\varphi : \mathcal{PA} \to \mathbb{S}$ such that

(i) $\varphi(p_I) = 1$ for every $I \subseteq [n]$, and

(ii) for $L \subseteq [n]$ and distinct $i, j, k \in [n] \setminus L$,

$$p_L a_{jk|L+i} - p_{L+i} a_{jk|L} + p_{ij|L} a_{ik|L} \in N_{\mathbb{S}}.$$

Example 6.2.13. Let $\Sigma := \begin{bmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 1/4 \\ 1/4 & 1/4 & 1 \end{bmatrix}$ be a real symmetric matrix. Then it is positive definite. $\det(\Sigma[12,13]) = 1/8$, $\det(\Sigma[12,23]) = -1/8$, and $\det(\Sigma[13,23]) = 7/16$. Note that there is a tract morphism $\mathbb{R} \to \mathbb{S}$ such that all positives go to $+1$ and all negatives go to -1. This implies that a map $\varphi : \mathcal{PA} \to \mathbb{S}$ such that $\varphi^{-1}(+1) = \mathcal{PA} \setminus \{a_{13|2}\}$ and $\varphi^{-1}(-1) = \{a_{13|2}\}$ is an oriented gaussoid.

Recall that $\det(\Sigma[X,Y]) = (-1)^{\sum_{i \in X}(i+|X \le i|)} \det(\Lambda[n, [n] - X + Y^*])$, where Λ is an $n \times ([n] \cup [n]^*)$ matrix $[I_n \mid \Sigma]$. Therefore, we can define an oriented gaussoid alternatively in terms of transversals and almost-transversals.

Definition 6.2.14. An *oriented gaussoid* is a map $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to \mathbb{S}$ such that

(i) $\varphi([n] - X + X^*) = (-1)^{\sum_{i \in X}(i+|X \le i|)}$ for every $X \subseteq [n]$, and

(ii) for every transversal $S \cup \{a,b,c\}$ with $S \cap \{a,b,c\} = \emptyset$,

$$(-1)^{|L<a|}\varphi(S \cup abc)\varphi(S \cup bb^*c^*) + (-1)^{|L<b^*|}\varphi(S \cup abc^*)\varphi(S \cup bb^*c) \\ + (-1)^{|L<c^*|}\varphi(S \cup abb^*)\varphi(S \cup bcc^*) \in N_{\mathbb{S}}$$

where $L = \{a, c, b^*, c^*\}$.

Note that the formula in Definition 6.2.14(ii) is the edge relations over $\mathbb{S}$. Hence $\varphi^{-1}(0)$ is a gaussoid for every oriented gaussoid $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to \mathbb{S}$. Also, the following proposition is straightforward.

Proposition 6.2.15. *Let $[\varphi]$ be an oriented antisymmetric matroid on $[n] \cup [n]^*$. If $\varphi([n] - X + X^*) = (-1)^{\sum_{i \in X}(i+|X \le i|)}$ for every $X \subseteq [n]$, then φ is an oriented gaussoid.* □

An oriented gaussoid $\varphi : \mathcal{PA} \to \mathbb{S}$ is *realizable* if there is a positive definite real symmetric matrix Σ such that $\varphi(a_{ij|K}) \in \{0, \pm 1\} = \mathbb{S}$ equals the sign of $\det(\Sigma[K+i, K+j])$ for all $K \subseteq [n]$ and distinct $i, j \in [n] \setminus K$. A *positive gaussoid* is an oriented gaussoid $\varphi : \mathcal{PA} \to \mathbb{S}$ such that $\varphi(a_{ij|K}) \in \{0, 1\}$ for every $a_{ij|K}$. Equivalently, a positive gaussoid is an oriented gaussoid $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to \mathbb{S}$ such that $\varphi([n] - X + Y^*) = 0$ or $(-1)^{\sum_{i \in X}(i+|X \le i|)}$ for all $[n] - X + Y^* \in \mathcal{A}_n$.

Ardila, Rincón, and Williams [3] showed that every positively orientable matroid is representable over the real field $\mathbb{R}$. Boege et al. [13] showed an analogous result for positive gaussoids.

Theorem 6.2.16 ([13, Theorem 4])**.** *Every positive gaussoid is realizable.*

By the previous theorem together with Proposition 6.2.15, we deduce the following.

Corollary 6.2.17. *Let $M = [\varphi]$ be an oriented antisymmetric matroid. Suppose that*

(i) $\varphi([n] - X + X^*) = (-1)^{\sum_{i \in X}(i+|X \le i|)}$ *for every* $X \subseteq [n]$*, and*

(ii) $\varphi([n] - X + Y^*) = 0$ *or* $(-1)^{\sum_{i \in X}(i+|X \le i|)}$ *for all* $[n] - X + Y^* \in \mathcal{A}_n$*.*

Then there is an antisymmetric matroid N over $\mathbb{R}$ such that $f_* N = M$, where $f : \mathbb{R} \to \mathbb{S}$ is a tract morphism sending all positives to 1 and all negatives to -1. □

It is interesting to show whether the condition (i) can be weakened to that $\varphi([n] - X + X^*) = 0$ or $(-1)^{\sum_{i \in X}(i+|X \le i|)}$ for every $X \subseteq [n]$.

6.3 Proof of Homotopy Theorem 4.2.29

We prove the Homotopy Theorem for antisymmetric matroids (Theorem 4.2.29), which is a key tool for showing the cryptomorphism on antisymmetric F-matroids (Theorem 6.4.1).

Let M be an antisymmetric matroid. For a subgraph G of the transversal base graph $\mathcal{G}_M$ and two vertices B, B' in G, let $\mathrm{dist}_G(B, B')$ be the smallest sum $\eta(P) = \sum_{e\in E(P)} \eta(e)$ among all paths P from B to B' in G. For convenience, we write $\mathrm{dist}_M(B, B')$ for $\mathrm{dist}_{\mathcal{G}_M}(B, B')$.

Lemma 6.3.1. *The following hold.*

(i) $\mathrm{dist}_M(B, B') = |B \setminus B'|$.

(ii) *For every cycle C in $\mathcal{G}_M$, the weight $\eta(C) = \sum_{e\in E(C)} \eta(e)$ is even.*

Proof. (i) Clearly, $\mathrm{dist}_M(B, B') \geq |B \setminus B'|$. We claim $\mathrm{dist}_M(B, B') \leq |B \setminus B'|$ by induction on $|B \setminus B'|$, which trivially holds for $|B \setminus B'| = 0$. We may assume $B \neq B'$. By (Exch), there are $e \in B \setminus B'$ and $f \in B' \setminus B$ such that $B'' := B - e + f \in \mathcal{B}(M)$. By the induction hypothesis, $\mathrm{dist}_M(B, B') \leq \mathrm{dist}_M(B, B'') + \mathrm{dist}_M(B'', B') \leq 1 + |B'' \setminus B'| = |B \setminus B'|$.

(ii) Fix a vertex B_0 of $\mathcal{G}_M$ and for each nonnegative integer i, let V_i be the set of vertices B such that $|B \setminus B_0| = i$. By interchainging j and j^* for some $j \in [n] \cup [n]^*$, we may assume that $B_0 = [n]$. Then $V_i = \{B \in \mathcal{B} : |B \cap [n]^*| = i\}$. Then it is easily observed that for every edge BB' in $\mathcal{G}_M$ such that $|B \setminus B_0| \leq |B' \setminus B_0|$,

(a) $\eta(BB') = 1$ if and only if $B \in V_i$ and $B' \in V_{i+1}$ for some i, and

(b) $\eta(BB') = 2$ if and only if $B \in V_i$ and $B' \in V_i \cup V_{i+2}$ for some i.

Thus, if a path P is from B_0 to a vertex in V_i, then $\eta(P) \equiv i \pmod 2$. Then it is straightforward that for every cycle C, its weight $\eta(C)$ is even. □

We denote the homology group of $\mathcal{G}_M$ by H_M. We call a cycle C in $\mathcal{G}_M$ is *reducible* if, in H_M, it can be generated by the cycles of weight smaller than $\eta(C)$. Otherwise, we say C is *irreducible*.

Lemma 6.3.2. *Let C be an irreducible cycle in $\mathcal{G}_M$. Then for every pair of vertices B, B' in C,* $\mathrm{dist}_C(B, B') = \mathrm{dist}_M(B, B')$.

Proof. Suppose not. We take a pair of distinct vertices B, B' in C and a path P from B to B' in C such that

(i) $\eta(P) \leq \eta(C)/2$,

(ii) $\eta(P) > \mathrm{dist}_M(B, B')$, and

(iii) subject to (i) and (ii), $\mathrm{dist}_M(B, B')$ is minimized.

Let P' be the path from B to B' in C other than P, and let Q be a path from B to B' in $\mathcal{G}_M$ such that $\eta(Q) = \mathrm{dist}_M(B, B')$. By (iii), no internal vertex of Q is in C. Then the cycle induced by paths P and Q have weight $\eta(P) + \eta(Q) < 2\eta(P) \leq \eta(C)$, and the cycle induced by paths P' and Q have weight $\eta(P') + \eta(Q) < \eta(P') + \eta(P) = \eta(C)$. It contradicts that C is irreducible. □

Lemma 6.3.3. *Let C be a cycle of weight 2ℓ in $\mathcal{G}_M$ and let $B_0 \in V(C)$. If C is irreducible, then either*

- *there is $B \in V(C)$ such that* $\mathrm{dist}_M(B_0, B) = \ell$ *or*

- *there is* $BB' \in E(C)$ *such that* $\mathrm{dist}_M(B_0, B) = \mathrm{dist}_M(B_0, B') = \ell - 1$ *and* $\eta(BB') = 2$. □

Proof of Theorem 4.2.29. Suppose to the contrary that $\mathcal{G}_M$ has irreducible cycles of weight larger than 8. Among such cycles, we choose C such that

(i) its weight $\eta(C)$ is minimized, and

(ii) subject to (i), the number $|E(C) \cap \eta^{-1}(1)|$ of wieght 1 edges in C is minimized.

We denote by $\eta(C) = 2\ell > 8$. Then all cycles of weight $2\ell'$ with $4 < \ell' < \ell$ are reducible. We select an arbitrary vertex $B_0 \in V(C)$. There are two cases by Lemma 6.3.3.

Case I. There is a vertex B in C such that $\mathrm{dist}_M(B_0, B) = \ell$. Let B_1 and B_2 be two distinct neighbors of B in C, and let P be a path in C from B_1 to B_2 containing B_0.

Subcase I.1. $\eta(BB_1) = \eta(BB_2) = 1$. Then there are distinct elements $e, f \in B$ such that $B_1 = B \triangle \{f, f^*\}$ and $B_2 = B \triangle \{e, e^*\}$. By (ii), B_1B_2 is not an edge in $\mathcal{G}_M$ and hence $B + e^* - f$ is not a base of M. Thus, by the base exchange (Exch) applied to B_1 and B_2, $B' := B \triangle \{e, e^*, f, f^*\}$ is a base; see Figure 6.1(top left). Then B_1B', B_2B' are edges in $\mathcal{G}_M$ of weight 1 and $\mathrm{dist}_M(B_0, B') = \ell - 2$. If X is a vertex in C such that $\mathrm{dist}_M(B_0, X) = \ell - 2$, then $\mathrm{dist}_C(B_i, X) = 3$ for some $i \in \{1, 2\}$ and by Lemma 6.3.2, $\mathrm{dist}_M(B_i, X) = 3$. Thus, $B' \notin V(C)$. Let C' be a cycle concatenating two paths P and $B_1B'B_2$. Then $\mathrm{dist}_{C'}(B_0, B') = \ell$ and thus C' is reducible by Lemma 6.3.2. Let $C'' := BB_1B'B_2B$ be a cycle of weight 4. As $C = aC' + bC''$ in H_M for some $a, b \in \{1, -1\}$, it contradicts that C is irreducible.

Subcase I.2. $\eta(BB_1) = 2$ and $\eta(BB_2) = 1$. Then there are elements $e, f, g \in B$ such that $B_1 = B \triangle \{f, f^*, g, g^*\}$ and $B_2 = B \triangle \{e, e^*\}$. Because $\mathrm{dist}_M(B_1, B_2) = \mathrm{dist}_C(B_1, B_2) = 3$, the elements e, f, g are distinct. By (Exch) applied to B_1 and B_2, there is $h \in \{e, f^*, g^*\} = B_1 \setminus B_2$ such that $B_1 + e^* - h$ is a base of M.

Suppose that $B' := B_1 + e^* - e$ is a base; see Figure 6.1(top middle). Then $\mathrm{dist}_M(B_2, B') = 2$ and $\mathrm{dist}_M(B_0, B') = \ell - 3$. If X is a vertex in C such that $\mathrm{dist}_M(B_0, X) = \ell - 3$, then either $\mathrm{dist}_C(B_2, X) = 4$ or $\mathrm{dist}_C(B_1, X) = 5$ and by Lemma 6.3.2, $\mathrm{dist}_M(B_2, X) = 4$ or $\mathrm{dist}_M(B_5, X) = 5$. Thus, $B' \notin V(C)$. Let Q be a path from B_2 to B' of weight $\mathrm{dist}_M(B_2, B') = 2$. Then by Lemma 6.3.2, a cycle induced by P, Q, and $B'B_1$ is reducible. Since a cycle induced by two paths Q and $B'B_1BB_2$ has weight 6, we deduce that C is reducible, a contradiction. Therefore, $B_1 + e^* - e$ is not a base. Then $h \neq e$.

By symmetry, we can assume that $h = f^*$. By Lemma 4.2.5, one of two transversals $(B_1 + e^* - f^*) - e + f$ and $(B_1 + e^* - f^*) - e + f^*$ is a base. Hence $B'' := (B_1 + e^* - f^*) - e + f = B \triangle \{e, e^*, g, g^*\}$ is a base; see Figure 6.1(top right). Then B_1B'' is an edge of weight 2, B_2B'' is an edge of weight 1, and $\mathrm{dist}_M(B_0, B'') = \ell - 2$. Similar to before, we can deduce that $B'' \notin V(C)$. Let Q be a path from B_2 to B'' of weight $\mathrm{dist}_M(B_2, B'') = 2$. Then by Lemma 6.3.2, a cycle induced by P, Q, and $B''B_1$ is reducible. A cycle consisting of two paths Q and $B''B_1BB_2$ has weight 6. This contradicts that C is irreducible.

Subcase I.3. $\eta(BB_1) = \eta(BB_2) = 2$. Then there are four elements $e, f, g, h \in B$ such that $B_1 = B \triangle \{g, g^*, h, h^*\}$ and $B_2 = B \triangle \{e, e^*, f, f^*\}$. Because $\mathrm{dist}_M(B_1, B_2) = \mathrm{dist}_C(B_1, B_2) = 4$, these four elements are distinct; see Figure 6.1(bottom). By the base exchange (Exch), there is $i \in \{e, f, g^*, h^*\} = B_1 \setminus B_2$ such that $B_1 + e^* - i$ is a base.

Suppose that $B' := B_1 + e^* - e$ is a base. Let Q be a path from B' to B_2 of weight $\mathrm{dist}_M(B', B_2) = 3$. By Lemma 6.3.2 applied to C, B_2 is the only vertex in both paths P and Q. A cycle consisting of two paths Q and $B'B_1BB_2$ has weight 8. A cycle consisting of P, Q, and $B'B_1$ is reducible by Lemma 6.3.2. It contradicts that C is irreducible. Therefore, $B_1 + e^* - e$ is not a base and $i \neq e$.

Suppose that $i = f$. By Lemma 4.2.5, $(B_1 + e^* - f) - e + f$ or $(B_1 + e^* - f) - e + f^*$ is a base. Hence $B'' := (B_1 + e^* - f) - e + f^*$ is a base. Then $\mathrm{dist}_M(B_0, B'') = \ell - 4$ and $B'' \notin V(C)$. Let Q be a path from B'' to B_2 of weight $\mathrm{dist}_M(B_2, B'') = 2$. Then $V(P) \cap V(Q) = \{B_2\}$ by Lemma 6.3.2. Then the union of C, Q, and $B''B_1$ has two cycles other than C. One has weight 8 and the other is reducible by Lemma 6.3.2, a contradiction. Thus, $i \in \{g^*, h^*\}$.

By symmetry, we can assume that $i = g^*$. By Lemma 4.2.5, $(B_1+e^*-g^*)-e+g$ or $(B_1+e^*-g^*)-e+g^*$ is a base. Hence $B''' := (B_1 + e^* - g^*) - e + g = B \triangle \{e, e^*, h, h^*\}$ is a base. Then $\mathrm{dist}_M(B_0, B''') = \ell - 2$ and $B''' \notin V(C)$. Similarly, Lemma 6.3.2 yields a contradiction, and we skip details.

Case II. There is an edge BB' in C such that $\mathrm{dist}_M(B_0, B) = \mathrm{dist}(B_0, B') = \ell - 1$ and $\eta(BB') = 2$. Then $B' = B \triangle \{e, e^*, f, f^*\}$ for some distinct elements $e, f^* \in B$. Let B_1 be the neighbor of B in C which is not B'.

Subcase II.1. $\eta(BB_1) = 2$. Then $B_1 = B \triangle \{g, g^*, h, h^*\}$ for some distinct $g, h \in B \setminus \{e, f^*\}$; see Figure 6.2. Let P be a path in C from B_1 to B' containing B_0. By the base exchange (Exch), there is $i \in B_1 \setminus B' = \{e, f^*, g^*, h^*\}$ such that $B_1 + e^* - i$ is a base of M.

Suppose that $D := B_1 + e^* - e$ is a base. Then $\mathrm{dist}_M(B_0, D) = \ell - 4$ and $D \notin V(C)$. Let Q be a path from B' to D of weight $\mathrm{dist}_M(B', D) = 3$. Note that $V(P) \cap V(Q) = \{B'\}$ by Lemma 6.3.2 applied to C. Then a cycle induced by P, Q, and DB_1 is reducible by Lemma 6.3.2, and a cycle concatenating two paths Q and DB_1BB' has length 8. It contradicts that C is irreducible. Therefore, $B_1 + e^* - e$ is not a base and $i \neq e$.

Suppose that $i = f^*$. By Lemma 4.2.5, $(B_1 + e^* - f^*) - e + f$ or $(B_1 + e^* - f^*) - e + f^*$ is a base. Hence $D' := (B_1 + e^* - f^*) - e + f = B' \triangle \{g, g^*, h, h^*\}$ is a base. Then $\mathrm{dist}_M(B_0, B'') = \ell - 3$ and $D' \notin V(C)$. Using Lemma 6.3.2, we can similarly conclude that C is reducible, a contradiction. Thus, $i \neq f^*$ and so i is either g^* or h^*.

By symmetry, we may assume that $i = g^*$. By Lemma 4.2.5, $D'' := (B_1 + e^* - g^*) - e + g = B' \triangle \{f, f^*, h, h^*\}$ is a base. Then $\mathrm{dist}_M(B_0, D'') = \ell - 2$ and $D'' \notin V(C)$. Using Lemma 6.3.2, similarly one can deduce a contradiction.

Subcase II.2. $\eta(BB_1) = 1$. Let B_2 be the neighbor of B' in $V(C)$ other than B. By Subcase II.1, we may assume that $\eta(BB_2) = 1$. Then for some distinct elements $g, h \in B \cap B'$, we have $B_1 = B \triangle \{h, h^*\}$ and $B_2 = B' \triangle \{g, g^*\}$; see Figure 6.3. Let P be a path in C from B_1 to B_2 containing B_0. By the base exchange (Exch), there is $i \in B_1 \setminus B_2 = \{e, f^*, g, h^*\}$ such that $B_1 + e^* - i$ is a base of M.

Suppose that $D := B_1 + e^* - e$ is a base. Then $\mathrm{dist}_M(B_0, D) = \ell - 3$ and $D \notin V(C)$. Let Q be a path from B_2 to D of weight $\mathrm{dist}_M(B_2, D) = 3$. By Lemma 6.3.2, P and Q only meet at a vertex B_2. A cycle consisting of two paths $DB_1BB'B_2$ and Q has length 8, and a cycle consisting of P, Q, and DB_1 is reducible by Lemma 6.3.2. It contradicts that C is irreducible. Therefore, $B_1 + e^* - e$ is not a base and $i \neq e$.

Suppose that $i = f^*$. By Lemma 4.2.5, $D' := (B_1 + e^* - f^*) - e + f = B_2 \triangle \{g, g^*, h, h^*\}$ is a base. Then $\mathrm{dist}_M(B_0, B'') = \ell - 2$ and $D' \notin V(C)$. Similarly, we can deduce a contradiction using Lemma 6.3.2. Thus, we can assume that $i \neq f^*$.

Suppose that $i = h^*$. By Lemma 4.2.5, $D'' := (B_1 + e^* - h^*) - e + h = B_2 \triangle \{f, f^*, g, g^*\}$ is a base. Then $\mathrm{dist}_M(B_0, D'') = \ell - 2$ and $D'' \notin V(C)$. So, we can deduce a contradiction similarly as before.

Thus, we may assume that $i = g$. By Lemma 4.2.5, $D''' := (B_1 + e^* - g) - e + g^*$ is a base. Then $\mathrm{dist}_M(B_0, D''') = \ell - 4$ and $D''' \notin V(C)$. One can deduce a contradiction similarly as before. □

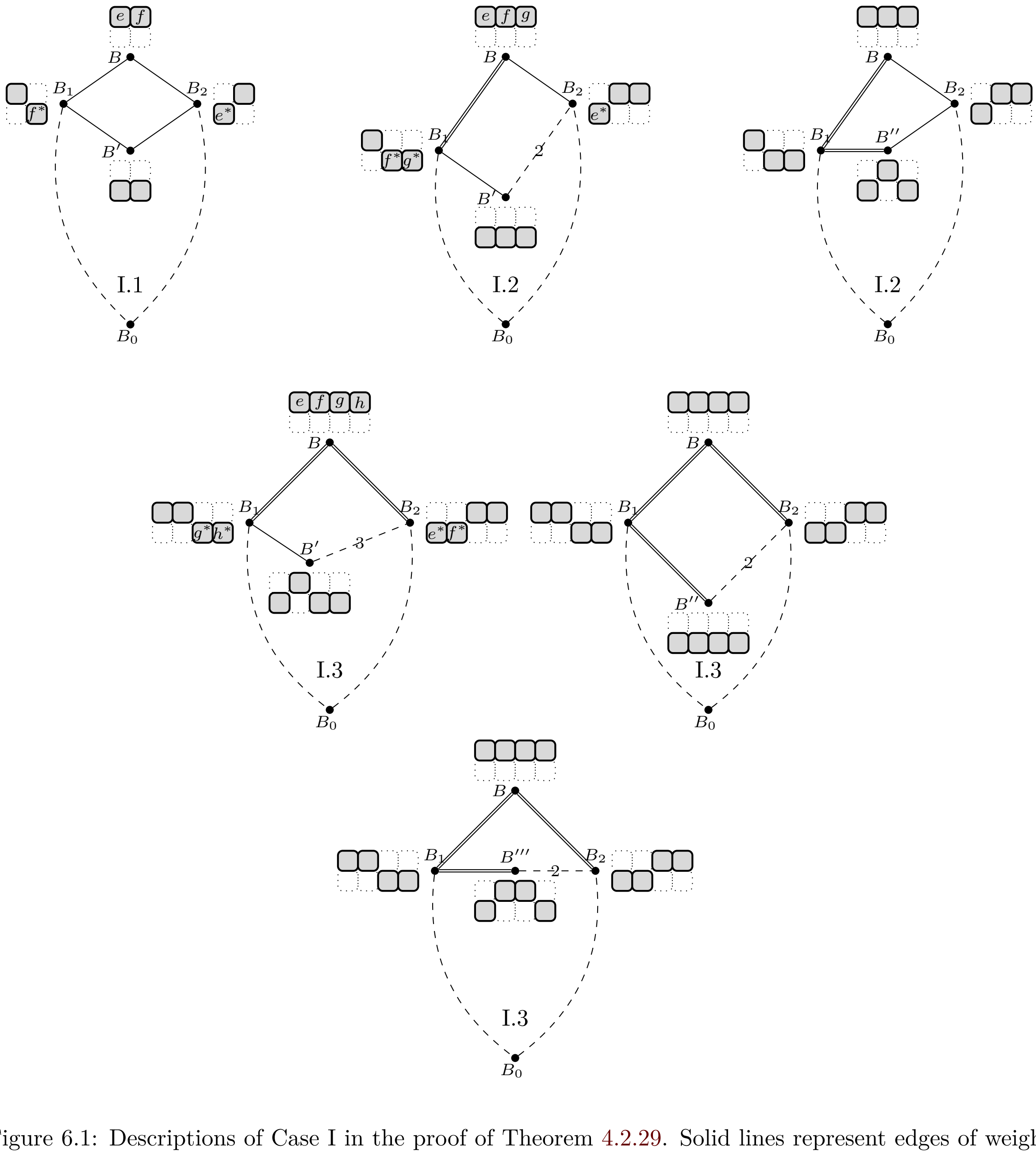


Figure 6.1: Descriptions of Case I in the proof of Theorem 4.2.29. Solid lines represent edges of weight 1, double lines represent edges of weight 2, and dashed lines represent paths in $\mathcal{G}_M$.

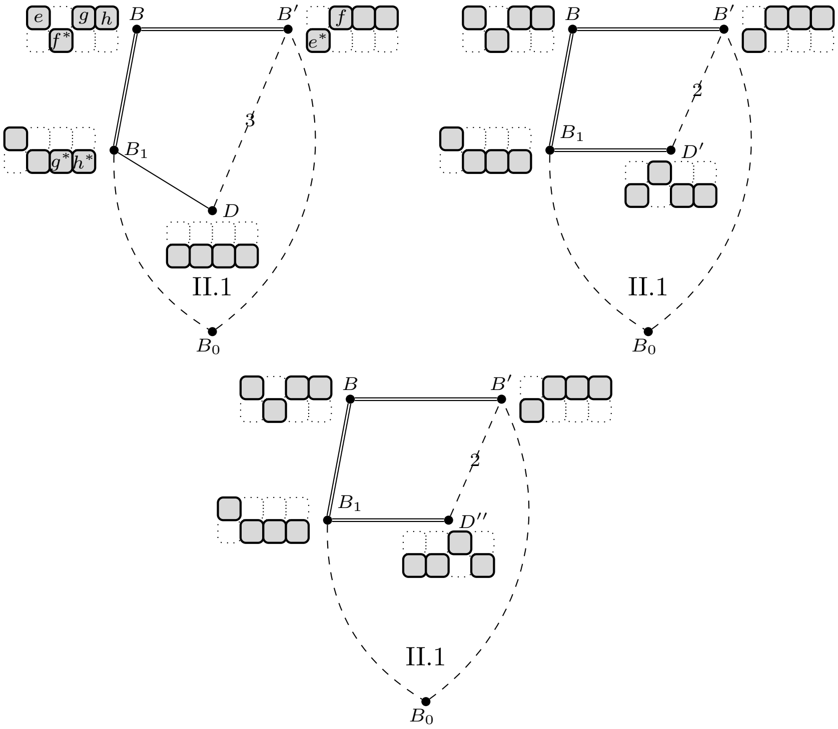


Figure 6.2: Descriptions of Case II.1 in the proof of Theorem 4.2.29.

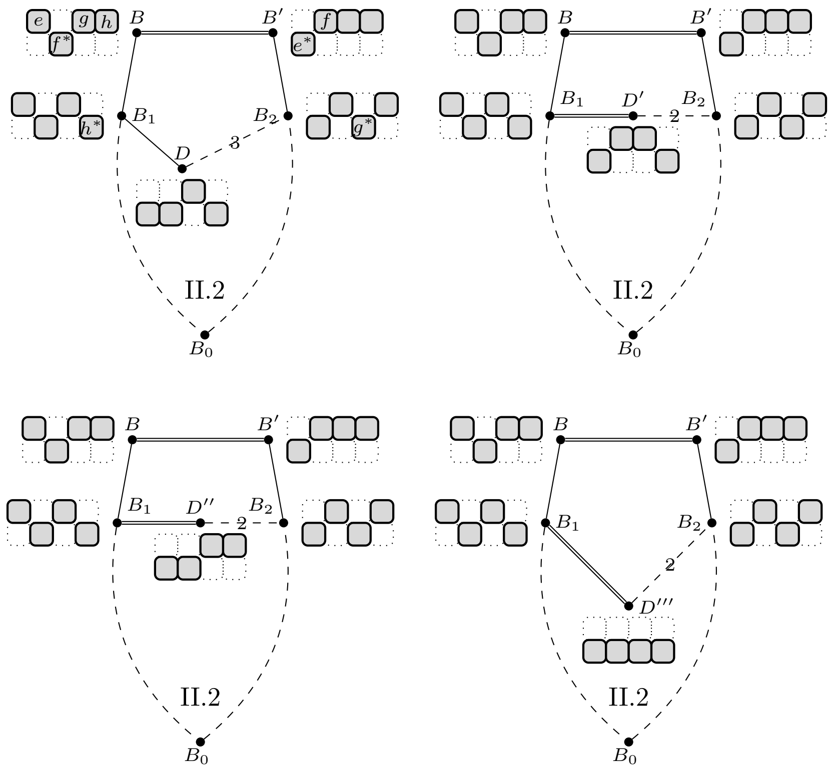


Figure 6.3: Descriptions of Case II.2 in the proof of Theorem 4.2.29.

6.4 Cryptomorphism

In Section 6.1, we introduced two concepts generalizing Lagrangian subspaces in the standard symplectic vector space. First, an antisymmetric F-matroid is defined by the restricted G-P relations over tracts. Second, an antisymmetric F-circuit set is defined as a maximal set of vectors that are orthogonal to each other, subject to a certain condition on their supports. We show that those two notions are equivalent.

Theorem 6.4.1. *There is a natural bijection between antisymmetric F-matroids and antisymmetric F-circuit sets.*

A proof of Theorem 6.4.1 is provided in Subsection 6.4.3. Theorem 6.4.1 generalizes not only the cryptomorphism on antisymmetric matroids but also the parametrization of the Lagrangian Grassmannian into the projective space of dimension $2^{n-2}(n + \binom{n}{2}) - 1$, as explained in Subsection 6.1.3. In Subsection 6.4.1, we construct an antisymmetric F-circuit set from an antisymmetric F-matroid. In Subsection 6.4.2, we oppositely build an antisymmetric F-matroid from an antisymmetric F-circuit set. Those two constructions are apparently a reverse step of each other, and hence we deduce Theorem 6.4.1.

6.4.1 Constructing an antisymmetric F-circuit set

In this subsection, we let $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ be a restricted G-P function on $E := [n] \cup [n]^*$ with coefficients in a tract F. We denote the underlying antisymmetric matroid of φ by $M = (E, \mathcal{B})$. The goal is to construct an antisymmetric F-circuit set from φ.

Let $S \subseteq E$ be a subset of size $n+1$ which contains exactly one skew pair, say $\{x, x^*\}$. Suppose that $S - x$ or $S - x^*$ is a base. Let $X_S \in F^E$ be a vector defined as follows:

- $\mathrm{supp}(X_S) \subseteq S$ and
- $X_S(y) = (-1)^{\chi(y)+|S<y|}\varphi(S - y)$ for each $y \in S$.

Let $\mathcal{C}$ be the set of all cX_S such that $c \in F^\times$ and $S = B + x^*$ with $B \in \mathcal{B} \cap \mathcal{T}_n$ and $x \in B$.

Lemma 6.4.2. *$\underline{\mathcal{C}} = \{\underline{X} : X \in \mathcal{C}\}$ is the set of circuits of M.*

Proof. Let B be a transversal base of M and let $e \in B^*$. Then by Lemma 4.2.13, the support of X_{B+e} is the fundamental circuit with respect to B and e. By Lemma 4.2.12, every circuit is a fundamental circuit with respect to some transversal base and element, and thus $\underline{\mathcal{C}} = \mathcal{C}(M)$. □

Theorem 6.4.3. *$\mathcal{C}$ is an antisymmetric F-circuit set.*

Proof. It is clear that $\mathcal{C}$ satisfies Definition 6.1.3(i)–(iii). By Lemma 6.4.2, it satisfies (ΔMax*). Next we examine (ΔOrth*), i.e., $\omega(X, Y) \in N_F$ for all $X, Y \in \mathcal{C}$.

Fix $X, Y \in \mathcal{C}$. Let S, T be subsets of size $n + 1$ in E such that $\underline{X} \subseteq S$, $\underline{Y} \subseteq T$, and for some $x, y \in E$, $S - x^*$ and $T - y^*$ are transversal bases. Applying the restricted G-P relation (rGP3) to S and $T' := T \setminus \{y, y^*\}$, we have

$$\sum_{z \in S \setminus T'} (-1)^{|S<z|+|T'<z|} \cdot \varphi(S - z) \cdot \varphi(T' + z) \in N_F.$$

Note that $S \setminus T' = (S \cap \{y, y^*\}) \cup (S \setminus T)$. Let $m := |T' < y| + |T' < y^*|$. Then for each $z \in \{y, y^*\}$, we have $|T < z^*| = |T' < z^*| + \chi(z^*) \equiv |T' < z| + m + \chi(z^*) \pmod 2$ and thus $Y(z^*) = (-1)^{|T<z^*|+\chi(z^*)}\varphi(T - z^*) = (-1)^{|T'<z|+m}\varphi(T' + z)$.

For $z \in S \setminus T$, we have $1 + \sum_{w \in \{y, y^*, z, z^*\}} |(T+z) < w| \equiv |T' < z| + |T < z^*| + \chi(z^*) + m \pmod 2$. Then by (rGP2), $Y(z^*) = (-1)^{\chi(z^*)+|T<z^*|}\varphi(T-z^*) = (-1)^{|T'<z|+m}\varphi(T'+z)$. Therefore,

$$\begin{aligned}\omega(X,Y) &= \sum_{z \in S\setminus T'} (-1)^{\chi(z)} X(z) Y(z^*) \\ &= (-1)^m \sum_{z \in S \setminus T'} (-1)^{|S<z|+|T'<z|} \cdot \varphi(S-z) \cdot \varphi(T'+z) \in N_F.\end{aligned}$$

Finally, we check Definition 6.1.3(iv). Let $X, Y \in \mathcal{C}$ such that $\underline{X} \subseteq \underline{Y}$. By Lemma 6.4.2, $\underline{X} = \underline{Y}$. Choose $e \in \underline{X}$ such that $\underline{X} - e$ is a subtransversal. For each $f \in \underline{X} - e$, M has a circuit C such that $\underline{X} \cap C^* = \{e, f\}$ by Lemma 4.2.17, and let $Z \in \mathcal{C}$ such that $\underline{Z} = C$. Then by (ΔOrth*), $\frac{X(f)}{X(e)} = (-1)^{1+\chi(e)+\chi(f)} \frac{Z(e^*)}{Z(f^*)} = \frac{Y(f)}{Y(e)}$. Since f is arbitrary, we conclude that $X = cY$ for some $c \in F^\times$. □

6.4.2 Constructing an antisymmetric F-matroid

Let $\mathcal{C}$ be an antisymmetric F-circuit set on $E = [n] \cup [n]^*$, and let M be its underlying antisymmetric matroid, i.e., $\mathcal{C}(M) = \underline{\mathcal{C}}$. We will construct a restricted G-P function by approaching a reverse step of Section 6.4.1.

Definition 6.4.4. Let B_1 and B_2 be bases of M such that $S := B_1 \cup B_2$ has exactly one skew pair. We denote by $\{x\} = S \setminus B_1$ and $\{y\} = S \setminus B_2$, and we define

$$\gamma(B_1, B_2) := (-1)^{\chi(x)+\chi(y)+|S<x|+|S<y|} \frac{X(y)}{X(x)},$$

where X is a vector in $\mathcal{C}$ such that $\underline{X} \subseteq S$.

Definition 6.4.5. The *base graph* of M is a graph G_M on $\mathcal{B}(M)$ such that two vertices B and B' are adjacent if and only if $|B \setminus B'| = 1$ and at least one of B and B' is a transversal.

Lemma 6.4.6. *The base graph G_M is connected.*

Proof. It is an immediate consequence of Lemma 4.2.5 and 6.3.1. □

One natural candidate of a restricted G-P function $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ can be constructed as follows.

(i) Fix a base $B_0 \in \mathcal{B}(M)$ and let $\varphi(B_0) = 1 \in F^\times$.

(ii) For each $B \in \mathcal{B}(M)$, let $\varphi(B) = \prod_{i=0}^{k-1} \gamma(B_i, B_{i+1}) \in F^\times$ where $B_0 B_1 \dots B_k$ be a path from B_0 to $B_k := B$ in the base graph G_M.

(iii) For $B \in (\mathcal{T}_n \cup \mathcal{A}_n) \setminus \mathcal{B}(M)$, let $\varphi(B) = 0$.

By similar proof of Theorem 6.4.3, we can show that φ is a restricted G-P function on E with coefficients in F (Theorem 6.4.16). However, the hardest part is to show that φ is well defined, i.e., for different paths $P = B_0 \dots B_k$ and $P' = B'_0 \dots B'_\ell$ from $B_0 = B'_0$ to $B_k = B_{\ell'} = B$, we should prove that $\prod_{i=0}^{k-1} \gamma(B_i, B_{i+1}) = \prod_{i=0}^{\ell-1} \gamma(B'_i, B'_{i+1})$. Henceforth, we devoted most of the subsection to show that φ is well defined using the Homotopy Theorem (Theorem 4.2.29).

Lemma 6.4.7. $\gamma(B_1, B_2) = \gamma(B_2, B_1)^{-1}$ *for each* $B_1 B_2 \in E(G_M)$. □

Lemma 6.4.8. *Let B be a transversal base, and let A and A' be distinct almost-transversal bases such that $A = B + x^* - y$ and $A' = B - x + y^*$ for some $x, y \in B$. Then*

$$\gamma(B, A) = (-1)^m \gamma(B, A'),$$

where $m := 1 + \sum_{z \in \{x, x^*, y, y^*\}} |(B + x^* + y^*) < z|$.

Proof. Let $S = B+x^*$, $T = B+y^*$, and $U = B+x^*+y^*$. Let X, Y be vectors in $\mathcal{C}$ such that $\text{supp}(X) \subseteq S$ and $\text{supp}(Y) \subseteq T$. Then $\omega(X,Y) = (-1)^{\chi(x^*)}X(x^*)Y(x) + (-1)^{\chi(y)}X(y)Y(y^*) \in N_F$. Note that $|S < x^*| + |T < y^*| = |U < x^*| + |U < y^*| - 1$, $|S < y| = |U < y| - \chi(y)$, and $|T < x| = |U < x| - \chi(x)$. Therefore,

$$\gamma(B,A)\gamma(B,A')^{-1} = (-1)^{|S<x^*|+|T<y^*|+|S<y|+|T<x|}\frac{X(y)Y(y^*)}{X(x^*)Y(x)} = (-1)^m. \qquad \square$$

Lemma 6.4.9. *Let $B_1B_2B_3B_4B_1$ be a 4-cycle in G_M. Then $\prod_{i=1}^4 \gamma(B_i, B_{i+1}) = 1$, where $B_5 := B_1$.*

Proof. If B_i is an almost-transversal, then B_{i-1} and B_{i+1} are transversals. Hence, by relabelling, we may assume that B_1 and B_3 are transversals. As $B_1B_2B_3B_4B_1$ is a 4-cycle in G_M, $|B_2 \setminus B_1| = |B_4 \setminus B_1| = 1$ and $|B_3 \setminus B_1| = 2$. Then $B_3 = B_1 \triangle \{x, x^*, y, y^*\}$ for some $x, y \in B$.

Case I. Both B_2 and B_4 are almost-transversals. Then by symmetry, we may assume that $B_2 = B_1+x^*-y = B_3+x-y^*$ and $B_4 = B_1-x+y^* = B_3-x^*+y$. Therefore, $\gamma(B_1,B_2)\gamma(B_2,B_3) = \gamma(B_1,B_4)\gamma(B_4,B_3)$ by Lemma 6.4.8.

Case II. B_2 is a transversal and B_4 is an almost-transversal. Then by symmetry, we may assume that $B_2 = B_1 \triangle \{x, x^*\}$. Then B_4 is either $B_1 - y + x^*$ or $B_1 - x + y^*$. By Case I, we can assume that $B_4 = B_1 - y + x^*$. We denote by $B_4' := B_1 - x + y^*$.

Let $S := B_1 + x^*$ and $T := B_3 + y$. Let $X, Y \in \mathcal{C}$ be vectors such that $\underline{X} \subseteq S$ and $\underline{Y} \subseteq T$. Then

$$\gamma(B_1,B_2)\gamma(B_2,B_3) = (-1)^{|S<x|+|S<x^*|+|T<y|+|T<y^*|}\frac{X(x)}{X(x^*)}\frac{Y(y)}{Y(y^*)},$$
$$\gamma(B_1,B_4)\gamma(B_4',B_3) = (-1)^{|S<y|+|S<x^*|+|T<y|+|T<x^*|}\frac{X(y)}{X(x^*)}\frac{Y(y)}{Y(x^*)}.$$

As $\omega(X,Y) \in N_F$, we have $\frac{X(x)}{Y(y^*)} = (-1)^{\chi(x)+\chi(y)+1}\frac{X(y)}{Y(x^*)}$. Hence $\gamma(B_1,B_2)\gamma(B_2,B_3)\gamma(B_3,B_4')\gamma(B_4,B_1) = (-1)^{1+\chi(x)+\chi(y)+m}$ where $m := |S < x| + |S < y| + |T < y^*| + |T < x^*|$. If $U := S + y^* = T + x$, then $m = \sum_{e\in\{x,x^*,y,y^*\}} |U < e| - (1 + \chi(x^*) + \chi(y))$. Therefore, by Lemma 6.4.8, we obtain the desired equality.

Case III. Both B_2 and B_4 are transversals. Then by symmetry, $B_2 = B_1 \triangle \{x, x^*\} = B_3 \triangle \{y, y^*\}$ and $B_4 = B_1 \triangle \{y, y^*\} = B_3 \triangle \{x, x^*\}$. If $B_1 - x + y^*$ is a base, then applying Case II twice, we can deduce the desired equality. Therefore, we can assume that $B_1 - x + y^*$ is not a base.

Let $X, Y, Z, W \in \mathcal{C}$ be vectors such that $\underline{X} \subseteq B_1 \cup B_2$, $\underline{Y} \subseteq B_2 \cup B_3$, $\underline{Z} \subseteq B_1 \cup B_4$, and $\underline{W} \subseteq B_4 \cup B_3$. Because neither $B_1 - x + y^*$ nor $B_1 + x^* - y$ is a base, $X(y) = Y(x^*) = Z(x) = W(y^*) = 0$. Then $\underline{W} \subseteq (B_4 \cup B_3) - y^* \subseteq B_1 \cup B_2$ and thus $\underline{X} = \underline{W}$ by Lemma 4.2.13. Similarly, $\underline{Y} = \underline{Z}$. Hence $\frac{X(x)}{X(x^*)} = \frac{W(x)}{W(x^*)}$ and $\frac{Y(y)}{Y(y^*)} = \frac{Z(y)}{Z(y^*)}$. Therefore, $\gamma(B_1,B_2)\gamma(B_2,B_3) = \gamma(B_1,B_4)\gamma(B_4,B_3)$. $\square$

Definition 6.4.10. For two transversal bases B_1 and B_2 of M such that $|B_1 \setminus B_2| = 2$, let

$$\gamma(B_1,B_2) := \gamma(B_1,B)\gamma(B,B_2)$$

where B is an arbitrary base such that $BB_1, BB_2 \in E(G_M)$. It is well defined by Lemma 6.4.9.

Recall that the transversal base graph $\mathcal{G}_M$ is a graph on $\mathcal{B}(M) \cap \mathcal{T}_n$ together with weights $\eta(BB') = |B \setminus B'| \in \{1, 2\}$ on its edges BB'. We say two cycles C_1 and C_2 in $\mathcal{G}_M$ are 4-*homotopic*, denoted by $C_1 \simeq C_2$, if there is a sequence of cycles $D_1, \ldots, D_k$ such that

- $D_1 = C_1$,
- $D_k = C_2$, and

- each D_{i+1} is obtained from D_i either

  a. by replacing two edges B_1B_2 and B_2B_3 of weight 1 with an edge B_1B_3 of weight 2, or

  b. by replacing an edge B_1B_3 of weight 2 with two edges B_1B_2 and B_2B_3 of weight 1.

Lemma 6.4.11. *Let C be an irreducible cycle of weight 6 in $\mathcal{G}_M$. Then $C \simeq B_1B_2 \dots B_kB_1$ such that*

(i) *$k = 3$ and $\eta(B_1B_2) = \eta(B_2B_3) = \eta(B_3B_1) = 2$, or*

(ii) *$k = 4$ and $\eta(B_1B_2) = \eta(B_3B_4) = 1$ and $\eta(B_2B_3) = \eta(B_4B_1) = 2$.*

Proof. Let C' be an irreducible cycle 4-homotopic to C, which maximizes the number $|E(C') \cap \eta^{-1}(2)|$ of edges $e \in E(C')$ such that $\eta(e) = 2$.

Claim 3. *For each $B \in V(C')$, there are no three consecutive vertices D_1, D_2, D_3 in C' such that $\eta(D_1D_2) = \eta(D_2D_3) = 1$, $\mathrm{dist}_M(B, D_1) = \mathrm{dist}_M(B, D_3) = 2$, and $\mathrm{dist}_M(B, D_2) = 3$.*

Proof. Suppose that such vertices D_1, D_2, D_3 exist. If $D_1D_3 \in E(\mathcal{G}_M)$, then it contradicts our choice of C'. Thus, $D_1D_3 \notin E(\mathcal{G}_M)$ and hence by Lemma 4.2.5, M has a transversal base $D' \neq D_2$ such that $|D' \setminus D_i| = 1$ for $i \in \{1, 3\}$. Then $|D' \setminus B| = 1$. Hence C' is generated by three cycles of weight 4, a contradiction. ■

By the claim, we can easily deduce that $C' = B_1 \dots B_kB_1$ satisfies either (i) or (ii). □

Lemma 6.4.12. *Let $C = B_1B_2 \dots B_kB_1$ be a cycle of weight 6 in $\mathcal{G}_M$. Then $\prod_{i=1}^{k} \gamma(B_i, B_{i+1}) = 1$, where $B_{k+1} := B_1$.*

Proof. We may assume that C is irreducible by Lemma 6.4.9. Then by Lemma 6.4.11 and rotational symmetry, we may assume that either

(i) $k = 3$ and $\eta(B_iB_{i+1}) = 2$ for each $1 \leq i \leq 3$, or

(ii) $k = 4$ and $\eta(B_1B_2) = \eta(B_3B_4) = 1$ and $\eta(B_2B_3) = \eta(B_4B_1) = 2$.

In the case (i), $B_i = T \triangle \{x_i, x_i^*\}$ for some transversal T and elements $x_1, x_2, x_3 \in T$. Then $A_i := T + x_i^* - x_{i+1}$ with $1 \leq i \leq 3$ are bases, where we read the subscripts modulo 3, and $B_1A_1B_2A_2B_3A_3B_1$ is a 6-cycle in G_M; see Figure 6.4(left). Since C is irreducible, T is not a base of M.

Let $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in \mathcal{C}$ be vectors such that $\underline{X_i} \subseteq B_i \cup A_i = T + x_i$ and $\underline{Y_i} \subseteq A_i \cup B_{i+1}$. Because T is not a base, by Lemma 4.2.13, $\underline{X_i} \subseteq T$ for each $1 \leq i \leq 3$ and thus $\underline{X_1} = \underline{X_2} = \underline{X_3}$. By multiplying elements in $F^\times$, we can assume that $X_1 = X_2 = X_3 =: X$.

Because $\omega(X, Y_i) \in N_F$, we have $\frac{Y_i(x_i^*)}{Y_i(x_{i+1}^*)} = (-1)^{1+\chi(x_i)+\chi(x_{i+1})} \frac{X(x_{i+1})}{X(x_i)}$. Note that $\sum_{i=1}^{3} \big(|(T + x_i^*) < x_i| + |(T + x_i^*) < x_{i+1}| + |(T \triangle \{x_{i+1}, x_{i+1}^*\} + x_i^*) < x_{i+1}^*| + |(T \triangle \{x_{i+1}, x_{i+1}^*\} + x_i^*) < x_i^*| \big) \equiv 1 \pmod 2$. Therefore,

$$\gamma(B_1, B_2)\gamma(B_2, B_3)\gamma(B_3, B_1) = -\frac{X(x_2)}{X(x_1)} \frac{Y_1(x_1^*)}{Y_1(x_2^*)} \frac{X(x_3)}{X(x_2)} \frac{Y_2(x_2^*)}{Y_2(x_3^*)} \frac{X(x_1)}{X(x_3)} \frac{Y_3(x_3^*)}{Y_3(x_1^*)} = 1.$$

Now, we prove the case (ii). For some $x_1, x_2, x_3 \in B_1$, we have $B_2 = B_1 \triangle \{x_1, x_1^*\}$, $B_3 = B_1 \triangle \{x_1, x_1^*, x_2, x_2^*, x_3, x_3^*\}$, and $B_4 = B_1 \triangle \{x_2, x_2^*, x_3, x_3^*\}$; see Figure 6.4(right).

Suppose that $D := B_1 \triangle \{x_1, x_1^*, x_2, x_2^*\}$ is a base. Then $B_1 \triangle \{x_2, x_2^*\}$ is not a base since C is irreducible. By Lemma 4.2.5, DB_1 and DB_4 are edges of weight 2 in $\mathcal{G}_M$. Then by Lemma 6.4.9 and the case (i), we have $\prod_{i=1}^{k} \gamma(B_i, B_{i+1}) = 1$. Thus, we can assume that $B_1 \triangle \{x_1, x_1^*, x_2, x_2^*\}$ is not a base.

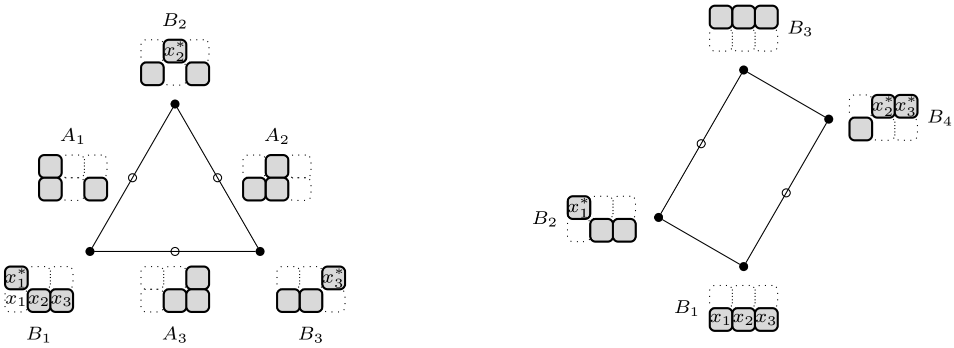


Figure 6.4: Two descriptions of the cycle C of weight 6 in $\mathcal{G}_M$ in the proof of Lemma 6.4.12. The cycle C can be identified with a 6-cycle in G_M, where solid dots represent transversal bases and hollow dots represent almost-transversal bases of M.

Similarly, we can assume that none of $B_1 \triangle \{x_1, x_1^*, x_3, x_3^*\}$, $B_1 \triangle \{x_2, x_2^*\}$, and $B_1 \triangle \{x_3, x_3^*\}$ is a base. Then for each $(i, j) \in [3]^2 \setminus \{(1,1), (2,3), (3,2)\}$, neither $B_1 - x_i + x_j^*$ nor $B_3 + x_i - x_j^*$ is a base.

Let $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in \mathcal{C}$ be vectors such that $\underline{X_1} \subseteq B_1 + x_1$, $\underline{X_2} \subseteq B_2 + x_2^*$, $\underline{X_3} \subseteq B_3 + x_2$, $\underline{Y_1} \subseteq B_3 + x_1^*$, $\underline{Y_2} \subseteq B_4 + x_2$, and $\underline{Y_3} \subseteq B_1 + x_2^*$. Because of the previous observations on non-bases of M, we have that

$$\omega(X_1, Y_1) = (-1)^{\chi(x_1)} X_1(x_1) Y_1(x_1^*) + (-1)^{\chi(x_1^*)} X_1(x_1^*) Y_1(x_1) \in N_F,$$
$$\omega(X_2, Y_2) = (-1)^{\chi(x_2^*)} X_2(x_2^*) Y_2(x_2) + (-1)^{\chi(x_3)} X_2(x_3) Y_2(x_3^*) \in N_F,$$
$$\omega(X_3, Y_3) = (-1)^{\chi(x_2)} X_3(x_2) Y_3(x_2^*) + (-1)^{\chi(x_3^*)} X_3(x_3^*) Y_3(x_3) \in N_F.$$

Therefore, $\gamma(B_1, B_2)\gamma(B_2, B_3) = \gamma(B_1, B_4)\gamma(B_4, B_3)$. □

Lemma 6.4.13. *Let C be an irreducible cycle of weight 8 in $\mathcal{G}_M$. Then $C \simeq B_1B_2B_3B_4B_1$ for some bases $B_1, \ldots, B_4$ such that*

(i) *$B_2 = U + x_1^* + x_2^* + x_3 + x_4$, $B_3 = U + x_1^* + x_2^* + x_3^* + x_4^*$, and $B_4 = U + x_1 + x_2 + x_3^* + x_4^*$, where $x_1, \ldots, x_4$ are distinct elements of B_1 and $U := B_1 \setminus \{x_1, x_2, x_3, x_4\}$, and*

(ii) *$V(\mathcal{G}_M) \cap \{U \cup X : X \subseteq \{x_1, \ldots, x_4, x_1^*, \ldots, x_4^*\}\} = \{B_1, \ldots, B_4\}$.*

Proof. Let C' be an irreducible cycle 4-homotopic to C, which maximizes the number of edges $e \in E(C')$ such that $\eta(e) = 2$. We will show that C' satisfies the desired properties (i) and (ii).

Claim 4. *For each $B \in V(C')$, there are no three consecutive vertices D_1, D_2, D_3 in C' such that $\eta(D_1D_2) = \eta(D_2D_3) = 1$, $\mathrm{dist}_M(B, D_1) = \mathrm{dist}_M(B, D_3) = 3$, and $\mathrm{dist}_M(B, D_2) = 4$.*

Proof. Suppose that such vertices D_1, D_2, D_3 exist. By our choice of C', D_1D_3 is not an edge in $\mathcal{G}_M$. Thus, there is a base $D' \neq D_2$ such that $\mathrm{dist}_M(D_1, D') = \mathrm{dist}_M(D_3, D') = 1$. Then $\mathrm{dist}_M(B, D') = 2$. Let C'' be a cycle obtained from C' by replacing a vertex D_2 with D'. Then C'' is 4-homotopic to C' and it is reducible by Lemma 6.3.2, contradicting that C' is irreducible. ■

Claim 5. *There are no edges D_1D_2, D_3D_4 of C' such that $\eta(D_1D_2) = 1$, $\eta(D_3D_4) = 2 = \mathrm{dist}_M(D_4, D_1)$, and $\mathrm{dist}_M(D_2, D_3) = 3$.*

Proof. Suppose such edges exist. Then for some $y_1, \dots, y_4 \in D_1$, we have $D_2 = D_1 \triangle \{y_3, y_3^*\}$, $D_3 = D_1 \triangle \{y_1, y_1^*, \dots, y_4, y_4^*\}$, and $D_4 = D_1 \triangle \{y_1, y_1^*, y_2, y_2^*\}$. Let $S := D_1 + y_3^*$ and $T := D_4 + y_3^*$. Then by Lemma 4.2.13, there are circuits $c \subseteq S$ and $c' \subseteq T$.

Suppose that $D' := T - y_3 = D_4 \triangle \{y_3, y_3^*\}$ is a base. Then $|D' \setminus D_2| = |\{x_1^*, x_2^*\}| = 2$. Let O be a cycle obtained from C' by replacing an edge D_3D_4 with a path $D_3D'D_4$. Then O is 4-homotopic to C' and it is reducible by Lemma 6.3.2. It contradicts that C' is irreducible. Thus, $T - y_3$ is not a base.

Suppose that $D'' := (T - y_3) \triangle \{y_1, y_1^*\} = D_4 \triangle \{y_1, y_1^*, y_3, y_3^*\}$ is a base. Then $|D' \setminus D_2| = 1$ and $|D' \setminus D_i| = 2$ for $i \in \{1, 3, 4\}$. Hence C' is generated by four cycles of weight at most six, a contradiction. Thus, $(T - y_3) \triangle \{y_1, y_1^*\}$ is not a base.

By Lemma 4.2.5, $T - y_1^*$ is not a base. Similarly, $T - y_2^*$ is not a base. Therefore, $c' \subseteq T - \{y_1^*, y_2^*, y_3\} \subseteq S$ by Lemma 4.2.13. Note that $y_3 \in c \subseteq S$ and so $c \neq c'$. It contradicts Lemma 4.2.13. ■

By Claims 4 and 5, $C' = B_1B_2B_3B_4B_1$ such that $\eta(B_iB_{i+1}) = 2$ for all i, where $B_5 := B_1$. Then one can denote by $B_2 = U + x_1^* + x_2^* + x_3 + x_4$, $B_3 = U + x_1^* + x_2^* + x_3^* + x_4^*$, and $B_4 = U + x_1 + x_2 + x_3^* + x_4^*$ for some elements $x_1, \dots, x_4$ of B_1 and $U := B_1 \setminus \{x_1, x_2, x_3, x_4\}$. As C' is irreducible, $B_1 \triangle \{x_i, x_i^*, x_j, x_j^*\}$ is not a base for each $1 \le i \le 2$ and $3 \le j \le 4$.

Claim 6. *$B_1 \triangle \{x_1, x_1^*\}$ is not a base.*

Proof. Suppose to the contrary that $B_1 \triangle \{x_1, x_1^*\}$ is a base. Since C' is irreducible, $B_4 \triangle \{x_1, x_1^*\}$ is not a base. Let $S = B_1 + x_1^*$ and $T = B_4 + x_1^*$. Let c and c' be circuits of M such that $c \subseteq S$ and $c' \subseteq T$. Then $x_1 \in c$ because $S - x_1 = B_1 \triangle \{x_1, x_1^*\}$ is a base. Since $T - x_1 = B_4 \triangle \{x_1, x_1^*\}$ is not a base, $x_1 \notin c'$. Hence $c \neq c'$. Because $(T - x_1) \triangle \{x_3, x_3^*\} = B_1 \triangle \{x_1, x_1^*, x_4, x_4^*\}$ is not a base, $T - x_3^*$ is not a base by Lemma 4.2.5. Similarly, $T - x_4^*$ is not a base. Hence $c' \subseteq T - x_3^* - x_4^* \subseteq S$, contradicting Lemma 4.2.13. ■

Simiarly, none of $B_i \triangle \{x_j, x_j^*\}$ with $i, j \in [4]$ is a base. Therefore, (ii) holds. □

Lemma 6.4.14. *Let $C = B_1B_2 \dots B_kB_1$ be a cycle of weight 8 in $\mathcal{G}_M$. Then $\sum_{i=1}^{k} \gamma(B_i, B_{i+1}) = 1$, where $B_{k+1} := B_1$.*

Proof. We may assume that C is irreducible by Lemmas 6.4.9 and 6.4.12. By Lemma 6.4.13, we can assume that $C = B_1B_2B_3B_4B_1$ and it satisfies the following:

(i) $B_2 = U + x_1^* + x_2^* + x_3 + x_4$, $B_3 = U + x_1^* + x_2^* + x_3^* + x_4^*$, and $B_4 = U + x_1 + x_2 + x_3^* + x_4^*$, where x_1, x_2, x_3, x_4 are distinct elements in B_1 and $U := B_1 \setminus \{x_1, x_2, x_3, x_4\}$, and

(ii) $V(\mathcal{G}_M) \cap \{U \cup X : X \subseteq \{x_1, \dots, x_4, x_1^*, \dots, x_4^*\}\} = \{B_1, \dots, B_4\}$.

Let $S_1 = B_1 + x_1^*$, $S_2 = B_2 + x_1$, $S_3 = B_2 + x_3^*$, and $S_4 = B_3 + x_3$. Let $T_1 = B_1 + x_3^*$, $T_2 = B_4 + x_3$, $T_3 = B_4 + x_1^*$, and $T_4 = B_3 + x_1$; see Figure 6.5. Then $S_1 \triangle T_3 = S_2 \triangle T_4 = \{x_3, x_3^*, x_4, x_4^*\}$ and $S_3 \triangle T_1 = S_4 \triangle T_2 = \{x_1, x_1^*, x_2, x_2^*\}$. Let X_i and Y_i be vectors in $\mathcal{C}$ such that $\underline{X_i} \subseteq S_i$ and $\underline{Y_i} \subseteq T_i$. By (ii), neither $S_1 - x_3$ nor $S_1 - x_4$ is a base. Then $\underline{X}_1 \subseteq T_3$ and thus by Lemma 4.2.13, $\underline{X}_1 = \underline{Y}_3$. Similarly, $\mathrm{supp}(X_i) = \mathrm{supp}(Y_{i+2})$ for each $1 \le i \le 4$, where the subscripts are read modulo 4. Thus, $X_i = c_iY_{i+2}$ for some $c_i \in F^\times$. Therefore, for some $m \in \{0, 1\}$,

$$\begin{aligned}\gamma(B_1, B_2)\gamma(B_2, B_3) &= (-1)^m \frac{X_1(x_2)}{X_1(x_1^*)} \frac{X_2(x_1)}{X_2(x_2^*)} \frac{X_3(x_4)}{X_3(x_3^*)} \frac{X_4(x_3)}{X_4(x_4^*)} \\ &= (-1)^m \frac{Y_1(x_4)}{Y_1(x_3^*)} \frac{Y_2(x_3)}{Y_2(x_4^*)} \frac{Y_3(x_2)}{Y_3(x_1^*)} \frac{Y_4(x_1)}{Y_4(x_2^*)} = \gamma(B_1, B_4)\gamma(B_4, B_3).\end{aligned}$$ □

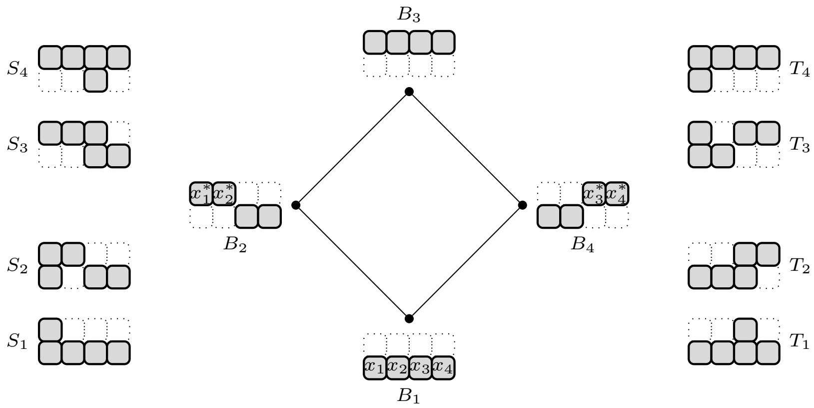


Figure 6.5: An illustration of a cycle $C = B_1B_2B_3B_4B_1$ of weight 8 in $\mathcal{G}_M$ together with S_i and T_i with $i \in [4]$ in the proof of Lemma 6.4.14.

Proposition 6.4.15. *For two bases B and B' of M, let P_1 and P_2 be two paths in the base graph G_M from B to B'. Then $\gamma(P_1) = \gamma(P_2)$.*

Proof. Let C be a cycle in G_M consisting of P_1 and P_2, and let C' be the corresponding cycle in $\mathcal{G}_M$. By Homotopy Theorem 4.2.29 and Lemmas 6.4.9, 6.4.12, 6.4.14, $\gamma(C') = 1$. Then $\gamma(C) = \gamma(C') = 1$ by Lemma 6.4.9. Thus, $\gamma(P_1) = \gamma(P_2)$ by Lemma 6.4.7. □

By Proposition 6.4.15, the function $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ described below Definition 6.4.5 is well defined. We finally show that φ satisfies the restricted Grassmann-Plücker relations (rGP*).

Theorem 6.4.16. *φ is a restricted G-P function.*

Proof. By Lemma 6.4.8, φ satisfies (rGP2).

Let $S \in \binom{E}{n+1}$ and $T \in \binom{E}{n-1}$ be sets such that S contains exactly one skew pair, say $\{x, x^*\}$, and T has no skew pair. Let $\{y, y^*\}$ be the unique skew pair not contained in T, and let $T' := T + \{y, y^*\}$. We claim that φ satisfies (rGP*). We can assume that for some $z \in S \setminus T$, both $S - z$ and $T + z$ are bases of M. Then by Lemma 4.2.5, $S - x$ or $S - x^*$ is a transversal base of M. Also, $T + y$ or $T + y^*$ is a transversal base. Hence there are $X, Y \in \mathcal{C}$ such that $\underline{X} \subseteq S$ and $\underline{Y} \subseteq T'$.

By symmetry, we can assume that $S-x$ and $T+y$ are bases. Then $\varphi(S-x)$, $\varphi(T+y)$, $X(x)$, and $Y(y^*)$ are nonzero in F. For each $z \in S \setminus T$, we have $\frac{\varphi(S-z)}{\varphi(S-x)} = \gamma(S-x, S-z) = (-1)^{\chi(x)+\chi(z)+|S<x|+|S<z|}\frac{X(z)}{X(x)}$. We also have

$$\frac{\varphi(T+y^*)}{\varphi(T+y)} = \gamma(T+y, T+y^*) = (-1)^{\chi(y)+\chi(y^*)+|T'<y|+|T'<y^*|}\frac{Y(y)}{Y(y^*)} = (-1)^{|T<y|+|T<y^*|}\frac{Y(y)}{Y(y^*)}.$$

For $z \in S \setminus T'$, let $U_z := T + y + z$ and let Y_z be a vector in $\mathcal{C}$ such that $\underline{Y_z} \subseteq U_z$. Since $\omega(Y, Y_z) = (-1)^{\chi(y^*)}Y(y^*)Y_z(y) + (-1)^{\chi(z^*)}Y(z^*)Y_z(z) \in N_F$, we have

$$\frac{\varphi(T+z)}{\varphi(T+y)} = (-1)^{\chi(y)+\chi(z)+|U_z<y|+|U_z<z|}\frac{Y_z(y)}{Y_z(z)} = (-1)^{|T<y|+|T<z|}\frac{Y(z^*)}{Y(y^*)}.$$

Let $c := (-1)^{\chi(x)+|S<x|+|T<y|}\frac{\varphi(S-x)\varphi(T+y)}{X(x)Y(y^*)} \in F^\times$. Note that $X(z) = 0$ if $z \in E \setminus S$, and $Y(z^*) = 0$ if $z \in T$. Therefore,

$$\begin{aligned}\sum_{z \in S\setminus T}(-1)^{|S<z|+|T<z|}\varphi(S-z)\varphi(T+z) &= c\sum_{z\in S\setminus T}(-1)^{\chi(z)}X(z)Y(z^*)\\ &= c\sum_{z\in E}(-1)^{\chi(z)}X(z)Y(z^*) \in N_F.\end{aligned}$$ □

6.4.3 Equivalence

In Section 6.4.1, we constructed an antisymmetric F-circuit set from an antisymmetric F-matroid. Conversely, we built an antisymmetric F-matroid from an antisymmetric F-circuit set in Section 6.4.2. By definition, these two constructions are the reverse step of each other, and thus we deduce Theorem 6.4.1 as follows.

Proof of Theorem 6.4.1. Let $M = [\varphi]$ be an antisymmetric F-matroid on $E = [n] \cup [n]^*$, and let $\mathcal{C}$ be the antisymmetric F-circuit set constructed from φ in the sense of Section 6.4.1. Let φ' be a restricted G-P function constructed from $\mathcal{C}$ in the sense of Section 6.4.2. Then the underlying matroids of φ, φ', and $\mathcal{C}$ are the same. Let B_1 and B_2 be bases such that $|B_1 \setminus B_2| = 1$, and let $X \in \mathcal{C}$ be a vector whose support $\underline{X}$ is a subset of $S := B_1 \cup B_2$. We denote by $\{x\} = S \setminus B_1$ and $\{y\} = S \setminus B_1$. Then

$$\frac{\varphi(B_2)}{\varphi(B_1)} = (-1)^{\chi(x)+\chi(y)+|S<x|+|S<y|} \frac{X(y)}{X(x)} = \frac{\varphi'(B_2)}{\varphi'(B_1)}.$$

Therefore, $M = [\varphi']$.

Let $\mathcal{C}'$ be the antisymmetric F-circuit set constructed from φ'. Then similarly we deduce that $\mathcal{C} = \mathcal{C}'$. Thus, there is a natural bijection between antisymmetric F-matroids and antisymmetric F-circuit sets. □

6.5 Analogue of Tutte's theorem

We show an analog of Theorem 3.1.4 for antisymmetric matroids and Lagrangian Grassmannians.

Definition 6.5.1. A *weak restricted G-P function on $E = [n] \cup [n]^*$ with coefficients in a tract F* is a nontrivial function $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ such that the support of φ form the set of bases of an antisymmetric matroid on E and φ satisfies (rGP2) and the following weaker replacement of (rGP3):

(rGP3′) For $S \in \binom{E}{n+1}$ and $T \in \binom{E}{n-1}$ such that S contains exactly one skew pair and T has no skew pair, if $|S \setminus T| \le 4$, then

$$\sum_{x \in S \setminus T} (-1)^{|S<x|+|T<x|} \varphi(S - x)\varphi(T + x) \in N_F. \qquad \text{(rGP*)}$$

A *weak antisymmetric F-matroid* is an equivalence class of weak restricted G-P functions.

Theorem 6.5.2. *For a field $\mathbb{F}$, let $\varphi \in \mathbb{F}^{2^n + \binom{n}{2} 2^{n-2}}$. Then the following are equivalent.*

(i) *φ is a restricted G-P function.*

(ii) *φ is a weak restricted G-P function.*

(iii) *There is an $n \times 2n$ matrix $\Lambda = [A_1 \mid A_2]$ over $\mathbb{F}$ such that $A_1 A_2^t$ is symmetric and $X_B = \det(A[B])$ with $B \in \mathcal{T}_n \cup \mathcal{A}_n$.*

Note that the row-space of an n-by-$2n$ matrix $[A_1 \mid A_2]$ is Lagrangian if and only if $A_1 A_2^t$ is symmetric. We show two lemmas before proving Theorem 6.5.2.

Lemma 6.5.3. *Let M be an antisymmetric matroid on $[n] \cup [n]^*$ such that $[n]$ is a base. Let $X, Y \subseteq [n]$ be sets such that $|X| = |Y| \ge 2$ and $|X \setminus Y| \le 1$. If $[n] - X + Y^*$ is a base, then there is $Z \subseteq X \cap Y$ such that $|X \setminus Z| = |Y \setminus Z| \in \{1, 2\}$ and $[n] - Z + Z^*$ is a base.*

Proof. Suppose that $X = Y$. By (Exch), $[n] - (X - e) + (Y - f)^*$ is a base for some $e, f \in X$. We may assume that $e \neq f$. Then by Lemma 4.2.5, $[n] - (X - e) + (X - e)^*$ or $[n] - (X - e - f) + (X - e - f)^*$ is a base. Therefore, we may assume that $X \neq Y$. We denote by $\{x\} = X - Y$ and $\{y\} = Y - X$. By (Exch), $[n] - (X - x) + (Y - g)^*$ is a base for some $g \in Y$. We may assume that $g \neq y$. Then by Lemma 4.2.5, $[n] - (X - x) + (Y - y)^*$ or $[n] - (X - x - g) + (Y - y - g)^*$ is a base. □

Lemma 6.5.4. *For $S \subseteq [n]$, let $\Psi_s : F^{[n]\cup[n]^*} \to F^{[n]\cup[n]^*}$ be a linear map such that for each $i \in [n]$,*

$$\mathbf{e}_i \mapsto \begin{cases} \mathbf{e}_i & \text{if } i \notin S, \\ \mathbf{e}_{i^*} & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{e}_{i^*} \mapsto \begin{cases} \mathbf{e}_{i^*} & \text{if } i \notin S, \\ -\mathbf{e}_i & \text{otherwise.} \end{cases}$$

Then Ψ_S induces a bijection from $\mathrm{SpGr}_k(n, 2n)$ to itself such that for every $W \in \mathrm{SpGr}_k(n, 2n)$, a set $B \in \mathcal{T}_n \cup \mathcal{A}_n$ is a base of $M(W)$ if and only if $T \triangle (S \cup S^*)$ is a base of $M(\Psi_S(W))$. □

Proof of Theorem 6.5.2. Obviously, (i) implies (ii). By Theorem 4.2.2, (i) and (iii) are equivalent. Hence it remains to prove that (ii) implies (iii).

Let φ be a weak restricted G-P function on $E := [n] \cup [n]^*$ and let M be its underlying antisymmetric matroid. By Lemma 6.5.4, we may assume that $B_0 := [n]$ is a base of M. Let $a_{ij} = (-1)^{n-i}\varphi(B_0 - i + j^*)/\varphi(B_0)$ for all $i, j \in [n]$. By (rGP2), we have $a_{ij} = a_{ji}$ and thus $\Sigma := (a_{ij})_{1\leq i,j\leq n}$ is a symmetric matrix.

We claim that

$$\det(\Sigma[X, Y]) = (-1)^{tn+\binom{t}{2}+\sum_{x\in X} x}\varphi(B_0 - X + Y^*).$$

for all $X, Y \subseteq [n]$ such that $|X| = |Y| =: t$ and $|X \setminus Y| \leq 1$. Note that the row-space of an $n \times E$ matrix $A := \left[I_n \,|\, \Sigma\right]$ is Lagrangian, and $\det(A[n, B_0 - X + Y^*]) = (-1)^{tn+\binom{t}{2}+\sum_{x\in X} x}\det(\Sigma[X, Y])$. Thus, the claim suffices to conclude (iii).

We prove the claim by induction on $|X|$. By our choice of Σ, we may assume that $|X| \geq 2$

Case I. $X = Y$. By relabelling, we can assume that $X = \{1, 2, \ldots, t\}$. Let $m := tn + \binom{t}{2} + \binom{t+1}{2}$.

Suppose that none of $B_0 - (X - 1) + (X - i)^*$ with $i \in [m]$ is a base. Then by (Exch), $B_0 - X + X^*$ is not a base. Also, $\det(A[X - 1, X - i]) = 0$ for all $i \in X$ by the induction hypothesis, and thus $\det(A[X, X]) = 0 = \varphi(B_0 - X + X^*)/\varphi(B_0)$. Therefore, we may assume that $B_0 - (X - 1) + (X - j)^*$ is a base for some $j \in [m]$. By Lemma 6.5.3, there is $S \subseteq [t] - \{1, j\}$ such that $|\{2, \ldots, t\} \setminus S| \in \{1, 2\}$ and $[n] - S + S^*$ is a base. By relabelling, we may assume that S is either $[t] \setminus \{1, 2\}$ or $[t] \setminus \{1, 2, 3\}$.

Subcase I.1. $S = [t] \setminus \{1, 2\}$. Applying the 3-term restricted G-P relation to $\{1, 1^*, 2^*\} + (B_0 - X + S^*)$ and $\{2\} + (B_0 - X + S^*)$, we have

$$\begin{aligned}\varphi(B_0 - S + S^*)\varphi(B_0 - X + X^*) + \varphi(B_0 - (S+1) + (S+1)^*)\varphi(B_0 - (S+2) + (S+2)^*) \\ - \varphi(B_0 - (S+1) + (S+2)^*)\varphi(B_0 - (S+2) + (S+1)^*) = 0\end{aligned}$$

By the induction hypothesis, for $i, j \in [n]$, we have

$$\begin{aligned}\frac{\varphi(B_0 - S + S^*)}{\varphi(B_0)} &= (-1)^{m-2n-(2k-3)-3}\det(\Sigma[S, S]), \\ \frac{\varphi(B_0 - (S+i) + (S+j)^*)}{\varphi(B_0)} &= (-1)^{m-n-(k-1)-i}\det(\Sigma[S+i, S+j]).\end{aligned}$$

Then by the generalized Laplace expansion,

$$\begin{aligned}\frac{\varphi(B_0 - X + X^*)}{\varphi(B_0)} &= \frac{(-1)^m}{\det(\Sigma[S,S])}\big(\det(\Sigma[S+1,S+1])\det(\Sigma[S+2,S+2]) \\ &\qquad - \det(\Sigma[S+1,S+2])\det(\Sigma[S+2,S+1])\big) \\ &= (-1)^m \det(\Sigma[X,X]).\end{aligned}$$

Subcase I.2. $S = [t] \setminus \{1,2,3\}$. By the induction hypothesis, for proper subsets I, J of [3] such that $|I| = |J|$ and $|I \setminus J| \le 1$, we have $\varphi(B_0 - (S+I) + (S+J)^*)/\varphi(B_0) = (-1)^{m-\eta(I)} \det(A[S+I, S+J])$, where $\eta(I) := (n+t)(3-|I|) + \binom{|I|+1}{2} + (6 - \sum_{i \in I} i)$. Then by the 4-term restriced G–P relation applied to $\{1, 1^*, 2^*, 3^*\} + (B_0 - X + S^*)$ and $\{2,3\} + (B_0 - X + S^*)$ and the generalized Laplace expansion, we have

$$\begin{aligned}\frac{\varphi(B_0 - X + X^*)}{\varphi(B_0)} &= \frac{(-1)^m}{\det(\Sigma[S,S])}\big(\det(\Sigma[S+1,S+1])\det(\Sigma[S+2+3,S+2+3]) \\ &\qquad - \det(\Sigma[S+1,S+2])\det(\Sigma[S+2+3,S+1+3]) \\ &\qquad + \det(\Sigma[S+1,S+3])\det(\Sigma[S+2+3,S+1+2])\big) \\ &= (-1)^m \det(\Sigma[X,X]).\end{aligned}$$

Case II. $|X \setminus Y| = 1$. By relabelling, we may assume that $X = [t] \setminus \{1\}$ and $Y = [t] \setminus \{2\}$. Let $m := (t-1)n + \binom{t-1}{2} + \binom{t}{2} - 1$.

Suppose that none of $[n] - (X - i) + (Y - 1)^*$ with $i \in X$ is a base. By (Exch), $B_0 - X + Y^*$ is not a base. For each $i \in X$, $\det(A[X - i, Y - 1]) = 0$ by the induction hypothesis. Hence $\det(A[X,Y]) = 0 = \varphi(B_0 - X + Y^*)/\varphi(B_0)$. Therefore, we may assume that $B_0 - (X - j) + (Y - 1)^*$ is a base for some $j \in X$. By Lemma 6.5.3, there is $S \subseteq [t] - \{1, 2, j\}$ such that $|(X - j) \setminus S| \in \{1, 2\}$ and $[n] - S + S^*$ is a base. By relabelling, we may assume that S is either $[t] \setminus \{1,2,3\}$ or $[t] \setminus \{1,2,3,4\}$.

Subcase II.1. $S = [t] \setminus \{1,2,3\}$. Then we have $\varphi(B_0 - S + S^*) = (-1)^{m-2n-(2t-3)-5} \det(\Sigma[S,S])$ and $\varphi(B_0 - (S+i) + (S+j)^*) = (-1)^{m-n-(t-1)+i} \det(\Sigma[S+i, S+j])$ for each $i, j \in [3]$ by the induction hypothesis. Then by the 3-term restricted G-P relation applied to $\{1, 2, 1^*, 3^*\} + (B_0 - [t] + S^*)$ and $\{1,3\} + (B_0 - [t] + S^*)$ and the generalized Laplace expansion, we deduce that

$$\begin{aligned}\frac{\varphi(B_0 - X + Y^*)}{\varphi(B_0)} &= \frac{(-1)^m}{\det(\Sigma[S,S])}\big(\det(\Sigma[S+2,S+1])\det(\Sigma[S+3,S+3]) \\ &\qquad + \det(\Sigma[S+2,S+3])\det(\Sigma[S+3,S+1])\big) \\ &= (-1)^m \det(\Sigma[X,Y]).\end{aligned}$$

Subcase II.2. $S = [t] \setminus \{1,2,3,4\}$. Then we have $\varphi(B_0 - S + S^*) = (-1)^{m-3n-(3t-6)-9} \det(\Sigma[S,S])$ by the induction hypothesis. Also, $\varphi(B_0 - (S+2) + (S+i)^*) = (-1)^{m-2n-(2t-3)+7} \det(\Sigma[S+2, S+i])$ for each $i \in [4]$, and $\varphi(B_0 - (S+3+4) + (S+i+j)^*) = (-1)^{m-n-(t-1)+2} \det(\Sigma[S+3+4, S+i+j])$ for $i \in [4]$ and $j \in \{3,4\}$. Then by the 4-term restricted G-P relation applied to $\{1, 2, 1^*, 3^*, 4^*\} + (B_0 - [t] + S^*)$ and $\{1,3,4\} + (B_0 - [t] + S^*)$ and the generalized Laplace expansion, we have

$$\begin{aligned}\frac{\varphi(B_0 - X + Y^*)}{\varphi(B_0)} &= \frac{(-1)^m}{\det(\Sigma[S,S])}\big(\det(\Sigma[S+2,S+1])\det(\Sigma[S+3+4,S+3+4]) \\ &\qquad + \det(\Sigma[S+2,S+3])\det(\Sigma[S+3+4,S+1+4]) \\ &\qquad + \det(\Sigma[S+2,S+3])\det(\Sigma[S+3+4,S+1+3])\big) \\ &= (-1)^m \det(\Sigma[X,Y]).\end{aligned}$$

□

Note that Theorem 6.5.2 fails if we weaken the definition of weak restricted G-P functions by removing the condition that the support of φ forms an antisymmetric matroid. For example, let F be an arbitrary tract and $\varphi : \mathcal{T}_4 \cup \mathcal{A}_4 \to F$ be a function such that $\mathrm{supp}(\varphi) = \{[4], [4]^*\}$. Then $\mathrm{supp}(\varphi)$ is not the set of bases of an antisymmetric on $[4] \cup [4]^*$ and thus φ is not a restricted G-P function. However, φ satisfies all 3- and 4-term restricted G-P relations. Similarly, if we moderated the definition of weak restricted G-P functions by replacing the 3-/4-term restricted G-P relations with the 3-term restricted G-P relations, then Theorem 6.5.2 does not hold anymore; see Example 6.5.5. We remark that Tutte's theorem was extended for *perfect tracts* including all partial fields and the sign, tropical, Krasner hyperfields by Baker and Bowler [4]. Also, an analogue of Tutte's theorem holds for even symmetric matroids and Lagrangian orthogonal Grassmannians over partial fields [6] and the tropical hyperfield $\mathbb{T}$ [102].

Example 6.5.5. Let $M = ([4] \cup [4]^*, \mathcal{B})$ be an antisymmetric matroid such that

$$\mathcal{B} := \{[4]\} \cup \{i^*j^*k\ell : ijkl = [4]\} \cup \left\{ii^*jk : ijk \in \binom{[4]}{3}\right\} \cup \left\{ii^*jk^* : ijk \in \binom{[4]}{3}\right\}.$$

Let $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to \mathbb{F}_2$ be a function whose support is $\mathrm{supp}(\varphi) = \mathcal{B}$. Then φ satisfies all 3-term restricted G-P relations, but it does not satisfy 4-term restricted G-P relations. More precisely, it does not satisfy a 4-term restricted G-P relation (rGP) applied to $S = \{1, 2, 2^*, 3^*, 4^*\}$ and $T = \{1^*2, 3\}$. Both $S - x$ and $T + x$ are bases for each $x \in \{2^*, 3^*, 4^*\}$, and $S - 1$ is not a base of M. Hence $\sum_{x \in S-T} \varphi(S - x)\varphi(T + x) = 3 \neq 0$.

Remark 6.5.6. The same proof holds even if we replace a field $\mathbb{F}$ in Theorem 6.5.2 with a partial field.

Chapter 7. Variants of Tutte's Wheel Theorem for graphs with vertex-/pivot-minors

Tutte [110, (4.1)] showed that every simple 3-connected graph G has an edge e such that $G \setminus e$ or G/e is simple 3-connected. It is called Tutte's Wheel Theorem and is a useful inductive tool in graph theory. We prove analogous theorems for binary even delta-matroids.

Theorem 7.1.2. *Let M be a 3-connected binary even delta-matroid with $|E(M)| \geq 4$. Then M has two elements x_1, x_2 such that $M \setminus x_i$ or M/x_i is 3-connected for each i unless M has a cycle as a fundamental graph.*

Recall that Θ is the set of graphs consisting of at least two internally-disjoint paths between two fixed distinct vertices having no common neighbor.

Theorem 7.1.4. *Let M be a 3-connected binary even delta-matroid with $|E(M)| \geq 4$. Then M has two elements x_1, x_2, x_3 such that $M \setminus x_i$ or M/x_i is 3-connected for each i unless M has a fundamental graph in Θ.*

We examined connections between binary even delta-matroids with the minior relation and graphs with the pivot-minor relations in Section 4.1.4. Accordingly, the above results can be rephrased in terms of graphs with pivot-minors as follows.

Theorem 7.1.2 (restated)**.** *Every prime graph G with at least four vertices has two vertices x_1, x_2 such that $G \setminus x_i$ or $G \wedge x_i w_i \setminus x_i$ with $x_i y_i \in E(G)$ is prime for each i, unless G is pivot-equivalent to a cycle.*

Theorem 7.1.4 (restated)**.** *Every prime graph G with at least four vertices has three vertices x_1, x_2, x_3 such that $G \setminus x_i$ or $G \wedge x_i w_i \setminus x_i$ with $x_i y_i \in E(G)$ is prime for each i, unless G is pivot-equivalent to a graph in Θ.*

In Section 7.1, we prove Theorems 7.1.2 and 7.1.4 and their analogs for vertex-minors. We indeed write Theorem 7.1.4 as the 'if-and-only-if' condition, which is stronger than the above statement.

We focus on another topic in Section 7.2. A graph is *PU-orientable* if it admits an orientation whose adjacency matrix is principally unimodular. PU-orientability of graphs is preserved under the pivot-minor relation. Camion [40] showed that every PU-orientable bipartite graph has a unique PU-orientation up to cut-switching. Note that a graph is bipartite if and only if it has no K_3-pivot-minor. We show the following theorem for graphs with no K_4-pivot-minor, which implies Camion's result.

Proposition 7.2.2. *Every PU-orientable graph without K_4-pivot-minor has a unique PU-orientation up to negation and cut-switching.*

Structure of this chapter. In Section 7.1, we prove Theorems 7.1.2 and 7.1.4 and its analogue for vertex-minors. The proofs are relies on properties of isotropic systems that are a linear algebraic notion capturing local equivalences of graphs. Hence, we will review isotropic systems in Subsections 7.1.2–7.1.3. We show Proposition 7.2.2 in Section 7.2.

7.1 Strong Tutte's Wheel Theorem for vertex-/pivot-minors

An edge e of a graph G is *non-essential* if the deletion of e in G, denoted by $G \setminus e$, or the contraction of e in G, denoted by G/e, is simple and 3-connected. Tutte's Wheel Theorem [110, (4.1)] states that every simple 3-connected graph has a non-essential edge unless it is isomorphic to a wheel graph. Oxley and Wu [100] showed a stronger result, that is, every simple 3-connected graph has at least two non-essential edges unless it is isomorphic to a wheel graph, and they [99] determined all simple 3-connected graphs having exactly two non-essential edges. Moreover, they [101] investigated all simple 3-connected graphs having exactly three non-essential edges. We remark that all of these results except for the last have corresponding results for matroids; see [112, 100, 99].

We aim to prove analogous theorems for the vertex-minor and pivot-minor relations (Theorems 7.1.1–7.1.4). In this section, except for the first paragraph, all graphs are assumed to be simple, meaning that they have neither loops nor parallel edges. The definitions of vertex-/pivot-minors of graphs and related notions can be found in Chapter 2. For a vertex v and distinct neighbors w and w' of v, $G \wedge vw \wedge ww' = G \wedge vw'$; see Oum [92, Proposition 2.5]. Hence $G \wedge vw \setminus v$ and $G \wedge vw' \setminus v$ are pivot-equivalent (so locally equivalent) because $G \wedge vw \setminus v \wedge ww' = G \wedge vw' \setminus v$. Let G/v denote $G \wedge vw \setminus v$ for an arbitrary neighbor w of v if v has a neighbor and $G \setminus v$ otherwise. Note that G/v is well defined up to pivot equivalence (and up to local equivalence). Bouchet [24, (9.2)] proved that for a graph G and a vertex v, every vertex-minor of G on $V(G) - \{v\}$ is locally equivalent to $G \setminus v$, $G * v \setminus v$, or G/v.

A vertex v of a graph G is *non-essential* if at least two of $G \setminus v$, $G * v \setminus v$, and G/v are prime, and a vertex v of a graph G is *non-pivotal* if $G \setminus v$ or G/v is prime. Obviously, every non-essential vertex is non-pivotal, but the converse does not hold; see Figure 7.1. Allys [1, Theorem 4.3] proved that every prime graph with more than 4 vertices has a non-essential vertex unless it is locally equivalent to a cycle. Oum and the author [75] prove that, indeed, such graphs have at least two non-essential vertices, and we show the corresponding result concerning non-pivotal vertices.

Theorem 7.1.1 (Strong Tutte's Wheel Theorem for vertex-minors)**.** *Every prime graph with at least four vertices has at least two non-essential vertices unless it is locally equivalent to a cycle.*

Theorem 7.1.2 (Strong Tutte's Wheel Theorem for pivot-minors)**.** *Every prime graph with at least four vertices has at least two non-pivotal vertices unless it is pivot-equivalent to a cycle.*

We also characterize prime graphs with at least 3 non-essential (or non-pivotal) vertices as follows. Recall that Θ is the set of graphs consisting of at least two internally-disjoint paths between two fixed distinct vertices having no common neighbor, and every cycle of length at least four is in Θ.

Theorem 7.1.3. *A prime graph with at least four vertices has at least three non-essential vertices if and only if it is not locally equivalent to any graph in Θ.*

Theorem 7.1.4. *A prime graph with at least four vertices has at least three non-pivotal vertices if and only if it is not pivot-equivalent to any graph in Θ.*

Bouchet [22, Theorem in page 244] showed that every prime graph with at least six vertices has a prime vertex-minor with one fewer vertex, which was used in a recognition algorithm and the proof of obstructions for circle graphs [22, 31]. From Theorem 7.1.3, we obtain the following strengthening of Bouchet's result.

Corollary 7.1.5 ([75])**.** *Let G be a prime graph with at least six vertices and let x and y be vertices of G. Then there is a prime vertex-minor H of G such that $|V(H)| = |V(G)| - 1$ and $x, y \in V(H)$.*

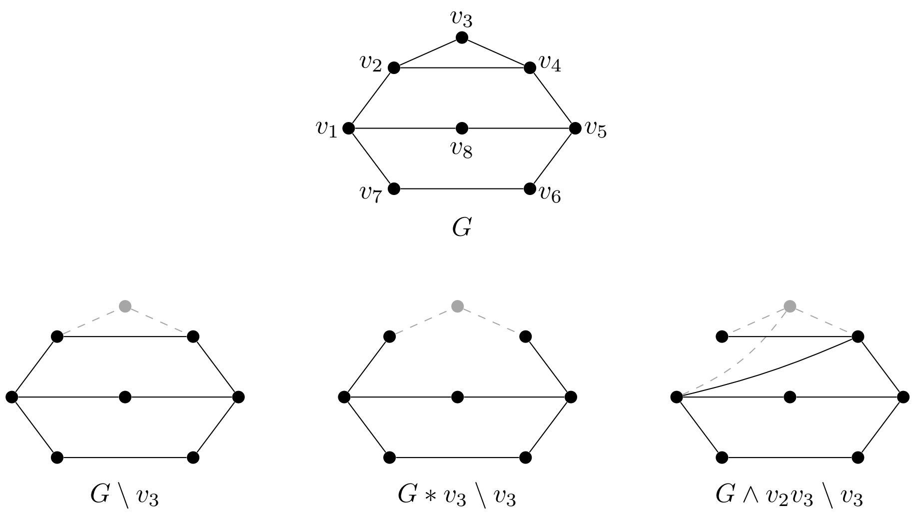


Figure 7.1: The set of non-essential vertices of G is $\{v_1, v_5, v_8\}$ and the set of non-pivotal vertices of G is $\{v_1, v_2, v_3, v_4, v_5, v_8\}$. For instance, $G \setminus v_3$ is prime and neither $G * v_3 \setminus v_3$ nor $G \wedge v_2v_3 \setminus v_3$ is prime.

We note that several other corollaries of the above theorems are also written in [75, Pages 2–4]. From Theorems 7.1.2 and 7.1.4, we deduce the following corollaries regarding bipartite graphs, of which proof will be provided in Section 7.1.6.

Corollary 7.1.6. *Every prime bipartite graph with at least four vertices has at least two non-pivotal vertices unless it is pivot-equivalent to an even cycle.*

Corollary 7.1.7. *A bipartite prime graph with at least four vertices has at least three non-pivotal vertices if and only if it is not pivot-equivalent to any bipartite graph in* Θ.

We remark that Corollary 7.1.6 is equivalent to Corollary 3.5 of Oxley and Wu [100] restricted to binary matroids, and Corollary 7.1.7 implies Theorems 1.3 and 1.4 of Oxley and Wu [99] restricted to binary matroids. These implication are written in [75, Appendix B]. Note that Tutte's Wheel Theorem for graphs can be easily deduced from that for binary matroids, and therefore, Theorem 7.1.1 implies original Tutte's Wheel Theorem [110].

7.1.1 Graphs with vertex-/pivot-minors

We discuss several properties of prime graphs and vertex-/pivot-minors. We also see some notions about 3-uniform hypergraphs, which will be used in the proof of our main Theorem 7.1.1–7.1.4.

Prime graphs Note that a partition $(X, V \setminus X)$ of V is a split of a graph G if and only if $\min\{|X|, |V \setminus X|\} \geq 2$ and $\rho_G(X) \leq 1$. It is easy to observe the following.

Lemma 7.1.8. *Every prime graph with at least* 4 *vertices is* 2*-connected and has no twins.* □

Thus one may observe that every prime graph with at least 4 vertices has neither isolated vertices nor pendant vertices. It also follows that there is no prime graph with exactly 4 vertices.

Vertex-minors Recall that the local complementation at a vertex v to a graph G results in the following graph $G * v = (V(G), E(G) \triangle \{xy : x \text{ and } y \text{ are two distinct neighbors of } v\})$. The pivoting an edge vw of G results in the graph $G \wedge vw = G * v * w * v$.

Proposition 7.1.9 (Oum [92, Proposition 2.1]). *For a graph G and an edge vw of G, let G' be a graph on $V(G)$ such that*

$$\begin{aligned} E(G') = E(G) \triangle \{xy : x \in N_G(v) - (N_G(w) \cup \{w\}),\ y \in N_G(w) - (N_G(v) \cup \{v\})\} \\ \triangle \{xy : x \in N_G(v) - (N_G(w) \cup \{w\}),\ y \in N_G(v) \cap N_G(w)\} \\ \triangle \{xy : x \in N_G(w) - (N_G(v) \cup \{v\}),\ y \in N_G(v) \cap N_G(w)\}. \end{aligned}$$

Then $G \wedge vw$ is equal to the graph obtained from G' by exchanging the labels of v and w; see Figure 2.3.

The following proposition can be seen easily from the theory of isotropic systems [24], and Geelen and Oum [66] presented a short graph-theoretic proof.

Proposition 7.1.10 ([66, Lemma 3.1]). *Let G be a graph and v, w be its vertices.*

(i) *If $v \neq w$ and vw is not an edge of G, then $G * w \setminus v$, $G * w * v \setminus v$, $G * w/v$ are locally equivalent to $G \setminus v$, $G * v \setminus v$, G/v, respectively.*

(ii) *If $v \neq w$ and vw is an edge of G, then $G * w \setminus v$, $G * w * v \setminus v$, $G * w/v$ are locally equivalent to $G \setminus v$, G/v, $G * v \setminus v$, respectively.*

(iii) *If $v = w$, then $G * w \setminus v$, $G * w * v \setminus v$, $G * w/v$ are locally equivalent to $G * v \setminus v$, $G \setminus v$, G/v, respectively.*

As a corollary, we deduce the following result.

Corollary 7.1.11. *Locally equivalent graphs have the same set of non-essential vertices.* □

It is known from [92, Proposition 2.5] that $G \wedge xz = G \wedge xy \wedge yz$ for two edges xy and xz in a graph G, and therefore we deduce the following corollary.

Corollary 7.1.12. *Pivot-equivalent graphs have the same set of non-pivotal vertices.* □

3-uniform hypergraphs and tight paths A pair $H = (V, E)$ is a *hypergraph* if V is a finite set and E is a set of nonempty subsets of V, and we denote the vertex set of H by $V(H) = V$ and denote the edge set of H by $E(H) = E$. A hypergraph is 3-*uniform* if every edge has cardinality 3. A hypergraph H' is a *partial hypergraph* of a hypergraph H if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$.

A *tight path* P in a 3-uniform hypergraph H is a partial hypergraph that admits an ordering $v_0, v_1, \ldots, v_{k+1}$ of $V(P)$ where $k \geq 1$ and $E(P) = \{\{v_{i-1}, v_i, v_{i+1}\} : 1 \leq i \leq k\}$. We usually denote P by a sequence $v_0v_1v_2 \ldots v_{k+1}$ of distinct vertices. The *length* of a tight path is its number of edges. An *end* of a tight path is a vertex incident with exactly one edge of the tight path. Note that if $k \geq 2$, P has exactly two ends v_0 and v_{k+1}, and if $k = 1$, then P has exactly three ends v_0, v_1, and v_2. A tight path P in H is *maximal* if there is no tight path Q in H such that $E(P) \subsetneq E(Q)$.

Lemma 7.1.13. *A tight path $P = v_0v_1 \ldots v_{k+1}$ of length $k \geq 1$ in a 3-uniform hypergraph H is not maximal if and only if at least one of the following holds:*

(i) *There is a vertex w such that $v_0v_1 \ldots v_{k+1}w$ or $wv_0v_1 \ldots v_{k+1}$ is a tight path in H.*

(ii) *$k = 1$ and there is a vertex w such that $v_1v_0v_2w$ is a tight path in H.*

(iii) *$k = 2$ and there is a vertex w such that $v_0v_2v_1v_3w$ or $wv_0v_2v_1v_3$ is a tight path in H.*

Proof. If P is not maximal, then there is a tight path Q of length $k+1$ containing P. In a vertex ordering of Q certifying that Q is a tight path, a vertex ordering of P certifying that P is a tight path can be obtained by deleting the first or the last vertex which is not a vertex of P. It remains to enumerate all vertex ordering of P guaranteeing that P is a tight path.

If $k \geq 3$, then P admits a unique vertex ordering $v_0 v_1 \dots v_{k+1}$ up to reversing. If $k = 1$, then P admits three vertex ordering $v_0v_1v_2$, $v_0v_2v_1$, and $v_1v_0v_2$ up to reversing. If $k = 2$, then P admits two vertex orderings $v_0v_1v_2v_3$ and $v_0v_2v_1v_3$ up to reversing. □

Recall that an end of a tight path P is a vertex incident with exactly one edge of P. An *internal vertex* of P is a vertex incident with at least two edges of P. Equivalently, an internal vertex is a vertex of P that is not an end. It is readily shown that the set of ends of $P = v_0 v_1 \dots v_{k+1}$ is $\{v_0, v_{k+1}\}$ if $k \geq 2$, and $V(P)$ if $k = 1$.

7.1.2 Isotropic systems

We review isotropic systems defined by Bouchet [21, 24, 26]. We follow notations in [91, 93].

Let $K = \{0, \alpha, \beta, \gamma\}$ be a 2-dimensional vector space over the binary field $\mathbb{F}_2$, and let $\langle \cdot, \cdot \rangle_K : K \times K \to \mathbb{F}_2$ be a bilinear form such that $\langle x, y \rangle_K = 1$ if and only if $0 \neq x \neq y \neq 0$. For a finite set V, let K^V be the set of functions from V to K, regarded as a $2|V|$-dimensional vector space over $\mathbb{F}_2$. Let $\langle \cdot, \cdot \rangle : K^V \times K^V \to \mathbb{F}_2$ be a bilinear form such that $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{v \in V} \langle \mathbf{a}(v), \mathbf{b}(v) \rangle_K$. For a subspace L of K^V, let $L^\perp := \{\mathbf{a} \in K^V : \langle \mathbf{a}, \mathbf{b} \rangle = 0 \text{ for all } \mathbf{b} \in L\}$. The *support* of a vector $\mathbf{a} \in K^V$, denoted by $\operatorname{supp}(\mathbf{a})$, is the set of elements v in V such that $\mathbf{a}(v) \neq 0$. A vector $\mathbf{a} \in K^V$ is *complete* if $\operatorname{supp}(\mathbf{a}) = V$. Two vectors $\mathbf{a}$ and $\mathbf{b}$ in K^V are *supplementary* if they are complete and $\mathbf{a}(v) \neq \mathbf{b}(v)$ for every $v \in V$.

A subspace L of K^V is *totally isotropic* if $\langle \mathbf{a}, \mathbf{b} \rangle_K = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ in L, equivalently, $L \subseteq L^\perp$. Note that for every subspace L of K^V, we have $\dim(L) + \dim(L^\perp) = \dim(K^V) = 2|V|$; see Lang [80, Theorem 6.4]. Hence for a totally isotropic subspace L of K^V, we have $\dim(L) \leq |V|$ where the equality holds if and only if $L^\perp = L$. An *isotropic system* is a pair (V, L) consisting of a finite set V and a subspace L of K^V such that L is totally isotropic and $\dim(L) = |V|$. For an isotropic system $S = (V, L)$, we call each element $v \in V$ a *vertex* of S.

Minors For a subset X of V, let p_X be a map from K^V to K^X such that $(p_X(\mathbf{a}))(v) = \mathbf{a}(v)$. For $\mathbf{a} \in K^V$ and $X \subseteq V$, let $\mathbf{a}[X]$ be a vector in K^V such that

$$\mathbf{a}[X](v) = \begin{cases} \mathbf{a}(v) & \text{if } v \in X, \\ 0 & \text{otherwise.} \end{cases}$$

For a subspace L of K^V, $v \in V$, and $x \in K - \{0\}$, let

$$L|^v_x := \{p_{V-\{v\}}(\mathbf{a}) \in K^{V-\{v\}} : \mathbf{a} \in L \text{ and } \mathbf{a}(v) \in \{0, x\}\}.$$

For an isotropic system $S = (V, L)$, let $S|^v_x := (V - \{v\}, L|^v_x)$ be the *elementary minor* of S at $v \in V$ with respect to $x \in K - \{0\}$. The isotropic system S has three elementary minors $S|^v_\alpha$, $S|^v_\beta$, $S|^v_\gamma$ at v. Bouchet [21, (8.1)] proved that every elementary minor of an isotropic system is an isotropic system. An isotropic system S is a *minor* of an isotropic system S' if $S = S'|^{v_1}_{x_1} \dots |^{v_t}_{x_t}$ for some vertices $v_1, \dots, v_t$ of S' and $x_1, \dots, x_t \in K - \{0\}$.

Connectivity For a subspace L of K^V and a subset X of V, let

$$L|_{\subseteq X} := \{p_X(\mathbf{a}) : \mathbf{a} \in L \text{ and } \operatorname{supp}(\mathbf{a}) \subseteq X\} \text{ and}$$
$$L|_X := \{p_X(\mathbf{a}) : \mathbf{a} \in L\}.$$

The *connectivity function* of an isotropic system $S = (V, L)$ is a function $c_S : 2^V \to \mathbb{Z}$ such that $c_S(X) = |X| - \dim(L|_{\subseteq X})$. We omit the subscript S in c_S if it is clear from the context.

Lemma 7.1.14 (Oum [93, Lemma 5.1])**.** *Let L be a totally isotropic subspace of K^V and X be a subset of V. Then $(L|_{\subseteq X})^\perp = L^\perp|_X$.*

For an isotropic system $S = (V, L)$ and $X \subseteq V$, by Lemma 7.1.14, we have $(L|_{\subseteq X})^\perp = L^\perp|_X = L|_X$. Hence $\dim(L|_{\subseteq X}) + \dim(L|_X) = \dim(K^X) = 2|X|$ and $c_S(X) = \dim(L|_X) - |X|$.

Proposition 7.1.15 (Bouchet [26]; see Allys [1, Proposition 2.3.1])**.** *Let $S = (V, L)$ be an isotropic system with the connectivity function c. Then for all subsets X, $Y \subseteq V$, the following hold.*

(i) $0 \leq c(X) \leq |X|$.

(ii) $c(X) = c(V - X)$.

(iii) $c(X) + c(Y) \geq c(X \cup Y) + c(X \cap Y)$.

The following two lemmas display handy properties of the connectivity function.

Lemma 7.1.16 (Allys [1, Lemma 3.1])**.** *Let $S = (V, L)$ be an isotropic system with the connectivity function c. For a subset $X \subseteq V$ and a vertex $v \in V - X$, the following hold.*

(i) $c(X) - 1 \leq c(X \cup \{v\}) \leq c(X) + 1$.

(ii) $c(X \cup \{v\}) \leq c(X)$ *if and only if* $L|_{\subseteq X \cup \{v\}}$ *has a vector* $\mathbf{a}$ *such that* $\mathbf{a}(v) \neq 0$.

(iii) $c(X \cup \{v\}) = c(X) - 1$ *if and only if* $L|_{\subseteq X \cup \{v\}}$ *has vectors* $\mathbf{a}$, $\mathbf{b}$ *such that* $0 \neq \mathbf{a}(v) \neq \mathbf{b}(v) \neq 0$.

Lemma 7.1.17 (Allys [1, Proposition 3.2])**.** *Let $S = (V, L)$ be an isotropic system and $S|_x^v$ be its elementary minor such that L has no vector whose support is $\{v\}$. Let c and c' be the connectivity functions of S and $S|_x^v$, respectively. Then for a subset X of $V - \{v\}$, the following hold.*

(i) $c(X) - 1 \leq c'(X) \leq c(X)$ *and* $c(X \cup \{v\}) - 1 \leq c'(X) \leq c(X \cup \{v\})$.

(ii) $c'(X) = c(X) - 1$ *if and only if* $L|_{\subseteq X \cup \{v\}}$ *has a vector* $\mathbf{a}$ *such that* $\mathbf{a}(v) = x$.

For an isotropic system S and a positive integer k, a partition (X, Y) of the vertex set of S is a *k-separation* of S if $\min\{|X|, |Y|\} \geq k$ and $c_S(X) < k$. An isotropic system is *k-connected* if it has no k'-separation with $1 \leq k' < k$.

The following lemma is straightforward from the definition.

Lemma 7.1.18. *If $S = (V, L)$ is a 3-connected isotropic system with $|V| \geq 4$, then $|\operatorname{supp}(\mathbf{a})| \geq 3$ for every nonzero vector $\mathbf{a} \in L$.*

Proof. Suppose that there is a nonzero vector $\mathbf{a} \in L$ with $|\operatorname{supp}(\mathbf{a})| \leq 2$. Let $X = \operatorname{supp}(\mathbf{a})$. Then $\dim(L|_{\subseteq X}) \geq 1$ because of $\mathbf{a}$, so $c_S(X) = |X| - \dim(L|_{\subseteq X}) \leq |X| - 1$. Therefore, S has an $|X|$-separation $(X, V - X)$, which contradicts the assumption that S is 3-connected. □

For an isotropic system S, a vertex v is *non-essential* if at least two of $S|_\alpha^v$, $S|_\beta^v$, $S|_\gamma^v$ are 3-connected, and *essential* otherwise.

Fundamental graphs For a graph $G = (V, E)$ and two supplementary vectors $\mathbf{a}$ and $\mathbf{b}$ in K^V, let L_G be the subspace of K^V spanned by $\{\mathbf{a}[N_G(v)] + \mathbf{b}[\{v\}] : v \in V\}$. Bouchet [24, (3.1)] proved that $S = (V, L_G)$ is an isotropic system. We call a triple $(G, \mathbf{a}, \mathbf{b})$ a *graphic presentation* of S.

A vector $\mathbf{a} \in K^V$ is an *Eulerian* vector of an isotropic system $S = (V, L)$ if $\mathbf{a}$ is complete and $\mathbf{a}[X] \notin L$ for every nonempty subset X of V.

Lemma 7.1.19 (Bouchet [24, (4.1)]). *Let S be an isotropic system. For every complete vector $\mathbf{c}$, there is an Eulerian vector $\mathbf{a}$ of S supplementary to $\mathbf{c}$.*

Proposition 7.1.20 (Bouchet [24, (4.3) and (4.4)]). *Let $\mathbf{a}$ be an Eulerian vector of an isotropic system $S = (V, L)$. Then for each $v \in V$, there is a unique vector $\mathbf{b}_v \in L$ such that*

(i) $\langle \mathbf{b}_v(v), \mathbf{a}(v)\rangle_K = 1$, *and*

(ii) $\langle \mathbf{b}_v(w), \mathbf{a}(w)\rangle_K = 0$ *for all $w \in V - \{v\}$.*

Moreover, $\mathbf{b}_v(w) \neq 0$ if and only if $\mathbf{b}_w(v) \neq 0$ for all distinct $v, w \in V$, and $\{\mathbf{b}_v : v \in V\}$ is a base of L.

The set of such vectors $\mathbf{b}_v$ for all $v \in V$ is called the *fundamental base* of L with respect to $\mathbf{a}$. The *fundamental graph* of an isotropic system $S = (V, L)$ with respect to an Eulerian vector $\mathbf{a}$ is a graph on V such that two vertices v and w are adjacent if and only if $\mathbf{b}_v(w) \neq 0$, where $\{\mathbf{b}_v : v \in V\}$ is the fundamental base of S with respect to $\mathbf{a}$. Let $\mathbf{b}$ be the complete vector such that $\mathbf{b}(v) = \mathbf{b}_v(v)$. Then $\mathbf{b}$ is supplementary to $\mathbf{a}$. The following proposition shows that $(G, \mathbf{a}, \mathbf{b})$ is a graphic presentation of S.

Proposition 7.1.21 (Bouchet [24, (4.5)]). *Let S be an isotropic system.*

(i) *If $(G, \mathbf{a}, \mathbf{b})$ is a graphic presentation of S, then $\mathbf{a}$ is an Eulerian vector of S.*

(ii) *For an Eulerian vector $\mathbf{a}$ of S, let G be the fundamental graph of S with respect to $\mathbf{a}$, let $\{\mathbf{b}_v : v \in V\}$ be the fundamental base of S with respect to $\mathbf{a}$, and let $\mathbf{b}$ be the complete vector such that $\mathbf{b}(v) = \mathbf{b}_v(v)$ for all $v \in V$. Then $(G, \mathbf{a}, \mathbf{b})$ is a graphic presentation of S. Furthermore, if $(G, \mathbf{a}, \mathbf{b}')$ is a graphic presentation of S, then $\mathbf{b}' = \mathbf{b}$.*

Bouchet [26] explains a relation between the connectivity function of an isotropic system and the cut-rank of its fundamental graph.

Proposition 7.1.22 (Bouchet [26, Theorem 6]). *Let G be a fundamental graph of an isotropic system S. Then $c_S(X) = \rho_G(X)$ for every subset X of the vertex set of G.*

Corollary 7.1.23 (Bouchet [26, Theorem 11]). *Let G be a fundamental graph of an isotropic system S with at least four vertices. Then S is 3-connected if and only if G is prime.*

Lemma 7.1.24 (Bouchet [26, Theorem 23]). *No isotropic system on 4 vertices is 3-connected.*

An isotropic system is *cyclic* if it has a cycle graph of length at least 5 as a fundamental graph.

Lemma 7.1.25 (Bouchet [26, Theorem 23]). *An isotropic system on 5 vertices is 3-connected if and only if it is cyclic.*

For a vertex v of C_n with $n \geq 5$, neither $C_n \setminus v$ nor C_n/v is prime, and therefore we deduce the following.

Lemma 7.1.26 (Allys [1, Lemma 4.2]). *If S is a cyclic isotropic system with at least 5 vertices, then S is prime and every vertex is essential.*

We dedicate the remainder of this subsection to explaining the relation between minors of an isotropic system and vertex-minors of its fundamental graph.

Lemma 7.1.27 (Bouchet [21, (9.4)]). *Let* $\mathbf{a}$ *be an Eulerian vector of an isotropic system* $S = (V, L)$*, and let* v *be a vertex of* S*. Let* $\mathbf{a}'$ *and* $\mathbf{a}''$ *be two complete vectors such that* $\mathbf{a}[V-\{v\}] = \mathbf{a}'[V-\{v\}] = \mathbf{a}''[V-\{v\}]$ *and* $\{\mathbf{a}(v), \mathbf{a}'(v), \mathbf{a}''(v)\} = K - \{0\}$*. Then exactly one of* $\mathbf{a}'$ *and* $\mathbf{a}''$ *is an Eulerian vector of* S*.*

We write $\mathbf{a} * v$ to denote such an Eulerian vector $\mathbf{a}'$ or $\mathbf{a}''$ in Lemma 7.1.27.

Lemma 7.1.28 (Bouchet [24, (7.1)]). *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be Eulerian vectors of an isotropic system* S*. Then there is a sequence of vertices* $v_1, v_2, \ldots, v_k$ *such that* $\mathbf{b} = \mathbf{a} * v_1 * v_2 * \cdots * v_k$*.*

Proposition 7.1.29 (Bouchet [24, (7.6) and (8.3)]). *Let* $(G, \mathbf{a}, \mathbf{b})$ *be a graphic presentation of* S*. For a vertex* u *and an edge* vw *of* G*,*

$$(G, \mathbf{a}, \mathbf{b}) * u := (G * u,\ \mathbf{a} + \mathbf{b}[\{u\}],\ \mathbf{a}[N_G(u)] + \mathbf{b})$$

and

$$(G, \mathbf{a}, \mathbf{b}) \wedge vw := (G \wedge vw,\ \mathbf{a}[V - \{v, w\}] + \mathbf{b}[\{v, w\}],\ \mathbf{a}[\{v, w\}] + \mathbf{b}[V - \{v, w\}])$$

are graphic presentations of S*.*

Therefore, fundamental graphs of an isotropic system are locally equivalent. We say that two graphic presentations $(G, \mathbf{a}, \mathbf{b})$ and $(H, \mathbf{c}, \mathbf{d})$ of an isotropic system are *locally equivalent* if $(H, \mathbf{c}, \mathbf{d}) = (G, \mathbf{a}, \mathbf{b}) * v_1 \cdots * v_m$ for some vertices $v_1, \ldots, v_m$. They are *pivot-equivalent* if $(H, \mathbf{c}, \mathbf{d}) = (G, \mathbf{a}, \mathbf{b}) \wedge e_1 \cdots \wedge e_m$ for some edges $e_1, \ldots, e_m$.

Proposition 7.1.30 (Bouchet [24, (9.1)]; see Oum [93, Proposition 3.7]). *Let* $(G, \mathbf{a}, \mathbf{b})$ *be a graphic presentation of* $S = (V, L)$*. Then one of the following is a graphic presentation of* $S|^v_x$*.*

(i) $(G \setminus v,\ p_{V-\{v\}}(\mathbf{a}),\ p_{V-\{v\}}(\mathbf{b}))$ *if either* $x = \mathbf{a}(v)$ *or* v *is an isolated vertex,*

(ii) $(G \wedge vw \setminus v,\ p_{V-\{v\}}(\mathbf{a}[V - \{v, w\}] + \mathbf{b}[\{v, w\}]),\ p_{V-\{v\}}(\mathbf{a}[\{v, w\}] + \mathbf{b}[V - \{v, w\}]))$ *if* $x = \mathbf{b}(v)$ *and* w *is a neighbor of* v*, and*

(iii) $(G * v \setminus v,\ p_{V-\{v\}}(\mathbf{a}),\ p_{V-\{v\}}(\mathbf{a}[N_G(v)] + \mathbf{b}))$ *otherwise.*

Corollary 7.1.31. *Let* G *be a fundamental graph of an isotropic system* S *with at least five vertices. A vertex of* S *is non-essential in* S *if and only if it is non-essential in* G*.* □

7.1.3 Triangles in 3-connected isotropic systems

Let $S = (V, L)$ be an isotropic system. A *triangle* in S is a vector in L such that the size of its support is 3. Let $H(S)$ be the 3-uniform hypergraph on V whose edge set is the set of supports of triangles in S. First, we present several lemmas of Allys [1] which show the existence of triangles whose supports contain some essential vertices in a 3-connected isotropic system.

Lemma 7.1.32 (Allys [1, Lemma 3.3]). *Let* $S = (V, L)$ *be a* 3*-connected isotropic system with* $|V| \geq 4$*. If* $\mathbf{t}$ *and* $\mathbf{t}'$ *are triangles in* S*, then one of the following holds.*

(i) $\mathrm{supp}(\mathbf{t})$ *and* $\mathrm{supp}(\mathbf{t}')$ *are disjoint.*

(ii) $\mathrm{supp}(\mathbf{t}) \cap \mathrm{supp}(\mathbf{t}') = \{v\}$ *and* $\mathbf{t}(v) = \mathbf{t}'(v)$ *for some* $v \in V$*.*

(iii) $\text{supp}(\mathbf{t}) \cap \text{supp}(\mathbf{t}') = \{v, w\}$, $\mathbf{t}(v) \neq \mathbf{t}'(v)$, *and* $\mathbf{t}(w) \neq \mathbf{t}'(w)$ *for some* $v, w \in V$.

(iv) $\mathbf{t} = \mathbf{t}'$.

By Lemma 7.1.32, in a 3-connected isotropic system S with at least 4 vertices, triangles have distinct supports, and thus there is a bijection from the set of triangles of S to the set of edges of $H(S)$. Now we investigate what vertices of a tight path in $H(S)$ are essential or non-essential in S. Recall that a vertex v of S is essential if at least two of $S|^v_\alpha$, $S|^v_\beta$, $S|^v_\gamma$ are 3-connected.

Lemma 7.1.33. *Let S be an isotropic system with at least 5 vertices, and let $\mathbf{t}$ be a triangle in S. Then for each v in the support of $\mathbf{t}$, $S|^v_{\mathbf{t}(v)}$ is not 3-connected.*

Proof. A minor $S|^v_{\mathbf{t}(v)}$ has a nonzero vector $p_{V-\{v\}}(\mathbf{t})$ whose support has size 2. By Lemma 7.1.18, $S|^v_{\mathbf{t}(v)}$ is not 3-connected. □

Lemma 7.1.34. *Let S be a 3-connected isotropic system with at least 5 vertices. Every internal vertex of a tight path in $H(S)$ is essential in S.*

Proof. It is enough to prove that if $\mathbf{t}$ and $\mathbf{t}'$ are triangles in S such that $\text{supp}(\mathbf{t}) \cap \text{supp}(\mathbf{t}') = \{v, w\}$, then v is essential in S. By Lemma 7.1.33, neither $S|^v_{\mathbf{t}(v)}$ nor $S|^v_{\mathbf{t}'(v)}$ is 3-connected. By Lemma 7.1.32, $\mathbf{t}(v) \neq \mathbf{t}'(v)$ and, therefore, v is essential in S. □

Lemma 7.1.35. *Let S be a 3-connected isotropic system. For distinct vertices u and v of $H(S)$, there are at most three edges of $H(S)$ incident with both u and v.*

Proof. Suppose that there are four distinct triangles $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, and $\mathbf{t}_4$ in S whose supports contain both u and v. Then $\mathbf{t}_k(u) \in K - \{0\} = \{\alpha, \beta, \gamma\}$ for each $1 \le k \le 4$. So $\mathbf{t}_i(u) = \mathbf{t}_j(u)$ for some distinct i and j. By Lemma 7.1.32, $\mathbf{t}_i = \mathbf{t}_j$, which is a contradiction. □

Lemma 7.1.36. *Let S be a 3-connected isotropic system. Let $\{u, v, w_1\}$ and $\{u, v, w_2\}$ be distinct edges in $H(S)$. If e is an edge incident with u in $H(S)$, then e is incident with at least one of v, w_1, and w_2.*

Proof. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be triangles in S whose supports are $\{u, v, w_1\}$ and $\{u, v, w_2\}$, respectively. Let $\mathbf{t}$ be a triangle whose support is e. By Lemma 7.1.32, $\mathbf{t}_1(u) \neq \mathbf{t}_2(u)$. Without loss of generality, $\mathbf{t}(u) \neq \mathbf{t}_1(u)$. By Lemma 7.1.32, $|\text{supp}(\mathbf{t}) \cap \text{supp}(\mathbf{t}_1)| = 2$, so e is incident with v or w_1. □

Lemma 7.1.37 (Allys [1, Lemma 3.5]). *Let $S = (V, L)$ be a 3-connected isotropic system with at least 4 vertices. For a vertex $v \in V$ and two distinct $x, y \in K - \{0\}$, if neither $S|^v_x$ nor $S|^v_y$ is 3-connected, then S has a triangle $\mathbf{t}$ such that $\mathbf{t}(v) \in \{x, y\}$.*

Lemma 7.1.37 implies that for a 3-connected isotropic system S with at least 4 vertices, if a vertex is essential, then it is incident with an edge of $H(S)$. The following lemma is proved by Allys [1, Lemma 3.5].[1]

Lemma 7.1.38 ([1]). *Let $\mathbf{t}$ be a triangle in a 3-connected isotropic system S with at least 4 vertices, where $\text{supp}(\mathbf{t}) = \{u, v, w\}$. For $x \in K - \{0, \mathbf{t}(u)\}$ and $y \in K - \{0, \mathbf{t}(w)\}$, if neither $S|^u_x$ nor $S|^w_y$ is 3-connected, then there are triangles $\mathbf{t}_1$ and $\mathbf{t}_2$ (possibly $\mathbf{t}_1 = \mathbf{t}_2$) such that $\mathbf{t}_1(u) = x$, $\mathbf{t}_2(w) = y$, and $\mathbf{t}_1(v) = \mathbf{t}_2(v)$.*

Lemma 7.1.39. *For a 3-connected isotropic system S with at least 4 vertices, if an edge e of $H(S)$ is incident with at least two essential vertices in S, then $H(S)$ has an edge e' such that $|e \cap e'| = 2$.*

[1] In [1], there are two Lemmas 3.5, and this is the second Lemma 3.5.

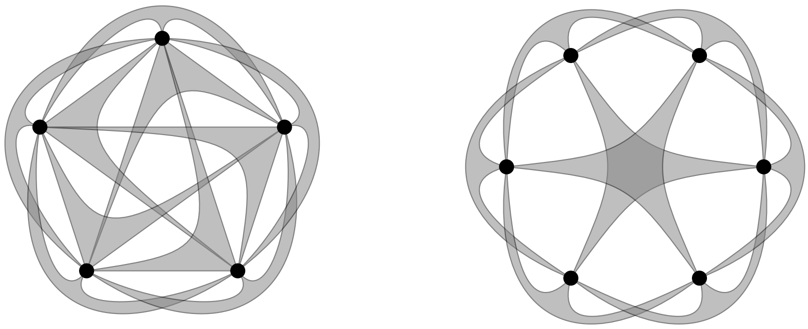


Figure 7.2: Illustrations of $H(S)$ for an isotropic system S whose fundamental graph is C_5 or C_6.

Proof. Let $u, w \in e$ be distinct essential vertices in S and let $\mathbf{t}$ be a triangle in S such that $\operatorname{supp}(\mathbf{t}) = e$. Because u and w are essential, we have $x \in K - \{0, \mathbf{t}(u)\}$ and $y \in K - \{0, \mathbf{t}(w)\}$ such that neither $S|_x^u$ nor $S|_y^w$ is 3-connected. By Lemma 7.1.38, S has a triangle $\mathbf{t}_1$ such that $\mathbf{t}_1(u) = x$. Then an edge $e' := \operatorname{supp}(\mathbf{t}_1)$ of $H(S)$ satisfies that $|e \cap e'| = 2$ by Lemma 7.1.32. □

Lemma 7.1.39 provides a sufficient condition for extending a tight path of length 1. Now we aim to prove that two ends of a maximal tight path in $H(S)$ are non-essential unless S is cyclic.

In the next two lemmas, we show that under some assumptions, no internal vertex of a tight path of length at least 3 in $H(S)$ is incident with edges not on the path. By Figure 7.2, it is necessary to require that S does not have C_5 or C_6 as a fundamental graph. Our proof of the following lemma is motivated by the proof of Claim 3 in Allys [1, Theorem 4.3] proving a weaker statement.

Lemma 7.1.40. *Let $S = (V, L)$ be a 3-connected isotropic system and let $P = v_0v_1v_2v_3v_4$ be a tight path in $H(S)$. If neither C_5 nor C_6 is a fundamental graph of S, then P contains every edge of $H(S)$ incident with v_2.*

Proof. By Lemma 7.1.25, we may assume that $|V| \geq 6$. For each $1 \leq i \leq 3$, let $\mathbf{t}_i$ be a triangle whose support is $\{v_{i-1}, v_i, v_{i+1}\}$. By Lemma 7.1.32 applied to $\mathbf{t}_1$ and $\mathbf{t}_3$, we have $\mathbf{t}_1(v_2) = \mathbf{t}_3(v_2)$. By Lemma 7.1.32 applied to $\mathbf{t}_1$ and $\mathbf{t}_2$, we have $\mathbf{t}_1(v_1) \neq \mathbf{t}_2(v_1)$ and $\mathbf{t}_1(v_2) \neq \mathbf{t}_2(v_2)$. Also by applying Lemma 7.1.32 to $\mathbf{t}_2$ and $\mathbf{t}_3$, we have $\mathbf{t}_2(v_2) \neq \mathbf{t}_3(v_2)$ and $\mathbf{t}_2(v_3) \neq \mathbf{t}_3(v_3)$.

Suppose for contradiction that S has a triangle $\mathbf{t}$ such that $v_2 \in \operatorname{supp}(\mathbf{t})$ and $\mathbf{t} \neq \mathbf{t}_i$ for all $i \in \{1, 2, 3\}$.

We first claim that $\mathbf{t}(v_2) = \mathbf{t}_2(v_2)$. Suppose that $\mathbf{t}(v_2) \neq \mathbf{t}_2(v_2)$. By Lemma 7.1.32, $|\operatorname{supp}(\mathbf{t}) \cap \operatorname{supp}(\mathbf{t}_2)| = 2$. Therefore there is a unique $j \in \{1, 3\}$ such that $v_j \in \operatorname{supp}(\mathbf{t}) \cap \operatorname{supp}(\mathbf{t}_2)$ and furthermore $\mathbf{t}_2(v_j) \neq \mathbf{t}(v_j)$. By reversing the path if necessary, we may assume that $j = 3$ and $v_1 \notin \operatorname{supp}(\mathbf{t})$. Since $v_2, v_3 \in \operatorname{supp}(t) \cap \operatorname{supp}(\mathbf{t}_3)$, by applying Lemma 7.1.32, we deduce that $\mathbf{t}(v_3) \neq \mathbf{t}_3(v_3)$ and $\mathbf{t}(v_2) \neq \mathbf{t}_3(v_2)$. Since $\mathbf{t}_3(v_2) \neq \mathbf{t}_2(v_2)$, we deduce that $\mathbf{t}(v_2) = \mathbf{t}_2(v_2) + \mathbf{t}_3(v_2) = \mathbf{t}_2(v_2) + \mathbf{t}_1(v_2) \neq \mathbf{t}_1(v_2)$. By Lemma 7.1.32, $|\operatorname{supp}(\mathbf{t}) \cap \operatorname{supp}(t_1)| = 2$ and therefore $v_0 \in \operatorname{supp}(\mathbf{t})$. Thus we deduce that $\operatorname{supp}(\mathbf{t}) = \{v_0, v_2, v_3\}$. As $\mathbf{t}_1(v_3) = 0$ and $\mathbf{t}_2(v_3) \neq \mathbf{t}(v_3)$, we deduce that $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}$ are linearly independent, so $c(\{v_0, v_1, v_2, v_3\}) = 4 - \dim(L|_{\subseteq\{v_0,v_1,v_2,v_3\}}) \leq 1$, where c is the connectivity function of S. This contracts to the assumption that S is 3-connected. Therefore, $\mathbf{t}(v_2) = \mathbf{t}_2(v_2)$.

By Lemma 7.1.32 for $\mathbf{t}$ and $\mathbf{t}_2$, the support of $\mathbf{t}$ contains neither v_1 nor v_3. Since $\mathbf{t}(v_2) \neq \mathbf{t}_1(v_2)$, by Lemma 7.1.32, $\operatorname{supp}(\mathbf{t})$ contains v_0 and $\mathbf{t}(v_0) \neq \mathbf{t}_1(v_0)$. Similarly, as $\mathbf{t}(v_2) \neq \mathbf{t}_3(v_2)$, the support of $\mathbf{t}$ contains v_4 and $\mathbf{t}(v_4) \neq \mathbf{t}_3(v_4)$. Hence $\mathrm{supp}(\mathbf{t}) = \{v_0, v_2, v_4\}$ and $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, $\mathbf{t}$ are linearly independent, so $c(\{v_0, v_1, v_2, v_3, v_4\}) \leq 5 - \dim(L|_{\subseteq\{v_0,v_1,v_2,v_3,v_4\}}) = 1$. Since S is 3-connected, we have $|V - \{v_0, v_1, v_2, v_3, v_4\}| \leq 1$ and therefore $|V| = 6$. Let v_5 denote the vertex of V other than $v_0, \ldots, v_4$. Because of $\mathbf{t}_1$, we have $\dim(L|_{\subseteq\{v_0,v_1,v_2\}}) \geq 1$. By Proposition 7.1.15(ii), $c(\{v_3, v_4, v_5\}) = c(\{v_0, v_1, v_2\})$ and thus $\dim(L|_{\subseteq\{v_3,v_4,v_5\}}) = \dim(L|_{\subseteq\{v_0,v_1,v_2\}}) \geq 1$. Then S has a nonzero vector $\mathbf{t}_4$ such that $\mathrm{supp}(\mathbf{t}_4) \subseteq \{v_3, v_4, v_5\}$. By Lemma 7.1.18, $\mathbf{t}_4$ is a triangle whose support is $\{v_3, v_4, v_5\}$. Similarly, because of $\mathbf{t}_2$ and $\mathbf{t}_3$, there are triangles $\mathbf{t}_5$ and $\mathbf{t}_0$ whose supports are $\{v_4, v_5, v_0\}$ and $\{v_5, v_0, v_1\}$, respectively. Let $v_{-1} := v_5$, $v_6 := v_0$, $\mathbf{t}_6 := \mathbf{t}_0$, and $\mathbf{t}_7 := \mathbf{t}_1$. Let C_6 be the cycle graph on $\{v_0, v_1, \ldots, v_5\}$ in this order and let $\mathbf{a}$ and $\mathbf{b}$ be vectors in K^V such that $\mathbf{a}(v_i) = \mathbf{t}_{i+1}(v_i)$ and $\mathbf{b}(v_i) = \mathbf{t}_i(v_i)$ for each $0 \leq i \leq 5$. Then, for every $0 \leq i \leq 5$,

- $\mathbf{a}$ and $\mathbf{b}$ are supplementary by Lemma 7.1.32,
- $\mathbf{t}_i(v_{i+1}) = \mathbf{t}_{i+2}(v_{i+1}) = \mathbf{a}(v_{i+1})$ by Lemma 7.1.32,
- $\mathbf{t}_i(v_{i-1}) = \mathbf{a}(v_{i-1})$ by the definition of $\mathbf{a}$,
- $\mathbf{t}_i(v_i) = \mathbf{b}(v_i)$ by the definition of $\mathbf{b}$, and
- $\mathbf{t}_i(w) = 0$ for all $w \in V \setminus \mathrm{supp}(\mathbf{t}_i)$.

Then $\mathbf{t}_i = \mathbf{a}[N_{C_6}(v_i)] + \mathbf{b}[\{v_i\}]$ for each i. Therefore $(C_6, \mathbf{a}, \mathbf{b})$ is a graphic presentation of S and so C_6 is a fundamental graph of S, contradicting the assumption. □

Lemma 7.1.41. *Let S be a 3-connected isotropic system. Let $P = v_0v_1v_2v_3v_4v_5$ be a tight path in $H(S)$. If v_0 is non-essential in S, then P contains every edge of $H(S)$ incident with v_1.*

Proof. Suppose that there is an edge e of $H(S)$ incident with v_1 and not in P. By Lemma 7.1.36, e is incident with at least one of v_0, v_2, and v_3. Since v_0 is non-essential in S, C_6 is not a fundamental graph of S and therefore neither v_2 nor v_3 is incident with e by Lemma 7.1.40. Thus e is incident with both v_0 and v_1. Let $v \neq v_0, v_1$ be a vertex incident with e. Then $vv_0v_1v_2$ is a tight path and by Lemma 7.1.34, v_0 is essential in S, contradicting the assumption. □

By Lemmas 7.1.40 and 7.1.41, we deduce the following.

Proposition 7.1.42. *For a 3-connected isotropic system S, if $H(S)$ has a tight path $P = v_0v_1 \cdots v_{k+1}$ of length $k \geq 4$ such that v_0 and v_{k+1} are non-essential in S, then P contains every edge of $H(S)$ incident with at least one of $v_1, v_2, \ldots, v_k$.* □

Lemma 7.1.43. *Let $S = (V, L)$ be a 3-connected isotropic system. If X is a subset of V such that $\min\{|X|, |V - X|\} \geq 2$ and $\dim(L|_{\subseteq X}) \geq |X| - 2$, then $\dim(L|_X) = |X| + 2$.*

Proof. Recall that by Lemma 7.1.14, $\dim(L|_{\subseteq X}) + \dim(L|_X) = \dim(K^X) = 2|X|$ and $c(X) = |X| - \dim(L|_{\subseteq X}) = \dim(L|_X) - |X|$. Hence $\dim(L|_X) = 2|X| - \dim(L|_{\subseteq X}) \leq |X| + 2$. Since S is 3-connected and $\min\{|X|, |V - X|\} \geq 2$, we have $c(X) \geq 2$ and so $\dim(L|_X) = |X| + c(X) \geq |X| + 2$. Therefore, $\dim(L|_X) = |X| + 2$. □

Lemma 7.1.44. *Let $S = (V, L)$ be a 3-connected isotropic system. Let $v_0v_1 \cdots v_{k+1}$ be a tight path of length $k \geq 3$ in $H(S)$. If $\{v_k, v_{k+1}, v_0\}$ is an edge of $H(S)$, then C_{k+2} is a fundamental graph of S.*

Proof. For each $1 \le i \le k$, let $\mathbf{t}_i$ be a triangle in S whose support is $\{v_{i-1}, v_i, v_{i+1}\}$. Let $\mathbf{t}_{k+1}$ be a triangle in S whose support is $\{v_k, v_{k+1}, v_0\}$. By Lemma 7.1.32, we deduce that $\mathbf{t}_{k+1}(v_0) = \mathbf{t}_1(v_0)$ and $\mathbf{t}_{i-1}(v_i) = \mathbf{t}_{i+1}(v_i)$ for every $2 \le i \le k$. Also by Lemma 7.1.32, we have that $\mathbf{t}_i(v_i) \neq \mathbf{t}_{i+1}(v_i)$ and $\mathbf{t}_i(v_{i+1}) \neq \mathbf{t}_{i+1}(v_{i+1})$ for all $1 \le i \le k$.

Let $X = \{v_2, v_3, \dots, v_k\}$. We claim that $p_X(\mathbf{t}_1), p_X(\mathbf{t}_2), \dots, p_X(\mathbf{t}_{k+1})$ are linearly independent. Suppose that $\sum_{i=1}^{k+1} c_i p_X(\mathbf{t}_i) = 0$ for some $c_1, c_2, \dots, c_{k+1} \in \mathbb{F}_2(2)$. For $1 < j < k+1$, $\sum_{i=1}^{k+1} c_i \mathbf{t}_i(v_j) = c_{j-1}\mathbf{t}_{j-1}(v_j) + c_j \mathbf{t}_j(v_j) + c_{j+1}\mathbf{t}_{j+1}(v_j) = c_j \mathbf{t}_j(v_j) + (c_{j-1} + c_{j+1})\mathbf{t}_{j+1}(v_j)$ and thus $c_j = 0$ because $\mathbf{t}_j(v_j)$ and $\mathbf{t}_{j+1}(v_j)$ are linearly independent in K. So $c_1 p_X(\mathbf{t}_1) + c_{k+1} p_X(\mathbf{t}_{k+1}) = 0$. Since $0 = c_1 \mathbf{t}_1(v_2) + c_{k+1}\mathbf{t}_{k+1}(v_2) = c_1 \mathbf{t}_1(v_2)$, we deduce that $c_1 = 0$ and so $c_{k+1} = 0$. Therefore $p_X(\mathbf{t}_1), p_X(\mathbf{t}_2), \dots, p_X(\mathbf{t}_{k+1})$ are linearly independent. This also implies that $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{k+1}$ are linearly independent.

Hence $c(\{v_0, v_1, \dots, v_{k+1}\}) = k + 2 - \dim(L|_{\subseteq \{v_0, v_1, \dots, v_{k+1}\}}) \le 1$. Since S is 3-connected, $|V - \{v_0, v_1, \dots, v_{k+1}\}| \le 1$. Hence $|V| = k+3$ or $k+2$.

We have $\dim(L|_{\subseteq X}) \ge k - 3 = |X| - 2$ because $\mathbf{t}_3, \mathbf{t}_4, \dots, \mathbf{t}_{k-1}$ are linearly independent. By Lemma 7.1.43, $\dim(L|_X) = |X| + 2 = k + 1$. Therefore $p_X(\mathbf{t}_1), p_X(\mathbf{t}_2), \dots, p_X(\mathbf{t}_{k+1})$ form a base of $L|_X$.

Suppose $|V| = k + 3$. Let w be the vertex of V other than $v_0, v_1, \dots, v_{k+1}$. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in L such that $\{\mathbf{t}_1, \dots, \mathbf{t}_{k+1}, \mathbf{a}, \mathbf{b}\}$ is a base of L. Since $\{p_X(\mathbf{t}_1), p_X(\mathbf{t}_2), \dots, p_X(\mathbf{t}_{k+1})\}$ is a base of $L|_X$, we may assume that $p_X(\mathbf{a}) = 0$ and $p_X(\mathbf{b}) = 0$. Hence the supports of $\mathbf{a}$ and $\mathbf{b}$ are subsets of $V - X = \{v_0, v_1, v_{k+1}, w\}$. Then $\mathrm{supp}(\mathbf{a}) \cap \mathrm{supp}(\mathbf{t}_2) \subseteq \{v_1\}$, so $0 = \langle \mathbf{a}, \mathbf{t}_2 \rangle = \langle \mathbf{a}(v_1), \mathbf{t}_2(v_1) \rangle_K$. It implies that $\mathbf{a}(v_1) \in \{0, \mathbf{t}_2(v_1)\}$ and similarly $\mathbf{b}(v_1) \in \{0, \mathbf{t}_2(v_1)\}$. Then one of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} + \mathbf{b}$, say $\mathbf{c}$, satisfies $\mathbf{c}(v_1) = 0$. Then $\mathrm{supp}(\mathbf{c}) \subseteq \{v_0, v_{k+1}, w\}$. Since S is 3-connected and $\mathbf{c}$ is a nonzero vector, $\mathrm{supp}(\mathbf{c}) = \{v_0, v_{k+1}, w\}$ by Lemma 7.1.18. Then $\mathrm{supp}(\mathbf{c}) \cap \mathrm{supp}(\mathbf{t}_{k+1}) = \{v_0, v_{k+1}\}$. Since $v_0 \in \mathrm{supp}(\mathbf{t}_1)$, by Lemma 7.1.36, the support of $\mathbf{t}_1$ contains v_k, v_{k+1}, or w, contradicting that $\mathrm{supp}(\mathbf{t}_1) = \{v_0, v_1, v_2\}$. Therefore, $|V| = k + 2$.

Let $\mathbf{t}_0$ be a vector in L such that $\{\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{k+1}\}$ is a base of S. Since $\{p_X(\mathbf{t}_1), \dots, p_X(\mathbf{t}_{k+1})\}$ is a base of $L|_X$, we may assume that $p_X(\mathbf{t}_0) = 0$ and therefore the support of $\mathbf{t}_0$ is a subset of $V - X = \{v_{k+1}, v_0, v_1\}$. Since S is 3-connected and $\mathbf{t}_0$ is nonzero, by Lemma 7.1.18, $\mathrm{supp}(\mathbf{t}_0) = \{v_{k+1}, v_0, v_1\}$. Let $\mathbf{a}, \mathbf{b} \in K^V$ be vectors such that $\mathbf{a}(v_i) = \mathbf{t}_{i+1}(v_i)$ and $\mathbf{b}(v_i) = \mathbf{t}_i(v_i)$ for all $0 \le i \le k+1$, where $\mathbf{t}_{k+2} := \mathbf{t}_0$. Let C_{k+2} be the cycle graph on $\{v_0, v_1, \dots, v_{k+1}\}$ in this order. Let $v_{-1} := v_{k+1}$, $v_{k+2} := v_0$, and $\mathbf{t}_{k+3} := \mathbf{t}_1$. Then, for all $i \in \{0, 1, 2, \dots, k+1\}$,

- $\langle \mathbf{a}(v_i), \mathbf{b}(v_i) \rangle_K = \langle \mathbf{t}_{i+1}(v_i), \mathbf{t}_i(v_i) \rangle_K = 1$ by Lemma 7.1.32,
- $\mathbf{t}_i(v_{i+1}) = \mathbf{t}_{i+2}(v_{i+1}) = \mathbf{a}(v_{i+1})$ by Lemma 7.1.32,
- $\mathbf{t}_i(v_{i-1}) = \mathbf{a}(v_{i-1})$ by the definition of $\mathbf{a}$,
- $\mathbf{t}_i(v_i) = \mathbf{b}(v_i)$ by the definition of $\mathbf{b}$, and
- $\mathbf{t}_i(u) = 0$ for all $u \in V$ with $u \neq v_{i-1}, v_i, v_{i+1}$ because $\mathrm{supp}(\mathbf{t}_i) = \{v_{i-1}, v_i, v_{i+1}\}$.

Thus $\mathbf{t}_i = \mathbf{a}[N_{C_{k+2}}(v_i)] + \mathbf{b}[\{v_i\}]$ for all i and therefore $(C_{k+2}, \mathbf{a}, \mathbf{b})$ is a graphic presentation of S. So C_{k+2} is a fundamental graph of S. □

The following lemma provides a sufficient condition to extend a tight path.

Lemma 7.1.45. *Let $S = (V, L)$ be a 3-connected isotropic system. Let $v_0 v_1 \dots v_{k+1}$ be a tight path of length $k \ge 2$ in $H(S)$. If S is not cyclic and v_{k+1} is essential in S, then $H(S)$ has a vertex v_{k+2} such that*

(i) $v_0v_1v_2v_3v_4$ *or* $v_0v_2v_1v_3v_4$ *is a tight path in* $H(S)$ *if* $k=2$, *and*

(ii) $v_0v_1\dots v_{k+1}v_{k+2}$ *is a tight path in* $H(S)$ *if* $k\geq 3$.

Proof. Observe that $|V|\geq 6$ by Lemmas 7.1.24 and 7.1.25 because S is 3-connected and not cyclic and $|V|\geq k+2\geq 4$.

Let $\mathbf{a}$ and $\mathbf{b}$ be triangles in S whose supports are $\{v_{k-2},v_{k-1},v_k\}$ and $\{v_{k-1},v_k,v_{k+1}\}$, respectively. By Lemma 7.1.32, $\mathbf{a}(v_{k-1})\neq\mathbf{b}(v_{k-1})$ and $\mathbf{a}(v_k)\neq\mathbf{b}(v_k)$.

Since v_{k+1} is essential, there is $x\in K-\{0,\mathbf{b}(v_{k+1})\}$ such that $S|_x^{v_{k+1}}$ is not 3-connected. By Lemma 7.1.18, $S|_{\mathbf{a}(v_k)}^{v_k}$ is not 3-connected because the support of $p_{V-\{v_k\}}(\mathbf{a})$ has size 2. Applying Lemma 7.1.38 for $\mathbf{b}$, $S|_{\mathbf{a}(v_k)}^{v_k}$, and $S|_x^{v_{k+1}}$, we obtain a triangle $\mathbf{c}$ in S such that

$$\mathbf{c}(v_{k+1})=x\neq\mathbf{b}(v_{k+1}).$$

By Lemma 7.1.32 for $\mathbf{b}$ and $\mathbf{c}$, the support of $\mathbf{c}$ contains exactly one of v_{k-1} and v_k.

As S is 3-connected, $c(\{v_{k-2},v_{k-1},v_k,v_{k+1}\})\geq 2$ and therefore $\dim(L|_{\subseteq\{v_{k-2},v_{k-1},v_k,v_{k+1}\}})\leq 2$. Observe that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly independent. Since the supports of $\mathbf{a}$ and $\mathbf{b}$ are subsets of $\{v_{k-2},v_{k-1},v_k,v_{k+1}\}$, the support of $\mathbf{c}$ is not a subset of $\{v_{k-2},v_{k-1},v_k,v_{k+1}\}$ because otherwise $L|_{\subseteq\{v_{k-2},v_{k-1},v_k,v_{k+1}\}}$ contains three linearly independent vectors. Thus $v_{k-2}\notin\operatorname{supp}(\mathbf{c})$, because $|\operatorname{supp}(\mathbf{c})\cap\{v_{k-1},v_k,v_{k+1}\}|=2$. If $k=2$, then $\mathbf{c}=\{v_2,v_3,v_4\}$ or $\{v_1,v_3,v_4\}$ for some $v_4\in V\setminus\{v_0,v_1,v_2,v_3\}$ and therefore $v_0v_1v_2v_3v_4$ or $v_0v_2v_1v_3v_4$ is a tight path in $H(S)$. Hence we may assume that $k\geq 3$.

Since neither C_5 nor C_6 is a fundamental graph of S, by Lemma 7.1.40, a tight path $v_{k-3}v_{k-2}v_{k-1}v_kv_{k+1}$ contains every edge of $H(S)$ incident with v_{k-1} and so $v_{k-1}\notin\operatorname{supp}(\mathbf{c})$. This implies that $v_k,v_{k+1}\in\operatorname{supp}(\mathbf{c})$. For each $i\in\{0,\dots,k-3\}$, as a cycle C_{k-i+2} is not a fundamental graph of S, by Lemma 7.1.44 applied to a tight path $v_iv_{i+1}\cdots v_kv_{k+1}$, a set $\{v_i,v_k,v_{k+1}\}$ is not an edge of $H(S)$. Hence none of v_0, v_1, …, v_{k-3} is in the support of $\mathbf{c}$. Recall that $v_{k-2}\notin\operatorname{supp}(\mathbf{c})$. Therefore $\operatorname{supp}(\mathbf{c})=\{v_k,v_{k+1},v_{k+2}\}$ for some $v_{k+2}\in V\setminus\{v_0,\dots,v_{k+1}\}$ and $v_0v_1v_2\cdots v_{k+1}v_{k+2}$ is a tight path in S. □

For a 3-connected isotropic system S with at least five vertices, Lemma 7.1.34 states that if $v_0v_1\dots v_{k+1}$ is a tight path of length $k\geq 2$ in $H(S)$, then $v_1,v_2,\dots,v_k$ are essential in S. The following proposition provides a feature of ends of a maximal tight path when S is not cyclic. Recall that a tight path has exactly two ends if its length is at least 2.

Proposition 7.1.46. *Let* S *be a* 3*-connected isotropic system with at least* 5 *vertices. If* S *is not cyclic, then at least two ends of a maximal tight path in* $H(S)$ *are non-essential in* S.

Proof. Let $v_0v_1\dots v_{k+1}$ be a maximal tight path in $H(S)$. If $k\geq 2$, then v_0 and v_{k+1} are non-essential in S by Lemma 7.1.45. If $k=1$, then at least two of v_0,v_1,v_2 are non-essential in S by Lemma 7.1.39. □

It is straightforward to prove Theorem 7.1.1 from Proposition 7.1.46.

Proof of Theorem 7.1.1. Let $G=(V,E)$ be a prime graph with at least four vertices which is not locally equivalent to a cycle graph. As no graph on four vertices is prime, $|V|\geq 5$. Let $\mathbf{a}$ and $\mathbf{b}$ be supplementary vectors in K^V. Let S be an isotropic system having a graphic presentation $(G,\mathbf{a},\mathbf{b})$. Then S is not cyclic because all fundamental graphs of S are locally equivalent. By Corollary 7.1.23, S is 3-connected. By Corollary 7.1.31, $v\in V$ is non-essential in G if and only if it is non-essential in S. Therefore, it suffices to show that S has at least two non-essential vertices.

We may assume that S has an essential vertex. Then by Lemma 7.1.37, S has a triangle and so $H(S)$ has a maximal tight path. By Proposition 7.1.46, at least two ends of the maximal tight path are non-essential, and therefore S has at least two non-essential vertices. □

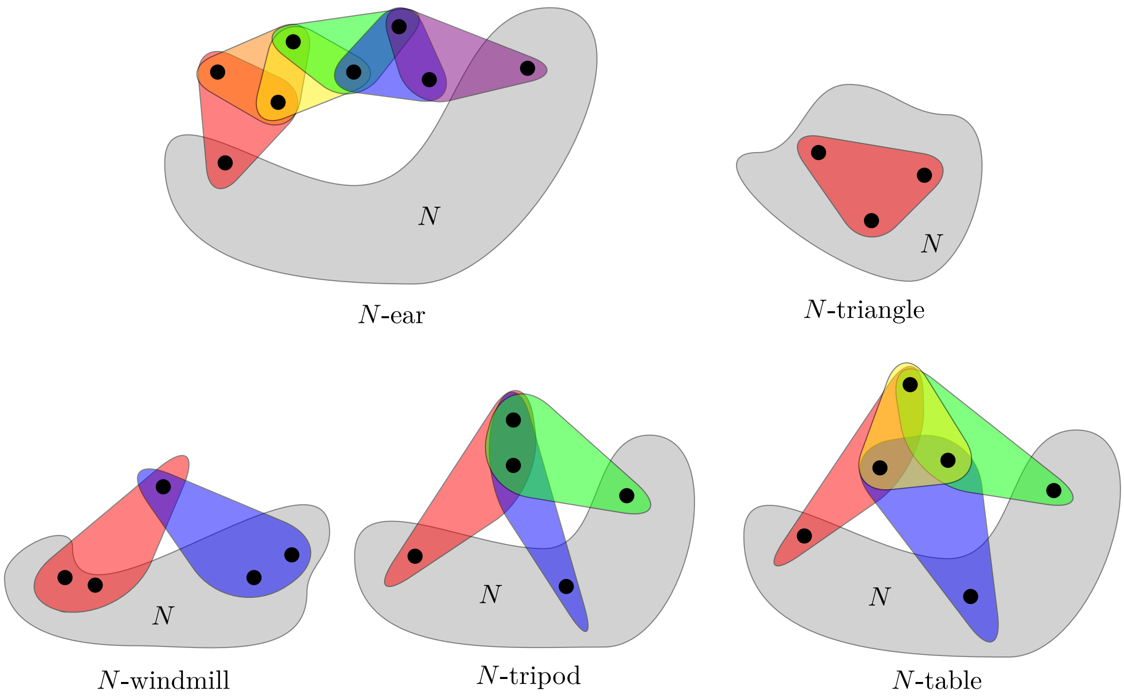


Figure 7.3: Five types of partial hypergraphs.

In the remainder, we describe a structure of a 3-connected isotropic system in terms of triangles, which will be a major ingredient to prove Theorem 7.1.3. We first define 5 types of partial hypergraphs. Let $H = (V, E)$ be a 3-uniform hypergraph and N be a subset of V.

- An N*-ear* in H is a tight path P of length at least 2 such that two ends are in N, all internal vertices are in $V - N$, and no edge in $E(H) - E(P)$ is incident with an internal vertex of P.
- An N*-triangle* in H is a partial hypergraph (V', E') of H without isolated vertices such that

(△1) $V' \subseteq N$,

(△2) $|E'| = 1$,

(△3) $|V' \cap e| \neq 2$ for all edges e of H.

- An N*-windmill* in H is a partial hypergraph (V', E') of H without isolated vertices for which there is a vertex $v \notin N$ such that

(W1) $V' - \{v\} \subseteq N$,

(W2) E' is the set of all edges of H incident with v,

(W3) $E' \neq \emptyset$,

(W4) $|e \cap e'| \neq 2$ for all edges e of H and all edges e' in E'.

- An N*-tripod* in H is a partial hypergraph (V', E') of H without isolated vertices for which there are two distinct vertices $v, w \notin N$ such that

(Y1) $V' - \{v, w\} \subseteq N$,

(Y2) E' is the set of all edges of H incident with both v and w, (Y3) $|E'| = 3$,

(Y4) no edge of H is incident with exactly one of v and w.

- An N-*table* in H is a partial hypergraph (V', E') of H without isolated vertices for which there are three distinct vertices $u, v, w \notin N$ such that

(T1) $V' - \{u, v, w\} \subseteq N$,

(T2) E' is the set of all edges of H incident with at least two of u, v, and w,

(T3) $|E'| = 4$,

(T4) $e \cap \{u, v, w\} \neq e' \cap \{u, v, w\}$ and $e - \{u, v, w\} \neq e' - \{u, v, w\}$ for distinct edges e, e' in E',

(T5) no edge of H is incident with exactly one of u, v, and w.

See Figure 7.3 for illustrations for these 5 types of hypergraphs. It is easy to observe the following two lemmas from the definition.

Lemma 7.1.47. *Let H be a 3-uniform hypergraph and $N \subseteq V(H)$. If each of H_1 and H_2 is an N-ear, an N-triangle, an N-windmill, an N-tripod, or an N-table in H, then $H_1 = H_2$ or $(V(H_1) - N) \cap (V(H_2) - N) = \emptyset$.* □

Lemma 7.1.48. *Let H be a 3-uniform hypergraph and $N \subseteq V(H)$. Let H' be an N-ear, an N-triangle, an N-windmill, an N-tripod, or an N-table of H.*

(i) *If P is a tight path of H such that $E(P) \cap E(H') \neq \emptyset$, then $E(P) \subseteq E(H')$.*

(ii) *If P is a maximal tight path of H contained in H', then $V(P) - N = V(H') - N$.* □

Theorem 7.1.49. *Let S be a 3-connected isotropic system with at least 5 vertices and let N be the set of non-essential vertices in S. If $N \neq \emptyset$, then the set of edge sets of all N-ears, N-triangles, N-windmills, N-tripods, and N-tables in $H(S)$ is a partition of the edge set of $H(S)$.*

Proof. As $N \neq \emptyset$, by Lemma 7.1.26, S is not cyclic. Note that for distinct x and y in N, there is at most one edge of $H(S)$ containing both x and y by Lemma 7.1.34. It suffices to show that each edge e of $H(S)$ is contained in an N-ear, an N-triangle, an N-windmill, an N-tripod, or an N-table, because if e is contained in two of such partial hypergraphs H_1 and H_2, then either both H_1 and H_2 are N-triangles, meaning that $H_1 = H_2$, or e is incident with a vertex not in N, implying that H_1 and H_2 share a vertex not in N, thus $H_1 = H_2$ by Lemma 7.1.47.

If all three vertices incident with e are non-essential in S, then a hypergraph $(e, \{e\})$ is an N-triangle in $H(S)$ because (△3) holds by Lemma 7.1.34. Therefore, we may assume that e is incident with an essential vertex in S. Let P be a maximal tight path in $H(S)$ containing e. We denote P by a sequence $av_1v_2 \cdots v_kb$ of distinct vertices, where k is the length of P. If $k \geq 2$, then the internal vertices $v_1, \ldots, v_k$ of P are essential in S by Lemma 7.1.34. If $k = 1$, then by relabelling, we may assume that v_1 is essential in S. By Proposition 7.1.46, two ends a and b of P are non-essential in S.

Suppose that P includes every edge of $H(S)$ incident with some of $v_1, v_2, \ldots, v_k$. If $k = 1$, then P is an N-windmill because (W4) holds by the assumption that P is maximal. If $k > 1$, then P is an N-ear. Therefore, we may assume that $H(S)$ has an edge $f \notin E(P)$ incident with some of $v_1, \ldots, v_k$. If the length of P is more than 3, then no such f exists by Proposition 7.1.42. Hence $k \leq 3$.

Case I. $k = 3$.

By Lemma 7.1.40, f is not incident with v_2. Without loss of generality, we may assume that f is incident with v_1. Since $v_1 \in f$ and $v_2 \notin f$, by Lemma 7.1.36 applied to edges $\{a, v_1, v_2\}$ and $\{v_1, v_2, v_3\}$, we deduce that f is incident with a or v_3. By Lemma 7.1.34, f is not incident with a because otherwise f and $\{a, v_1, v_2\}$ form a tight path, implying that a is essential in S. Therefore, $f = \{v_1, v_3, c\}$ for some $c \in V(H(S)) - V(P)$.

Now we show that $H' := (V(P) \cup \{c\}, E(P) \cup \{f\})$ is an N-table in $H(S)$. Note that $V(H') = \{a, b, c, v_1, v_2, v_3\}$ and $E(H') = \{\{a, v_1, v_2\}, \{b, v_2, v_3\}, \{c, v_1, v_3\}, \{v_1, v_2, v_3\}\}$. Thus (T3) holds and (T4) holds for $\{u, v, w\} = \{v_1, v_2, v_3\}$. By Lemma 7.1.40 applied to tight paths $av_1v_2v_3b$, $av_2v_1v_3c$, and $bv_2v_3v_1c$, we deduce that H' has all edges of H incident with v_1, v_2, or v_3. This implies not only (T2) and (T5), but also $c \in N$ by Proposition 7.1.46 because $av_2v_1v_3c$ is a maximal tight path. It follows that (T1) holds.

Case II. $k = 2$.

We prove that $H' := (V(P) \cup f, E(P) \cup \{f\})$ is an N-tripod in $H(S)$. Trivially (Y3) holds. To see (Y4) with $v := v_1$ and $w := v_2$, suppose that there is an edge g of $H(S)$ incident with exactly one of v_1 and v_2, say v_1 by symmetry. By Lemma 7.1.36 applied to $\{v_1, v_2, a\}$ and $\{v_1, v_2, b\}$, we deduce that g is incident with a or b. However, by Lemma 7.1.34, $g \cap \{a, v_1, v_2\} \neq \{a, v_1\}$ and $g \cap \{v_1, v_2, b\} \neq \{v_1, b\}$, contradicting our previous conclusion. This proves (Y4). This also implies that $f = \{v_1, v_2, c\}$ for some $c \in V(H(S)) - V(P)$.

By Lemma 7.1.35, H' satisfies (Y2). By (Y4), av_1v_2c is a maximal tight path in H. By Proposition 7.1.46, $c \in N$ and so $V(H') - \{v_1, v_2\} = \{a, b, c\} \subseteq N$, implying (Y1).

Case III. $k = 1$.

Let E' be the set of edges incident with v_1 and let V' be the set of all vertices incident with an edge in E'. We show that $H' := (V', E')$ is an N-windmill. By definition, (W2) holds and $e = \{a, v_1, b\} \in E'$, implying (W3).

To see (W1) with $v := v_1$, suppose that there is an edge g incident with both v_1 and a vertex not in $N \cup \{v_1\}$. Let Q be a maximal tight path containing g. By Proposition 7.1.46, the length of Q is at least 2 and v_1 is not an end of Q. Then Q has two edges e_1 and e_2 such that both are incident with v_1 and $|e_1 \cap e_2| = 2$. By Lemma 7.1.36 applied to e_1 and e_2, $|\{a, v_1, b\} \cap e_i| \geq 2$ for some $i \in \{1, 2\}$. We may assume that $|\{a, v_1, b\} \cap e_1| \geq 2$. Since P is a maximal tight path, we deduce that $|\{a, v_1, b\} \cap e_1| \neq 2$ and therefore $\{a, v_1, b\} = e_1$. Then P is a proper subpath of Q, contradicting the assumption that P is a maximal tight path. Thus, (W1) holds.

If there are edges $g \in E(H(S))$ and $g' \in E'$ such that $|g \cap g'| = 2$, then by Lemma 7.1.34, g' is incident with at least two essential vertices in S, contradicting (W1). Therefore, (W4) holds. □

Corollary 7.1.50. *Let S be a 3-connected isotropic system with at least 5 vertices and let N be the set of non-essential vertices in S. Let P_1 and P_2 be maximal tight paths of $H(S)$. If $N \neq \emptyset$, then $V(P_1) - N$ and $V(P_2) - N$ are equal or disjoint.*

Proof. Suppose that there is a vertex $v \in (V(P_1) - N) \cap (V(P_2) - N)$. For $i \in \{1, 2\}$, let e_i be an edge of P_i incident with v. By Theorem 7.1.49, for each $i \in \{1, 2\}$, there is a partial hypergraph H_i of H such that H_i includes e_i and H_i is an N-ear, an N-windmill, an N-tripod, or an N-table. We remark that since $v \notin N$, H_i is not an N-triangle. By Lemma 7.1.48(i), $E(P_i) \subseteq E(H_i)$. By Lemma 7.1.47, $H_1 = H_2$. Since P_1 and P_2 are maximal tight paths of H contained in $H_1 = H_2$, by Lemma 7.1.48(ii), $V(P_1) - N = V(P_2) - N$. □

7.1.4 Prime graphs with at most two non-essential vertices

We prove a part of Theorem 7.1.3, that is, if a prime graph with at least 5 vertices has at most 2 non-essential vertices, then it is locally equivalent to $\theta(\ell_1,\dots,\ell_m)$ for some m and ℓ_i. We first prove some lemmas in order to obtain a specific Eulerian vector of an isotropic system.

For two vectors $\mathbf{a}\in K^A$ and $\mathbf{b}\in K^B$ with disjoint sets A and B, let $\mathbf{a}\oplus\mathbf{b}$ be a vector in $K^{A\cup B}$ such that

$$(\mathbf{a}\oplus\mathbf{b})(v)=\begin{cases}\mathbf{a}(v) & \text{if } v\in A,\\ \mathbf{b}(v) & \text{otherwise.}\end{cases}$$

If $|B|=1$ and $\mathbf{b}(w)=x$ for $w\in B$, then we simply write $\mathbf{a}\oplus\mathbf{b}$ as $\mathbf{a}\oplus x$. Let 0_A be the zero vector in K^A.

Lemma 7.1.51. *Let $S=(V,L)$ be an isotropic system, v be a vertex in V, and $x\in K-\{0\}$. If $\mathbf{a}\in K^{V-\{v\}}$ is an Eulerian vector of $S|^v_x$ and $0_{V-\{v\}}\oplus x\notin L$, then $\mathbf{a}\oplus x$ is an Eulerian vector of S*

Proof. Suppose that $\mathbf{a}\oplus x$ is not an Eulerian vector of S. We have a nonempty subset X of V such that $(\mathbf{a}\oplus x)[X]\in L$. Then $\mathbf{a}[X-\{v\}]\in L|^v_x$. Since $\mathbf{a}$ is an Eulerian vector of $S|^v_x$, we deduce $X-\{v\}=\emptyset$. Hence $X=\{v\}$ and therefore $0_{V-\{v\}}\oplus x=(\mathbf{a}\oplus x)[X]\in L$, which is a contradiction. □

Lemma 7.1.52. *Let $W=\{w_1,w_2,\dots,w_k\}$ be a set of vertices in an isotropic system S, and let $x_i\in K-\{0\}$ for $1\le i\le k$. Let $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_k$ be vectors in L such that $\langle\mathbf{a}_i(w_i),x_i\rangle_K=1$ for every $1\le i\le k$, and $\langle\mathbf{a}_i(w_j),x_j\rangle_K=0$ for all $1\le j<i\le k$. Then S has an Eulerian vector $\mathbf{c}$ such that $\mathbf{c}(w_i)=x_i$ for every $1\le i\le k$.*

Proof. We proceed by induction on $k\ge 0$. For $k=0$, by Lemma 7.1.19, S has an Eulerian vector. Now we assume that $k\ge 1$. Observe that $p_{V-\{w_1\}}(\mathbf{a}_i)\in L|^{w_1}_{x_1}$ for all $2\le i\le k$, and $\langle p_{V-\{w_1\}}(\mathbf{a}_i)(w_i),x_i\rangle_K=\langle\mathbf{a}_i(w_i),x_i\rangle_K=1$ for all $2\le i\le k$, and $\langle p_{V-\{w_1\}}(\mathbf{a}_i)(w_j),x_j\rangle_K=\langle\mathbf{a}_i(w_j),x_j\rangle_K=0$ for all $2\le j<i\le k$. By the induction hypothesis, $S|^{w_1}_{x_1}$ has an Eulerian vector $\mathbf{c}'$ such that $\mathbf{c}'(w_i)=x_i$ for every $2\le i\le k$. A vector $0_{V-\{w_1\}}\oplus x_1$ is not in L, since $\langle\mathbf{a}_1,0_{V-\{w_1\}}\oplus x_1\rangle=\langle\mathbf{a}_1(w_1),x_1\rangle_K=1$. Thus, by Lemma 7.1.51, the proof is completed. □

Lemma 7.1.53. *Let $S=(V,L)$ be a 3-connected isotropic system with at least 5 vertices. Then for distinct $u,v\in V$ and nonzero $x,y\in K$, there is a vector $\mathbf{a}$ in L such that $\langle\mathbf{a}(u),x\rangle_K=1$ and $\langle\mathbf{a}(v),y\rangle_K=0$.*

Proof. Since S is 3-connected, by Lemma 7.1.18, $0_{V-\{v\}}\oplus x\notin L=L^\perp$ and therefore L has a vector $\mathbf{a}_1$ such that $\langle\mathbf{a}_1(u),x\rangle_K=1$.

Let $c\in K^V$ be a vector such that $c(u)=x$, $c(v)=y$, and $c(w)=0$ for all $w\in V-\{u,v\}$. Again by Lemma 7.1.18, $c\notin L=L^\perp$, and therefore L has a vector $\mathbf{a}_2$ such that $\langle\mathbf{a}_2(u),x\rangle_K\neq\langle\mathbf{a}_2(v),y\rangle_K$.

We may assume that $\langle\mathbf{a}_1(v),y\rangle_K=1$, since otherwise we finish the proof by taking $\mathbf{a}=\mathbf{a}_1$. We may assume that $\langle\mathbf{a}_2(u),x\rangle_K=0$ and $\langle\mathbf{a}_2(v),y\rangle_K=1$, since otherwise we finish the proof by taking $\mathbf{a}=\mathbf{a}_2$. Then $\mathbf{a}=\mathbf{a}_1+\mathbf{a}_2$ satisfies the desired condition. □

Proposition 7.1.54. *Let G be a prime graph with at least 5 vertices. If G has at most 2 non-essential vertices, then G is locally equivalent to a graph isomorphic to $\theta(\ell_1,\dots,\ell_m)$ for some m and ℓ_i.*

Proof. Since a cycle graph of length k is isomorphic to $\theta(1,k-1)$, we may assume that G is not locally equivalent to a cycle graph. Then by Theorem 7.1.1, G has exactly two non-essential vertices u and v.

Let $V = V(G)$ and let S be an isotropic system having G as a fundamental graph. Then S is not cyclic, since all fundamental graphs of S are locally equivalent. Furthermore, S is 3-connected and has exactly two non-essential vertices by Corollaries 7.1.23 and 7.1.31.

By Lemma 7.1.37, every essential vertex in S is in a tight path of $H(S)$. By Proposition 7.1.46, every maximal tight path has u and v as its ends. By Corollary 7.1.50, there are maximal tight paths $P_1, P_2, \ldots, P_m$ of $H(S)$ such that $V(P_1) - \{u, v\}, V(P_2) - \{u, v\}, \ldots, V(P_m) - \{u, v\}$ partition the set of essential vertices in S. Let us denote P_i as a sequence $uv_{i,1} \ldots v_{i,\ell(i)}v$. We may assume that $\ell(1) \leq \ell(2) \leq \cdots \leq \ell(m)$. Let $\mathbf{t}_{i,j}$ be a triangle in S whose support is $\{v_{i,j-1}, v_{i,j}, v_{i,j+1}\}$ for each $1 \leq i \leq m$ and $1 \leq j \leq \ell(i)$, where $v_{i,0} := u$ and $v_{i,\ell(i)+1} := v$ for each i.

Since all fundamental graphs of S are locally equivalent, it is enough to show that S has a fundamental graph isomorphic to $\theta(\ell(1)+1, \ell(2)+1, \ldots, \ell(m)+1)$ or $\theta(1, \ell(1)+1, \ell(2)+1, \ldots, \ell(m)+1)$.

Since u, v, and $v_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq \ell(i)$ are distinct, by symmetry of the nonzero elements in K, we can assume that $\mathbf{t}_{1,1}(u) = \mathbf{t}_{1,\ell(1)}(v) = \alpha$ and $\mathbf{t}_{i,j}(v_{i,j}) = \beta$ for all $1 \leq i \leq m$ and $1 \leq j \leq \ell(i)$. If $\ell(1) = \ell(2) = 1$, then $v_{1,1}uvv_{2,1}$ is a tight path properly containing P_1 and P_2, which contradicts that P_1 and P_2 are maximal tight paths. Thus, $2 \leq \ell(2) \leq \cdots \leq \ell(m)$. Then by Lemma 7.1.32, $\mathbf{t}_{i,1}(u) = \mathbf{t}_{1,1}(u) = \alpha$ and $\mathbf{t}_{i,\ell(i)}(v) = \mathbf{t}_{1,\ell(1)}(v) = \alpha$ for all $2 \leq i \leq m$. Applying Lemma 7.1.32, for all $1 \leq i \leq m$ and $2 \leq j \leq \ell(i) - 1$, we have $\mathbf{t}_{i,j-1}(v_{i,j}) = \mathbf{t}_{i,j+1}(v_{i,j}) \neq \mathbf{t}_{i,j}(v_{i,j}) = \beta$. By symmetry in $K - \{0\}$, we can assume that $\mathbf{t}_{i,j-1}(v_{i,j}) = \mathbf{t}_{i,j+1}(v_{i,j}) = \alpha$ for all $1 \leq i \leq m$ and $2 \leq j \leq \ell(i) - 1$. Similarly, for all $1 \leq i \leq m$ with $\ell(i) \geq 2$, we have $\mathbf{t}_{i,2}(v_{i,1}) \neq \mathbf{t}_{i,1}(v_{i,1}) = \beta$ and $\mathbf{t}_{i,\ell(i)-1}(v_{i,\ell(i)}) \neq \mathbf{t}_{i,\ell(i)}(v_{i,\ell(i)}) = \beta$, and thus we can assume that $\mathbf{t}_{i,2}(v_{i,1}) = \mathbf{t}_{i,\ell(i)-1}(v_{i,\ell(i)}) = \alpha$. In short, we assumed that

$$\mathbf{t}_{i,j}(v_{i,j-1}) = \mathbf{t}_{i,j}(v_{i,j+1}) = \alpha \text{ and } \mathbf{t}_{i,j}(v_{i,j}) = \beta$$

for all $1 \leq i \leq m$ and $1 \leq j \leq \ell(i)$.

By Lemma 7.1.53, there exist vectors $\mathbf{a}$ and $\mathbf{b}$ in L such that $\langle \mathbf{a}(u), \alpha \rangle_K = 1$, $\langle \mathbf{a}(v), \alpha \rangle_K = 0$, $\langle \mathbf{b}(u), \alpha \rangle_K = 0$, and $\langle \mathbf{b}(v), \alpha \rangle_K = 1$. Let us denote $T := \{\mathbf{t}_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq \ell(i)\}$. Since $\langle \mathbf{t}_{i,j}(u), \alpha \rangle_K = \langle \mathbf{t}_{i,j}(v), \alpha \rangle_K = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq \ell(m)$, $\{\mathbf{a}, \mathbf{b}\} \cup T$ is linearly independent. Then $\{\mathbf{a}, \mathbf{b}\} \cup T$ is a base of L because $|T| = \sum_{i=1}^{m} \ell(i) = |V| - 2$,

Let $w_1, w_2, \ldots, w_n$ be all vertices of V such that $w_1 = u$ and $w_2 = v$. Let $\mathbf{a}_1 = \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{b}$. For $k \geq 3$, let $\mathbf{a}_k = \mathbf{t}_{i,j}$ if $w_k = v_{i,j}$. Then $\langle \mathbf{a}_k(w_k), \alpha \rangle_K = 1$ for all k, and $\langle \mathbf{a}_k(w_s), \alpha \rangle_K = 0$ for $1 \leq s < k \leq n$. By applying Lemma 7.1.52 for $w_1, \ldots, w_n$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n$, we obtain an Eulerian vector $\mathbf{c}$ of S such that $\mathbf{c}(w_k) = \alpha$ for all $1 \leq k \leq n$. Let G' be the fundamental graph of S with respect to $\mathbf{c}$. Then $v_{i,j}$ is only adjacent to $v_{i,j-1}$ and $v_{i,j+1}$ in G' for each $1 \leq i \leq m$ and $1 \leq j \leq \ell(i)$ because $\mathbf{t}_{i,j}$ is a vector in the fundamental base of S with respect to $\mathbf{c}$. Therefore, G' is isomorphic to $\theta(\ell(1)+1, \ldots, \ell(m)+1)$ or $\theta(1, \ell(1)+1, \ldots, \ell(m)+1)$ depending on the adjacency between u and v. □

7.1.5 Graphs consisting of internally-disjoint paths

To complete the proof of Theorem 7.1.3, we investigate the condition that $\theta(\ell_1, \ldots, \ell_m)$ is prime and has at most 2 non-essential vertices.

The following lemma provides three ways to extend a prime graph. For a graph G and its induced subgraph H, a sequence $v_0, v_1, \ldots, v_\ell$ of distinct vertices of G is a *handle* of H if $\ell \geq 3$, $\{v_0, \ldots, v_\ell\} \cap V(H) = \{v_0, v_\ell\}$, and v_i is only adjacent to v_{i-1} and v_{i+1} in $G[V(H) \cup \{v_1, \ldots, v_{\ell-1}\}]$ for every $1 \leq i \leq \ell - 1$. We say that $G[V(H) \cup \{v_1, \ldots, v_{\ell-1}\}]$ is obtained from H by *adding a handle* of length ℓ.

Lemma 7.1.55 (Geelen [61]). *Let G be a graph with at least 5 vertices.*

(a) *If G has a vertex v of degree at least 2 such that $G \setminus v$ is prime and v has no twin in G, then G is prime.*

(b) *If G is obtained from its prime induced subgraph with at least 4 vertices by adding a handle, then G is prime.*

(c) *If G has an edge e such that both ends of e have degree 2 and G/e is prime, then G is prime.*

Proof. Both (a) and (b) were proved by Geelen in Lemma 5.3 and Proposition 5.5, respectively, of [61]. For (c), G/e is isomorphic to $G * v \setminus v$, where $e = vw$. Since $G * v \setminus v$ is prime with at least 4 vertices, it is easy to check that none of the two neighbors of v is a twin of v in $G * v$, and the neighbor of w other than v is not a twin of v in $G * v$. Hence v has no twin in $G * v$. By (a), $G * v$ is prime and therefore G is prime. □

Proposition 7.1.56. *Let $m \geq 2$ and $\ell_1, \ldots, \ell_m$ be positive integers, and $G = \theta(\ell_1, \ldots, \ell_m)$ be a graph with at least 5 vertices. Then G is prime if and only if $|\{i : \ell_i = 2\}| \leq 1$.*

Proof. If $|\{i : \ell_i = 2\}| \geq 2$, then G has twins and thus it is not prime. Now, let us prove the backward direction. By Lemma 7.1.55(c), it suffices to show that $\theta(1, 2, 3, \ldots, 3)$, $\theta(2, 3, \ldots, 3)$, $\theta(1, 3, \ldots, 3)$, and $\theta(3, \ldots, 3)$ are prime. By Lemma 7.1.55(b), it is enough to show that $\theta(1, 2, 3)$, $\theta(2, 3)$, $\theta(1, 3, 3)$, and $\theta(3, 3)$ are prime. Since $\theta(2, 3)$ and $\theta(3, 3)$ are cycles of length 5 and 6, respectively, they are prime. For the unique common neighbor v of two degree-3 vertices in $\theta(1, 2, 3)$, the graph $\theta(1, 2, 3) * v$ is isomorphic to C_5 and therefore $\theta(1, 2, 3)$ is prime. For an edge e in $\theta(1, 3, 3)$ whose both ends have degree 2, the graph $\theta(1, 3, 3) \wedge e$ is isomorphic to C_6 and so $\theta(1, 3, 3)$ is prime. □

The following lemma is useful for finding pivotal vertices in a graph. Remember that all pivotal vertices are essential.

Lemma 7.1.57. *If v and w are adjacent vertices of degree 2 in a graph G with at least 5 vertices, then both v and w are pivotal in G.*

Proof. By Lemma 7.1.8, neither $G \setminus v$ nor $G \wedge vw \setminus v$ is prime and therefore v is pivotal in G. Similarly, w is pivotal in G. □

To find non-essential vertices of $\theta(\ell_1, \ldots, \ell_m)$, we will use the next two lemmas. A graph is *outerplanar* if it has a planar embedding such that every vertex lies on the boundary of the outer face.

Lemma 7.1.58. *An outerplanar graph with at least 5 vertices is prime if and only if it is 2-connected.*

Proof. By Lemma 7.1.8, it is enough to prove the backward direction. Let G be a 2-connected outerplanar graph with at least 5 vertices. We fix an embedding of G into the plane such that the boundary of the outer face contains every vertex of G. Since G is 2-connected, there is a cycle C in G corresponding to the boundary of the outer face. Let $v_1, v_2, \ldots, v_n$ be the vertices of C in the clockwise order, starting at a vertex v_1. Suppose that G has a split (X, Y). We may assume that $|X| \geq 3$ by swapping X and Y if necessary. By rotational symmetry, we may assume that $v_{n-1} \in X$ and $v_n \in Y$.

We claim that if $v_i \in Y$ for some $i \in \{1, 2, \ldots, n-3\}$, then $v_{i+1} \in Y$. Suppose that $v_{i+1} \in X$. Since (X, Y) is a split, v_i is adjacent to v_{n-1}, and v_{i+1} is adjacent to v_n, contradicting the assumption that G is outerplanar. This proves the claim.

By the claim, $Y - \{v_{n-1}, v_n\} = Y - \{v_n\} = \{v_i : j \leq i \leq n-2\}$ for some $j \in \{1, 2, \ldots, n-1\}$. Since $|X| \geq 3$ and $|Y| \geq 2$, we deduce that $3 \leq j \leq n-2$. Since (X, Y) is a split, $v_1, v_{j-1} \in X$, $v_n, v_j \in Y$, and $v_1v_n, v_{j-1}v_j \in E(G)$, we deduce that $v_1v_j, v_{j-1}v_n \in E(G)$, contradicting the assumption that G is outerplanar. □

Lemma 7.1.59. *Let G be a graph and V_1, V_2, V_3 be disjoint subsets of $V(G)$ such that $|V_i| \geq 2$ for all i. If $G - V_i$ is prime for each i, then G is prime.*

Proof. Suppose that G is not prime. Then G has a split (X, Y). We call vertices in X *red* and vertices in Y *blue*.

If neither V_i nor V_j is monochromatic for some distinct i, j, then $G - V_k$ has a split for $k \neq i, j$, contradicting that $G - V_k$ is prime. Thus, there is at most one non-monochromatic V_i. We may assume that V_1 and V_2 are monochromatic. If V_1 and V_2 have different colors, then $G - V_3$ has a split, which is a contradiction. Therefore, we may assume that V_1 and V_2 are red. Then all blue vertices belong to $V(G) - (V_1 \cup V_2)$ and therefore $G - V_1$ has a split, which is a contradiction. □

Now we prove two lemmas presenting non-essential vertices of $\theta(\ell_1, \ldots, \ell_m)$.

Lemma 7.1.60. *Let $G = \theta(\ell_1, \ldots, \ell_m)$ such that $m \geq 3$, $\ell_1 \leq \cdots \leq \ell_m$, $\ell_1 \neq 2$, $\ell_2 \geq 3$, and if $m = 3$, then $\ell_1 \geq 3$ or $\ell_2 \geq 4$. For a vertex x of G, the following are equivalent: (i) x has degree larger than 2, (ii) x is non-pivotal, and (iii) x is non-essential. In particular, G has exactly two non-essential vertices.*

Proof. Trivially, (iii) implies (ii). Observe that $|V(G)| = 2 + \sum_{i=1}^{m}(\ell_i - 1) \geq 6$. By Lemma 7.1.57, (ii) implies (i). Therefore, it suffices to show that (i) implies (iii). Let u and v be the two distinct vertices of degree m in G, which are the only vertices of degree larger than 2. We prove that u and v are non-essential in G. By symmetry, it is enough to show that u is non-essential.

We claim that both $G * u \setminus u$ and $G \wedge uw \setminus u$ are prime. This claim implies that u is non-essential in G.

We proceed by induction on $m \geq 3$. Suppose that $\ell_1 = 1$ and $\ell_2 = 3$. Then $m \geq 4$. For the middle edge e of a path of length 3 between u and v in G, a graph $H := G \wedge e$ is isomorphic to $\theta(3, \ell_3, \ldots, \ell_m)$. By the inductive hypothesis, both $H * u \setminus u$ and $H \wedge uw \setminus u$ are prime. Hence both $G * u \setminus u = (H \wedge e) * u \setminus u = (H * u \setminus u) \wedge e$ and $G \wedge uw \setminus u = (H \wedge e) \wedge uw \setminus u = (H \wedge uw \setminus u) \wedge e$ are prime. Therefore we can assume that $\ell_1 \geq 3$ or $\ell_2 \geq 4$.

Let $P_1, \ldots, P_m$ be internally-disjoint paths between u and v of lengths $\ell_1, \ldots, \ell_m$, respectively, in G and let $V_i := V(P_i) - \{u, v\}$ for each i. Then $|V_i| = \ell_i - 1 \geq 2$ for all $i \geq 2$. Let w be the neighbor of u in P_1.

We first consider the case that $m = 3$ and $\ell_1 \geq 3$; see Figure 7.4. Then $|V_1| = \ell_1 - 1 \geq 2$. For each $i \in \{1, 2, 3\}$, $(G * u \setminus u) - V_i$ is a cycle of length $|V_j| + |V_k| + 1 \geq 5$, where $\{i, j, k\} = \{1, 2, 3\}$, and thus it is prime. Therefore, $G * u \setminus u$ is prime by Lemma 7.1.59. Note that $G \wedge uw \setminus u \setminus w$ is isomorphic to $\theta(\ell_1 - 2, \ell_2, \ell_3)$ and thus it is prime by Proposition 7.1.56. Since w has degree 2 and has no twin in $G \wedge uw \setminus u$, by Lemma 7.1.55(a), $G \wedge uw \setminus u$ is prime.

Next we consider the case that $m = 3$ and $\ell_1 = 1$; see Figure 7.5. Then $\ell_2 \geq 4$ and $w = v$ because v is the neighbor of u in P_1. Both $G * u \setminus u$ and $G \wedge uv \setminus u \setminus v$ are 2-connected outerplanar and thus they are prime by Lemma 7.1.58. Since v has degree 2 and has no twin in $G \wedge uv \setminus u$, by Lemma 7.1.55(a), $G \wedge uv \setminus u$ is prime.

Now it remains to consider the case that $m \geq 4$. For each $2 \leq i \leq 4$, as $G - V_i$ is isomorphic to $\theta(\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_m)$, we deduce that both $(G * u \setminus u) - V_i = (G - V_i) * u \setminus u$ and $(G \wedge uw \setminus u) - V_i = (G - V_i) \wedge uw \setminus u$ are prime by the inductive hypothesis. Thus by Lemma 7.1.59, both $G * u \setminus u$ and $G \wedge uw \setminus u$ are prime. □

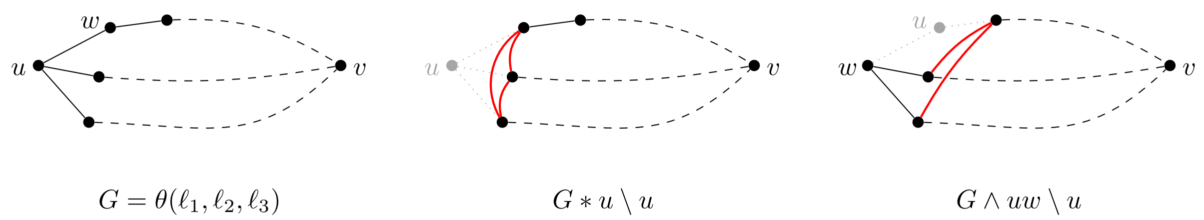


Figure 7.4: $\theta(\ell_1, \ell_2, \ell_3)$ with $3 \leq \ell_1 \leq \ell_2 \leq \ell_3$ and its vertex-minors.

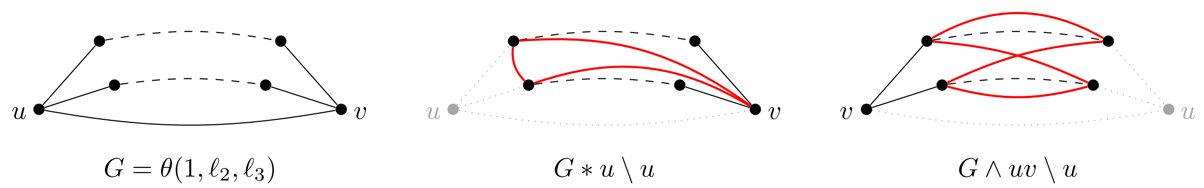


Figure 7.5: $\theta(1, \ell_2, \ell_3)$ with $4 \leq \ell_2 \leq \ell_3$ and its vertex-minors.

Lemma 7.1.61. *Let G be $\theta(1, 2, \ell_1, \ldots, \ell_m)$ or $\theta(2, \ell_1, \ldots, \ell_m)$ with $m \geq 2$ and $\min\{\ell_1, \ldots, \ell_m\} \geq 3$. For a vertex x of G, the following are equivalent: (i) x has degree larger than 2 or has no neighbor of degree 2, (ii) x is non-pivotal, and (iii) x is non-essential. In particular, G has exactly three non-essential vertices.*

Proof. Note that $|V(G)| = 3 + \sum_{i=1}^{m}(\ell_i - 1) \geq 7$. By definition, (iii) implies (ii) and by Lemma 7.1.57, (ii) implies (i). Thus, it suffices to show that (i) implies (iii). Let u and v be the two vertices of degree at least 3 in G and let w be the common neighbor of u and v. Note that w is the unique vertex that has degree 2 and has no neighbor of degree 2. We claim that u, v, and w are non-essential.

Since $\theta(1, 2, \ell_1, \ldots, \ell_m)$ and $\theta(2, \ell_1, \ldots, \ell_m)$ are locally equivalent by applying a local complementation at w, we may assume that $G = \theta(2, \ell_1, \ldots, \ell_m)$. Observe that $G \setminus w$ is isomorphic to $\theta(\ell_1, \ldots, \ell_m)$, and $G * w \setminus w$ is isomorphic to $\theta(1, \ell_1, \ldots, \ell_m)$. By Proposition 7.1.56, $G \setminus w$ and $G * w \setminus w$ are prime and therefore w is non-essential in G.

Since a cycle graph of length at least 5 has no non-essential vertex, G is not locally equivalent to a cycle graph. By Theorem 7.1.1, G has at least two non-essential vertices. Therefore, u or v is non-essential in G, and by symmetry, both u and v are non-essential in G. □

A graph $\theta(\ell_1, \ell_2)$ is a cycle of length $\ell_1 + \ell_2$, which is prime and has no non-essential vertex if $\ell_1 + \ell_2 \geq 5$. By Proposition 7.1.56, for positive integers $m \geq 3$ and $\ell_1 \leq \cdots \leq \ell_m$ with $\ell_2 \geq 2$, a graph $\theta(\ell_1, \ldots, \ell_m)$ is prime if and only if either (i) $\ell_2 \geq 3$ or (ii) $\ell_1 = 1$, $\ell_2 = 2$, and $\ell_3 \geq 3$. In the next proposition, we determine the number of non-essential vertices in $\theta(\ell_1, \ldots, \ell_m)$ when it is prime and $m \geq 3$.

Proposition 7.1.62. *Let m and $\ell_1, \ldots, \ell_m$ be positive integers with $m \geq 3$, $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_m$, and $\ell_3 \geq 3$ such that either $\ell_2 \geq 3$ or $(\ell_1, \ell_2) = (1, 2)$. Let $G = \theta(\ell_1, \ldots, \ell_m)$.*

(1) *If $\ell_1 = 1$, $m = 3$, and $\ell_2 \leq 3$, then G is locally equivalent to a cycle of length $\ell_2 + \ell_3$ and has no non-essential vertex.*

(2) *If $\ell_1 = 1$, $m = 3$, and $\ell_2 \geq 4$, then G has exactly 2 non-essential vertices.*

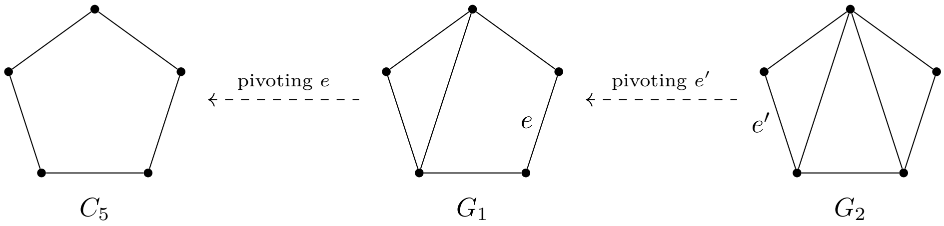


Figure 7.6: All graphs locally equivalent to C_5 up to isomorphism.

(3) *If $\ell_1 = 1$, $m \geq 4$, and $\ell_2 = 2$, then G has exactly 3 non-essential vertices.*

(4) *If $\ell_1 = 1$, $m \geq 4$, and $\ell_2 \geq 3$, then G has exactly 2 non-essential vertices.*

(5) *If $\ell_1 = 2$, then G has exactly 3 non-essential vertices.*

(6) *If $\ell_1 \geq 3$, then G has exactly 2 non-essential vertices.*

Proof. It is obvious to check (1). Lemma 7.1.60 implies (2), (4), and (6). Lemma 7.1.61 implies (3) and (5). □

By the preceding proposition, every graph in Θ has at most 2 non-essential vertices. We now prove Theorem 7.1.3 from Propositions 7.1.54 and 7.1.62.

Proof of Theorem 7.1.3. Let G be a prime graph with at least four vertices. Then $|V(G)| \geq 5$ because no graph on four vertices is prime. Suppose that G is locally equivalent to a graph $H \in \Theta$ consisting of m internally-disjoint paths between two fixed distinct vertices having no common neighbors, where $m \geq 2$. If $m = 2$, then H is a cycle and thus G has no non-essential vertex. If $m \geq 3$, then by Proposition 7.1.62, G has at most 2 non-essential vertices.

Now, we prove the backward direction. Suppose that G has at most 2 non-essential vertices. By Proposition 7.1.54, G is locally equivalent to a graph isomorphic to $\theta(\ell_1, \ldots, \ell_m)$ for some m and ℓ_i. Since G is prime, $m \geq 2$. We may assume that G is not locally equivalent to a cycle graph because a cycle graph of length k is isomorphic to $\theta(1, k-1)$. Thus $m \geq 3$. By Proposition 7.1.62, we conclude that $\ell_i \neq 2$ for all i. □

7.1.6 Pivot-minors and non-pivotal vertices

We prove Theorems 7.1.2 and 7.1.4, which are analogues of Theorems 7.1.1 and 7.1.3 for pivot-minors. We also prove Corollaries 7.1.6 and 7.1.7. We present useful results first.

Lemma 7.1.63. *A graph is locally equivalent to a cycle of length 5 if and only if it is pivot-equivalent to a cycle of length 5.*

Proof. The backward direction is trivial. The forward direction is easily seen by Figure 7.6 which depicts all graphs locally equivalent to a cycle of length 5 up to isomorphism. In Figure 7.6, $G_1 \wedge e$ is isomorphic to C_5, and $G_2 \wedge e'$ is isomorphic to G_1. Therefore C_5, G_1, and G_2 are pivot-equivalent. □

Theorem 7.1.64. *Let $S = (V, L)$ be an isotropic system, and let $(G, \mathbf{a}, \mathbf{b})$ and $(H, \mathbf{c}, \mathbf{d})$ be graphic presentations of S. Then there are nonnegative integers m, k, ℓ, vertices v_1, v_2, …, v_m, and edges e_1, e_2, …, e_k, e'_1, e'_2, …, e'_ℓ such that the following hold.*

(i) *For* $1 \le i \le k$, e_i *is an edge of* $G \wedge e_1 \wedge \cdots \wedge e_{i-1}$.

(ii) $\{v_1, v_2, \dots, v_m\}$ *is an independent set of vertices in* $G \wedge e_1 \wedge \cdots \wedge e_k$.

(iii) *For* $1 \le j \le \ell$, e'_j *is an edge of* $G \wedge e_1 \wedge \cdots \wedge e_k * v_1 * \cdots * v_m \wedge e'_1 \wedge \cdots \wedge e'_{j-1}$.

(iv) $(H, \mathbf{c}, \mathbf{d}) = (G, \mathbf{a}, \mathbf{b}) \wedge e_1 \wedge \cdots \wedge e_k * v_1 * \cdots * v_m \wedge e'_1 \wedge \cdots \wedge e'_\ell$

Theorem 7.1.64 is a slight strengthening of Fon-Der-Flaass [59] in Russian; see also [60, Theorem 3.4]. His theorem states that for two locally equivalent graphs G and H, there are vertices $v_1, \dots, v_m$ and edges $e_1, \dots, e_k, e'_1, \dots, e'_\ell$ satisfying (i)–(iii) and the following replacement of (iv):

(iv′) $H = G \wedge e_1 \wedge \cdots \wedge e_k * v_1 * \cdots * v_m \wedge e_1 \wedge \cdots \wedge e'_\ell$.

Our proof of Theorem 7.1.64 uses the *divergence*. This technique was introduced by Fon-Der-Flaass [58] for graphs, and used for isotropic systems by Bouchet [30].

Proof. Let $(G', \mathbf{a}', \mathbf{b}')$ and $(H', \mathbf{c}', \mathbf{d}')$ be graphic presentations pivot-equivalent to $(G, \mathbf{a}, \mathbf{b})$ and $(H, \mathbf{c}, \mathbf{d})$, respectively. For $v \in V$, let

$$d_{G',H'}(v) := \begin{cases} 0 & \text{if } \mathbf{c}'(v) = \mathbf{a}'(v), \\ 1 & \text{if } \mathbf{c}'(v) = \mathbf{b}'(v), \\ 2 & \text{otherwise.} \end{cases}$$

and let $D(G', H') := \sum_{v \in V} d_{G',H'}(v)$. For each $i \in \{0, 1, 2\}$, let A_i be the set of vertices $v \in V$ such that $d_{G',H'}(v) = i$.

We take G' and H' minimizing $D(G', H')$. To complete the proof, it suffices to show that $(H', \mathbf{c}', \mathbf{d}') = (G', \mathbf{a}', \mathbf{b}') * v_1 \cdots * v_m$ for some independent set $\{v_1, \dots, v_m\}$ in G'. We present four claims step by step.

Claim I. No vertex in A_1 is adjacent to vertices in $A_1 \cup A_2$ in G'. Suppose that there is $vw \in E(G')$ for some $v \in A_1$ and $w \in A_1 \cup A_2$. By Proposition 7.1.29,

$$\begin{aligned} &d_{G' \wedge vw, H'}(v) = 0, \\ &d_{G' \wedge vw, H'}(w) = \begin{cases} 0 & \text{if } w \in A_1, \\ 2 & \text{otherwise,} \end{cases} \\ &d_{G' \wedge vw, H'}(x) = d_{G',H'}(x) && \text{for all } x \in V - \{v, w\}. \end{aligned}$$

Therefore, $D(G' \wedge vw, H') \le D(G', H') - 1$, which contradicts our choice of G' and H'. This proves Claim I.

Claim II. $A_1 = \emptyset$. Suppose that A_1 has a vertex v. By Claim I, $N_{G'}(v) \subseteq A_0$. Since $(G', \mathbf{a}', \mathbf{b}')$ is a graphic presentation of $S = (V, L)$, L has a vector $\mathbf{b}'_v$ such that

$$\mathbf{b}'_v(w) = \begin{cases} \mathbf{b}'(v) & \text{if } w = v, \\ \mathbf{a}'(w) & \text{if } w \text{ is a neighbor of } v \text{ in } G', \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{c}'[\{v\} \cup N_{G'}(v)] = \mathbf{b}'_v \in L$, violating that $\mathbf{c}'$ is an Eulerian vector of S. This proves Claim II.

Note that for each $v \in A_2$, $\mathbf{d}'(v)$ is either $\mathbf{a}'(v)$ or $\mathbf{b}'(v)$ because $\mathbf{c}'(v) = \mathbf{a}'(v) + \mathbf{b}'(v)$. Let B be the set of vertices $v \in A_2$ such that $\mathbf{d}'(v) = \mathbf{a}'(v)$.

Claim III. The set A_2 is independent in H'. Suppose that there is $vw \in E(H')$ for some $v, w \in A_2$. By Proposition 7.1.29, for each $x \in \{v, w\}$,

$$d_{G',H'\wedge vw}(x) = \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{otherwise,} \end{cases}$$

and for each $y \in V - \{v, w\}$, $d_{G',H'\wedge vw}(y) = d_{G',H'}(y)$. Therefore, $D(G', H' \wedge vw) \leq D(G', H') - 2$, a contradiction. This proves Claim III.

Claim IV. $B = \emptyset$. Suppose that B has a vertex v. By Claims II and III, $N_{H'}(v) \subseteq A_0$. Since $(H', \mathbf{c}', \mathbf{d}')$ is a graphic presentation of S, there is a vector $\mathbf{d}'_v$ in L such that

$$\mathbf{d}'_v(w) = \begin{cases} \mathbf{d}'(v) & \text{if } w = v, \\ \mathbf{c}'(w) & \text{if } w \text{ is a neighbor of } v \text{ in } H', \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{a}'[\{v\} \cup N_{H'}(v)] = \mathbf{d}'_v \in L$, which contradicts that $\mathbf{a}'$ is an Eulerian vector of S. This proves Claim IV.

In conclusion, $V - A_0$ is independent in H' and for each $v \in V - A_0$, $\mathbf{c}'(v) = \mathbf{a}'(v) + \mathbf{b}'(v)$ and $\mathbf{d}'(v) = \mathbf{b}'(v)$. By Propositions 7.1.29 and 7.1.21(ii), $(G', \mathbf{a}', \mathbf{b}') = (H', \mathbf{c}', \mathbf{d}') * v_m \cdots * v_1$ where $\{v_1, \ldots, v_m\} = V - A_0$. Equivalently, $(H', \mathbf{c}', \mathbf{d}') = (G', \mathbf{a}', \mathbf{b}') * v_1 \cdots * v_m$. It is easy to check that $\{v_1, \ldots, v_m\}$ is independent in $G' = H' * v_m \cdots * v_1$. □

Corollary 7.1.65. *Let G be a prime graph with at least 5 vertices and let S be an isotropic system associated with a graphic presentation $(G, \mathbf{a}, \mathbf{b})$ such that $\{\mathbf{a}(v), \mathbf{b}(v)\} = \{\alpha, \beta\}$ for all $v \in V(G)$. For each graphic presentation of $(H, \mathbf{c}, \mathbf{d})$ of S, either $\{v \in V(G) : \mathbf{c}(v) = \gamma$ or $\mathbf{d}(v) = \gamma\}$ is empty or has at least 3 vertices.*

Proof. Denote $W := \{v \in V(G) : \mathbf{c}(v) = \gamma \text{ or } \mathbf{d}(v) = \gamma\}$. By Proposition 7.1.29 and Theorem 7.1.64, we may assume that $(H, \mathbf{c}, \mathbf{d}) = (G, \mathbf{a}, \mathbf{b}) * v_1 * \cdots * v_m$ for some distinct and pairwisely non-adjacent vertices $v_1, \ldots, v_m$ in G. By Proposition 7.1.29, $\mathbf{c}(v_i) = \gamma$ for all $i \in [m]$. Thus, we may assume that $m \leq 2$. Suppose that $m = 1$. Then for each $w \in N_G(v_1)$, $\mathbf{d}(w) = \gamma$ by Proposition 7.1.29 and therefore $W = \{v_1\} \cup N_G(v_1)$. Since G is prime and $|V(G)| \geq 5$, the minimum degree of G is 2 or more, which implies that $|W| \geq 3$. Hence we may assume that $m = 2$. Because G has no twins, $|N_G(v_1) \triangle N_G(v_2)| \geq 1$. By Proposition 7.1.29, $W = \{v_1, v_2\} \cup (N_G(v_1) \triangle N_G(v_2))$ and therefore $|W| \geq 3$. □

The following corollary was conjectured by Bouchet [29] and proved by Fon-Der-Flaass [59, 60]. It will be used to prove Corollary 7.1.6. We remark that this corollary is also a consequence of the following facts on binary matroids and their relation to bipartite graphs. Seymour [106] showed that if two connected binary matroids on the same ground set have identical connectivity functions, then they are equal up to duality. It is well known that a binary matroid is uniquely determined by its fundamental graph, which is bipartite. Oum [92] observed that the connectivity function of a binary matroid is precisely the cut-rank function of its fundamental graph and locally equivalent graphs have the same cut-rank functions.

Corollary 7.1.66 (Fon-Der-Flaass [59, 60])**.** *If two bipartite graphs are locally equivalent, then they are pivot-equivalent.*

For the convenience of readers, we include a proof by Fon-Der-Flaass using Theorem 7.1.64.

Proof. Let G and H be locally equivalent bipartite graphs. By Theorem 7.1.64, $H = G \wedge e_1 \wedge \cdots \wedge e_k * v_1 * \cdots * v_m \wedge e'_1 \wedge \cdots \wedge e'_\ell$ for some edges $e_1, \ldots, e_k, e'_1, \ldots, e'_\ell$ and an independent set $\{v_1, \ldots, v_m\}$ of $G \wedge e_1 \wedge \cdots \wedge e_k$. Let $G' := G \wedge e_1 \wedge \cdots \wedge e_k$ and $H' := H \wedge e'_\ell \wedge \cdots \wedge e'_1$. Observe that both G' and H' are bipartite and $H' = G' * v_1 * \cdots * v_m$. Now it is easy to see that $G' = H'$. □

Lemma 7.1.67 (Oum [93, Proposition 10.1]). *Let $(G_1, \mathbf{a}_1, \mathbf{b}_2)$ and $(G_2, \mathbf{a}_2, \mathbf{b}_2)$ be graphic presentations of an isotropic system. If $\{\mathbf{a}_1(v), \mathbf{b}_1(v)\} = \{\mathbf{a}_2(v), \mathbf{b}_2(v)\}$ for each vertex v, then G_1 and G_2 are pivot-equivalent.*

Now we prove Theorem 7.1.2 using Theorem 7.1.1 together with preceding results.

Proof of Theorem 7.1.2. Let G be a prime graph with at least four vertices which has at most 1 non-pivotal vertex. Denote $V := V(G)$. Then $|V| \geq 5$ since no graph on four vertices is prime. If $|V| = 5$, then G is locally equivalent to a cycle and therefore it is pivot-equivalent to a cycle by Lemma 7.1.63. Thus, we may assume that G has at least 6 vertices.

Since every non-essential vertex is non-pivotal, G has at most 1 non-essential vertex. By Theorem 7.1.1, G is locally equivalent to a cycle H. It is enough to show that G is pivot-equivalent to H.

Let S be an isotropic system associated with a graphic presentation $(G, \mathbf{a}, \mathbf{b})$ where $\mathbf{a}, \mathbf{b} \in K^V$ are supplementary vectors such that $\mathbf{a}(v) = \alpha$ and $\mathbf{b}(v) = \beta$ for each $v \in V$. Since H is locally equivalent to G, there exist supplementary vectors $\mathbf{c}$ and $\mathbf{d}$ such that $(H, \mathbf{c}, \mathbf{d})$ is a graphic presentation of S. A vertex v is non-pivotal in G if and only if $S|^v_\alpha$ or $S|^v_\beta$ is 3-connected by Proposition 7.1.30.

For every vertex v, a vertex-minor $H * v \setminus v$ is prime and therefore $S|^v_{\mathbf{c}(v)+\mathbf{d}(v)}$ is 3-connected by Proposition 7.1.30. So if $\mathbf{c}(v)+\mathbf{d}(v) \in \{\alpha, \beta\}$, then v is non-pivotal in G. Since G has at most 1 non-pivotal vertex, by Corollary 7.1.65, $\mathbf{c}(v) + \mathbf{d}(v) \notin \{\alpha, \beta\}$ for all $v \in V(G)$ and therefore $\{\mathbf{c}(v), \mathbf{d}(v)\} = \{\alpha, \beta\}$. By Lemma 7.1.67, G and H are pivot-equivalent. □

Proof of Theorem 7.1.4. By Lemma 7.1.60, every graph in Θ has at most 2 non-pivotal vertices. Thus, it suffices to show that if a prime graph G with at least 5 vertices has at most 2 non-pivotal vertices, then G is pivot-equivalent to a graph in Θ. Denote $V := V(G)$. Then $|V| \geq 5$ because no graph on four vertices is prime. If $|V| = 5$, then G is locally equivalent to a cycle and therefore it is pivot-equivalent to a cycle by Lemma 7.1.63. Thus, we may assume that G has at least 6 vertices.

Because G has at most 2 non-essential vertices, by Theorem 7.1.3, G is locally equivalent to a graph $H \in \Theta$ consisting of internally-disjoint paths $P_1, \ldots, P_m$ between two fixed vertices x and y such that $m \geq 2$ and no P_i has length 2.

Let S be an isotropic system associated with a graphic presentation $(G, \mathbf{a}, \mathbf{b})$ where $\mathbf{a}, \mathbf{b} \in K^V$ are supplementary vectors such that $\mathbf{a}(v) = \alpha$ and $\mathbf{b}(v) = \beta$ for each $v \in V$. Since G and H are locally equivalent, there are supplementary vectors $\mathbf{c}$ and $\mathbf{d}$ such that $(H, \mathbf{c}, \mathbf{d})$ is a graphic presentation of S. By Proposition 7.1.30, a vertex v is non-pivotal in G if and only if $S|^v_\alpha$ or $S|^v_\beta$ is 3-connected.

If v is a vertex of degree 2 in H, then $H * v \setminus v$ is prime by Proposition 7.1.56. Therefore, $S|^v_{\mathbf{c}(v)+\mathbf{d}(v)}$ is 3-connected by Proposition 7.1.30. So if $\mathbf{c}(v) + \mathbf{d}(v) \in \{\alpha, \beta\}$, then v is non-pivotal in G.

If H is a cycle, then since G has at most 2 non-pivotal vertices, by Corollary 7.1.65, $\mathbf{c}(v) + \mathbf{d}(v) \notin \{\alpha, \beta\}$ for all $v \in V$ and therefore $\{\mathbf{c}(v), \mathbf{d}(v)\} = \{\alpha, \beta\}$.

If H is not a cycle, then by Lemma 7.1.60, H has exactly 2 non-essential vertices that are x and y. Then x and y are non-essential in G and thus they are non-pivotal in G. Therefore no vertex of degree 2 in H is non-pivotal in G and so $\{\mathbf{c}(v), \mathbf{d}(v)\} = \{\alpha, \beta\}$ for every vertex v of degree 2 in H. By Corollary 7.1.65, $\{\mathbf{c}(x), \mathbf{d}(x)\} = \{\mathbf{c}(y), \mathbf{d}(y)\} = \{\alpha, \beta\}$.

In both cases, it implies that G is pivot-equivalent to $H \in \Theta$ by Lemma 7.1.67. □

Here is an easy observation on bipartite graphs.

Lemma 7.1.68. *If G is bipartite and uv is an edge of G, then $G \wedge uv$ is bipartite.* □

Finally, we are ready to prove Corollaries 7.1.6 and 7.1.7.

Proof of Corollary 7.1.6 using Theorem 7.1.2. Let G be a prime bipartite graph such that $|V(G)| \geq 4$ and G has fewer than two non-pivotal vertices. Then by Theorem 7.1.2, G is pivot-equivalent to a cycle C. By Lemma 7.1.68, C is bipartite and so it is an even cycle. □

Proof of Corollary 7.1.7 using Theorem 7.1.4. Let G be a prime bipartite graph such that $|V(G)| \geq 4$ and G has fewer than three non-pivotal vertices. Then by Theorem 7.1.4 and Lemma 7.1.68, G is pivot-equivalent to some bipartite graph in Θ. □

In the remainder of this section, we will show that Corollary 7.1.6 can be deduced directly from Theorem 7.1.1 without using Theorem 7.1.2. For that, we will need several properties of bipartite graphs.

Lemma 7.1.69 (Allys [1, Lemma 5.2])**.** *No bipartite graph is locally equivalent to an odd cycle of length at least five.*

Allys proved the preceding lemma by using isotropic systems. Lemma 7.1.69 is also implied by the following theorem of Fon-Der-Flaass [58], which was published earlier than [1]. An anternative proof of Lemma 7.1.69 was provided in [75, Appendix A].

Theorem 7.1.70 (Fon-Der-Flaass [58, Theorem 5.1])**.** *Every graph locally equivalent to a cycle of length at least five is Hamiltonian.*

Allys stated that Lemma 7.1.69 can be used to show that Theorem 4.3 of [1] implies the wheel and whirl theorem of Tutte [112, 8.2] for binary matroids. However, we believe that the next lemma is necessary to complete this implication.

Lemma 7.1.71. *A bipartite graph with at least five vertices is locally equivalent to a cycle if and only if it is pivot-equivalent to an even cycle.*

Proof. Since the backward direction is trivial, we now prove the forward direction. Let G be a bipartite graph locally equivalent to a cycle C of length at least 5. By Lemma 7.1.69, C must be an even cycle and by Corollary 7.1.66, G and C are pivot-equivalent. □

Proof of Corollary 7.1.6 using Theorem 7.1.1. It is straightforward from Theorem 7.1.1 and Lemma 7.1.71. □

7.2 PU-orientations of graphs without K_4-pivot-minor

A graph G is *PU-orientable* if it admits an orientation $\overrightarrow{G}$ such that the corresponding adjacency matrix $A_{\overrightarrow{G}}$ over $\mathbb{R}$ is principally unimodular. Here, 'PU' indicates 'principally unimodular.' A graph G is PU-orientable if and only if the binary even delta-matroid $D(A)$ is regular, where A is the adjacency matrix of G over the binary field. Therefore, PU-orientability is closed under taking pivot-minors because of the compatibility between pivot-minors of graphs and minors of even delta-matroids, examined in Subsection 4.1.4. There are two operations on oriented graphs preserving PU-orientedness. First, the *negation* is reversing the direction of every edge. It corresponds to multiplying adjacency matrices by -1. Second, the *cut-switching* at a vertex set X is an operation reversing the direction of each edge with one end in X and the other end not in X. It corresponds to multiplying X-rows and X-columns of adjacency matrices by -1.

Camion [40] showed the following result on PU-orientations of graphs.

Lemma 7.2.1 ([40]; see [68, Lemma 3]). *Every PU-orientable bipartite graph has a unique PU-orientation up to cut-switching.*

We show a strengthening of Camion's result.

Proposition 7.2.2. *Every PU-orientable graph without K_4-pivot-minor has a unique PU-orientation up to negation and cut-switching.*

Proposition 7.2.2 implies Lemma 7.2.1 by the following observations. A graph is bipartite if and only if it has no K_3-pivot-minor, and thus every bipartite graph has no K_4-pivot-minor. For a bipartite graph, the negation is equal to the cut-switching at one color class.

Note that K_4 has exactly two PU-orientations up to negation and cut-switching; see [36, Page 461 or Lemma 2.6], and every other graph with at most four vertices is uniquely PU-orientable. Bouchet, Cunningham, and Geelen [36] conclude the same result as Proposition 7.2.2 for PU-orientable prime graphs.

Theorem 7.2.3 ([36, Theorem 1.1]). *Every PU-orientable prime graph has a unique PU-orientation up to negation and cut-switching.*

We will use Theorem 7.2.3 in the proof of Proposition 7.2.2. We note that the class of PU-orientable prime graphs neither contains nor is contained in the class of PU-orientable graphs without K_4-pvito-minor.

Remark 7.2.4. A graph G is PU-orientable if and only if the binary even delta-matroid $D(A)$ is regular, where A is the adjacency matrix of G over $\mathbb{F}_2$. Moreover, if G is bipartite, then $D(A)$ is a twist of a matroid. Gerards [68] presented a simple proof of Tutte's excluded-minor characterization of regular matroids (Corollary 3.3.3) by making use of Lemma 7.2.1.

The proof of Proposition 7.2.2 will be given at the end of this subsection. We show several lemmas first.

Lemma 7.2.5. *Both the bowtie graph and the loose bowtie graph, depicted in Figure 7.7, have a pivot-minor isomorphic to K_4.*

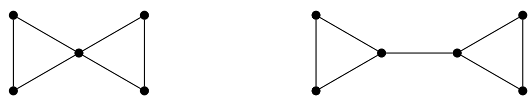


Figure 7.7: The *bowtie graph* (left) and the *loose bowtie graph* (right).

Recall that the 1-join of two graphs H_1 and H_2 with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$ is a graph obtained from the union of disjoint copies of $H_1 \setminus v_1$ and $H_2 \setminus v_2$ by adding all possible edges between $N_{H_1}(v_1)$ and $N_{H_2}(v_2)$. A *split decomposition* of a graph G is defined recursively as follows:

- $\{G\}$ is a split decomposition of G.
- If $\{H_1, \ldots, H_k\}$ is a split decomposition of G and H_1 is the 1-join of H_1' and H_1'', then $\{H_1', H_1'', H_2, \ldots, H_k\}$ is a split decomposition of G.

Lemma 7.2.6. *If a graph G is the 1-join of two graphs one of which is not bipartite, then G is not bipartite.* □

Lemma 7.2.7. *Let $\{H_1, \ldots, H_k\}$ be a split decomposition of a graph G. If two graphs H_i and H_j are non-bipartite, then G is the 1-join of two non-bipartite graphs.* □

Lemma 7.2.8. *Let G be a non-bipartite graph and $v \in V(G)$. Let C be a shortest odd cycle in G that minimizes the distance between v and $V(C)$ and let $P = v_0v_1 \ldots v_k$ be a shortest path from $v = v_0$ to $V(C)$. If $k \geq 1$, then $(V(C) \cup V(P), E(C) \cup E(P))$ is an induced subgraph of G.* □

Lemma 7.2.9. *If a graph is the 1-join of two non-bipartite graphs, then it has a K_4-pivot-minor.*

Proof. Let H_1 and H_2 be non-bipartite graphs and let $v_i \in V(H_i)$ for $i = 1, 2$. Let G be the 1-join of H_1 and H_2 with markers v_1 and v_2.

For each i, let C_i be a shortest odd cycle in H_i that minimizes the distance between v_i and C_i in H_i, and let P_i be a shortest path from v_i to C_i in H_i. By Lemma 7.2.8, $L_i := (V(C_i) \cup V(P_i), E(C_i) \cup E(P_i))$ is an induced subgraph of H_i. Let A be the 1-join of L_1 and L_2 with markers v_1 and v_2. Then A is an induced subgraph of G. If each of P_1 and P_2 has length 0, then A is isomorphic to an odd subdivision of K_4 and hence G has a pivot-minor isomorphic to K_4, a contradiction. Therefore, we may assume that one of P_i has a length larger than 0. Then A is isomorphic an odd subdivision of either the bowtie graph of the loose bowtie graph in Figure 7.7. By Lemma 7.2.5, G has a pivot-minor isomorphic to K_4, a contradiction. □

Bouchet, Cunningham, and Geelen [36] showed that the number of PU-orientations of a graph can be computed by the product of the numbers of PU-orientations of its componnets in a split decomposition. We denote by $\alpha(G)$ the number of PU orientations distinct up to cut-switching. For example, $\alpha(K_3) = 2$ witnessed by two different directed triangles.

Theorem 7.2.10 ([36, Theorem 3.5])**.** *Let $\{H_1, \ldots, H_k\}$ be a split decomposition of a graph G. Then $\alpha(G) = \prod_{i=1}^{k} \alpha(H_i)$.*

We prove Proposition 7.2.2 using Theorem 7.2.10 along with the previous observations.

Proof of Proposition 7.2.2. Let G be a PU-orientable graph without K_4-pivot-minor. It suffices to show that $\alpha(G) \leq 2$. Suppose to the contrary that $\alpha(G) > 2$. Let $\{H_1, \ldots, H_k\}$ be a split decomposition of G such that each H_i is bipartite, prime, or complete. Note that K_3 is prime. Hence, as G has no pivot-minor isomorphic to K_4, each H_i is bipartite or prime. By Lemma 7.2.1 and Theorem 7.2.3, $\alpha(H_i) \in \{1, 2\}$ for each i. By Theorem 7.2.10, we have $\prod_{i=1}^{k} \alpha(H_i) = \alpha(G) > 2$, and thus there are distinct i and j such that $\alpha(H_i) = \alpha(H_j) = 2$. Then H_i and H_j are non-bipartite by Lemma 7.2.1. Then Lemmas 7.2.7 and 7.2.9 implies that G has a pivot-minor isomorphic to K_4, a contradiction. □

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가장 먼저, 제가 한 명의 독립된 연구자가 될 수 있도록 지도해주신 엄상일 교수님께 감사드립니다. 엄 교수님께서는 배우는 과정이 느렸던 제가 연구자로서 성장할 수 있도록 많은 가르침을 주셨으며, 연구자가 가져야할 마음가짐과 추구해야할 가치에 대해 알려주셨습니다. 엄 교수님께서는 항상 연구실 문을 열어두시고 제가 방문하면 흔쾌히 시간을 내어 같이 연구해주셨습니다. 또한 어떤 주제에 대해 연구하던 적극적으로 지원해주셨으며, 논문 집필과 연구발표를 하는데 미숙했던 제가 어떠한 방향성을 가지고 글을 쓰고 발표를 해야 제 연구를 다른 연구자들에게 알릴 수 있는지 많은 조언을 주셨습니다. 그리고 우물 안 개구리에 그치지 않고 더 나아갈 수 있도록 타 연구자들과 교류하고 학회에 참석할 기회를 많이 주셨습니다. 이 모든 것들이 바탕이 되어 수학자 김동규가 완성될 수 있었습니다.

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