Baker-Bowler theory for Lagrangian Grassmannians

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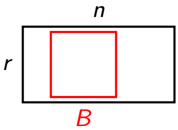
Matroid := Matrix + -oid

(*On the Abstract Properties of Linear Dependence* by Whitney '35)

$$\mathrm{Gr}(r,n) \xhookrightarrow{\qquad p \qquad} \mathbb{P}^{\binom{n}{r}-1}(K)$$

$V =$ row-sp(A)



$p_B(V) = \det(A[B])$

$\mathcal{C}^\perp = \{\text{min. supp. } X \in V - \mathbf{0}\}$ $\qquad$ $\mathcal{B} = \{B \in \binom{[n]}{r} : p_B(V) \neq 0\}$

Circuit elimination axiom:
$\forall C_1, C_2 \in \mathcal{C}^\perp \ \forall e \in C_1 \cap C_2 \ \exists C_3 \in \mathcal{C}^\perp$ s.t. $C_3 \subseteq (C_1 \cup C_3) - e$

(Strong) basis exchange axiom:
$\forall B_1, B_2 \in \mathcal{B} \ \forall e \in B_1 \setminus B_2 \ \exists f \in B_2 \setminus B_1$ s.t. $B_1 - e + f, B_2 + e - f \in \mathcal{B}$

$\mathrm{Gr}(r,n)$ is cut out by the *Grassmann-Plücker relations*: $\forall S \in \binom{[n]}{r+1} \ \forall T \in \binom{[n]}{r-1}$

$$\sum_{e \in S \setminus T} (-1)^{|S<e|+|T<e|} x_{S-e} x_{T+e} = 0$$

Parameterization of the Lagrangian Grassmannian

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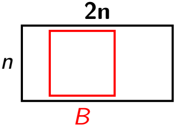
$$\mathrm{SpGr}(n, 2n) \xhookrightarrow{\quad\Phi\quad} \mathbb{P}^{2^n + \binom{n}{2} 2^{n-2} - 1}(K)$$

$W =$ row-sp(A)



$\Phi_B(W) = \det(A[B])$

$\mathcal{C}^* = \{$min. supp. $X \in V - \mathbf{0}$ containing ≤ 1 skew pair$\}$

$\mathcal{B} = \{B \in \mathcal{T}_n \cup \mathcal{A}_n : \Phi_B(V) \neq 0\}$

The coordinates are indexed by transversals $\mathcal{T}_n := \{T \in \binom{\mathbf{2n}}{n} : \nexists\{i, i^*\} \subseteq T\}$
and almost-transversals $\mathcal{A}_n := \{A \in \binom{\mathbf{2n}}{n} : \exists!\{i, i^*\} \subseteq A\}$
$x \in \mathbb{P}$ & subtransversal S of size $n - 2 \implies x_{S \cup \{i, i^*\}} = (-1)^{i+j} x_{S \cup \{j, j^*\}}$

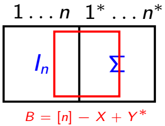
Theorem (Boege, D'Ali, Khale, Sturmfels 2019 & K. 2024)

Φ is a parameterization of $\mathrm{SpGr}(n, 2n)$ and the image is cut out by the restricted Grassmann-Plücker relations:

$$\sum_{e \in S \setminus T} (-1)^{|S<e|+|T<e|} x_{S-e} x_{T+e} = 0$$

for all $S \in \binom{2n}{n}$ and $T \in \binom{2n}{n}$ such that S contains exactly one skew pair $\{i, i^*\}$ and T contains no skew pair.

Note:
For $W =$ row-sp of n-by-$\mathbf{2n}$ matrix $A = [A_1|A_2]$, $\quad W = W^{\perp} \iff A_1 A_2^t$ is symmetric.

If $A = n$  then $\det(A[B]) = \pm\det(\Sigma[X, Y])$.

Thus $\det(A[B])$ with $B \in \mathcal{T}_n \cup \mathcal{A}_n$ are the principal and almost-principal minors of Σ.

Antisymmetric matroids

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Definition (K. 2024)

An antisymmetric matroid is a pair $M = (\mathbf{2n}, \mathcal{B})$ such that $\emptyset \neq \mathcal{B} \subseteq \mathcal{T}_n \cup \mathcal{A}_n$ and satisfies:

(Sym) $\forall$subtransversal $S \in \binom{\mathbf{2n}}{n}$ & distinct skew pairs $\{i, i^*\}, \{j, j^*\}$ non-intersect with S, $S \cup \{i, i^*\} \in \mathcal{B} \iff S \cup \{j, j^*\} \in \mathcal{B}$.

(Exch) $\forall S \in \binom{\mathbf{2n}}{n+1}$ $\forall T \in \binom{\mathbf{2n}}{n-1}$ satisfying S contains exactly one sp and T contains no sp there are NO or ≥ 2 elements $e \in S \setminus T$ s.t. both $S - e$ and $T - e$ are in $\mathcal{B}$.

We call each element in $\mathcal{B}$ a *basis*.

Ex. The supports of $\Phi(W)$ is the set of bases of an antisymmetric matroid.

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Ex. The supports of $\Phi(W)$ is the set of bases of an antisymmetric matroid.

(Exch') $\forall B, B' \in \mathcal{B}$ $\forall e \in B \setminus B'$ if $B - e$ contains no sp and $B' + e$ contains exactly one sp, then $\exists f \in B' \setminus B$ s.t. $B - e + f$, $B' + e - f \in \mathcal{B}$.

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Theorem (K. 2024)

Let $\mathcal{C}$ be a set of subsets C of $\mathbf{2n} = \{1, \ldots, n, 1^*, \ldots, n^*\}$ s.t. C contains ≤ 1 skew pair. Then $\mathcal{C}$ is the set of circuits of an antisymmetric matroid if and only if the following hold:

- *(C1)* $\emptyset \notin \mathcal{C}$
- *(C2)* $C_1, C_2 \in \mathcal{C}$ & $C_1 \subseteq C_2 \Longrightarrow C_1 = C_2$
- *(Orth)* $|C_1 \cap C_2^*| \neq 1 \quad \forall C_1, C_2 \in \mathcal{C}$
- *(Max)* $T \in \mathcal{T}_n$ & $e \in E \setminus T \Longrightarrow \exists C \in \mathcal{C}$ s.t. $C \subseteq T + e$

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- *(Max)* $T \in \mathcal{T}_n$ & $e \in E \setminus T \Longrightarrow \exists C \in \mathcal{C}$ s.t. $C \subseteq T + e$

Ex. The set of minimal supports of nonzero vectors $X \in W$ s.t. $\mathrm{supp}(X)$ contains ≤ 1 sp

(Add) $\forall C_1, C_2 \in \mathcal{C}\ \forall e \in C_1 \cap C_2$
if $(C_1 \cup C_2) - e$ contains ≤ 1 skew pair, then $\exists C_3 \in \mathcal{C}$ s.t. $C_3 \subseteq (C_1 \cup C_2) - e$.

Theorem (Minty 1966)

Let $\mathcal{C}$ and $\mathcal{D}$ be set of subsets of $[n]$. Then $\mathcal{C}$ and $\mathcal{D}$ are the sets of circuits and cocircuits of a matroid, respectively, if and only if the following hold:

(i) $\mathcal{C}$ *and* $\mathcal{D}$ *satisfy (C1) and (C2)*

(ii) $|C \cap D| \neq 1 \quad \forall C \in \mathcal{C}$ & $\forall D \in \mathcal{D}$

(iii) $\forall$*tripartition* $(P, Q, \{e\})$ *of* $[n]$
either $\exists C \in \mathcal{C}$ *s.t.* $e \in C \subseteq P + e$ *or* $\exists D \in \mathcal{D}$ *s.t.* $e \in D \subseteq Q + e$

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$V \in \mathrm{Gr}(r, n) \mapsto V \oplus V^{\perp} \in \mathrm{SpGr}(n, 2n)$

If $\mathcal{C}$ and $\mathcal{D}$ are the sets of circuits and cocircuits of a matroid on $[n]$,
then $\mathcal{C} \oplus \mathcal{D} := \mathcal{C} \cup \{D^* : D \in \mathcal{D}\}$ is the set of circuits of an antisymmetric matroid on $\mathbf{2n}$.

$$\{\text{Matroids}\} \subseteq \{\text{Antisymmetric matroids}\}$$

Proposition (K. 2024)

A matroid is representable over K in the usal sense if and only if it is representable over K as an antisymmetric matroid.

Theorem (Tutte 1958)

Let $p \in \mathbb{P}^{\binom{n}{r}-1}(K)$*. TFAE:*

(i) p satisfies all Grassmann-Plücker relations.

(ii) p satisfies all 3-term G–P relations and the support of p forms a matroid.

Theorem (K. 2024)

Let $x \in \mathbb{P}^{2^n+\binom{n}{2}2^{n-2}-1}(K)$*. TFAE:*

(i) x satisfies all restricted G–P relations.

(ii) x is all ≤ 4*-term restricted G–P relations and the support of x forms the bases of an antisymmetric matroid.*

Symmetric matroids vs. Antisymmetric matroids

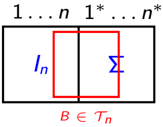
Definition

A symmetric matroid is a pair $M = (\mathbf{2n}, \mathcal{B})$ such that $\emptyset \neq \mathcal{B} \subseteq \mathcal{T}_n$ and

(SEA) $\forall B, B' \in \mathcal{B} \quad \forall \{i, i^*\} \subseteq B \triangle B'$
$\exists \{j, j^*\} \subseteq B \triangle B'$ (possibly $= \{i, i^*\}$) s.t. $B_1 \triangle \{i, i^*, j, j^*\} \in \mathcal{B}$.

Basis = each element of $\mathcal{B}$
Circuits = subtransversals not contained in any basis

Ex. $A = n$



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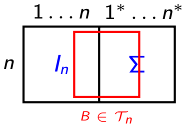
$A =$ 

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Proposition (K. 2024)

If $M = (\mathbf{2n}, \mathcal{B})$ is an antisymmetric matroid, then $(\mathbf{2n}, \mathcal{B} \cap \mathcal{T}_n)$ is a symmetric matroid.

The converse is open.
Note: Two distinct antisymmetric matroid may induce the same symmetric matroid.

Even symmetric matroids

A symmetric matroid on $\mathbf{2n}$ is *even* if $|B \cap [n]|$ have the same parity for all bases B.

Proposition (K. 2024)

If $M = (\mathbf{2n}, \mathcal{B})$ is an even symmetric matroid, then $\exists! \mathcal{B}' \subseteq \mathcal{A}_n$ s.t. $(\mathbf{2n}, \mathcal{B} \cup \mathcal{B}')$ is an antisymmetric matroid.

$$\{\text{Matroids}\} \subseteq \{\text{Even Sym Mat}\} \begin{array}{l} \subsetneq \{\text{Symmetric Mat}\} \\ \qquad\qquad \downarrow \\ \subsetneq \{\text{Antisym Mat}\} \end{array}$$

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Symmetric matroids (Late 1980s) $\equiv$ Delta-matroids, 2-matroids, Metroids, Pseudomatroids, Lagrangian (symplectic) matroids [Coxeter matroids of type C]

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Even symmetric matroids $\equiv$ Pfaffian structures (Kung 1978), Even delta-matroids, Tight 2-matroids, Lagrangian orthogonal matroids [Coxeter matroids of type D]

* Even symmetric matroids are 'nice' combinatorial structure for understanding Lagrangian *orthogonal* Grassmannian.

Matroids with coefficients

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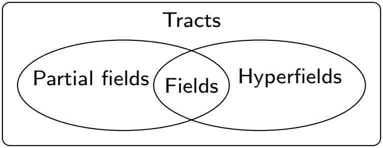
4. Baker and Bowler 2019: Matroids with coefficients in tracts
   Baker and Lorscheid 2020∼: The moduli space and Foundations of matroids

Antisymmetric matroids with coefficients

A tract is a pair $F = (F^\times, N_F)$ of a (multiplicative) abelian group $F^\times$ and a subset N_F of group semiring $\mathbb{N}[F^\times]$ satisfying certain axioms. Abusing notation: $F = F^\times \cup \{0\}$



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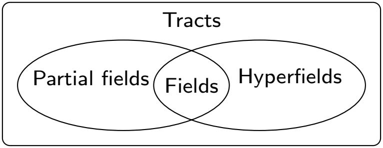

Definition

A restricted Grassmann-Plücker function on $\mathbf{2n}$ with coefficients in a tract F is a function $\varphi : \mathcal{T}_n \cup \mathcal{A}_n \to F$ such that $\varphi \not\equiv 0$ and satisfies:

(Sym) $\forall$subtransversal $S \in \binom{\mathbf{2n}}{n}$ & distinct skew pairs $\{i, i^*\}, \{j, j^*\}$ non-intersect with S

$$\varphi(S \cup \{i, i^*\}) = (-1)^{i+j} \varphi(S \cup \{j, j^*\})$$

(rGP) $\forall S \in \binom{\mathbf{2n}}{n+1}$ $\forall T \in \binom{\mathbf{2n}}{n-1}$ satisfying S contains exactly one sp and T contains no sp

$$\sum_{e \in S \setminus T} (-1)^{|S<e| + |T<e|} \varphi(S - e)\varphi(T + e) \in N_F$$

An antisymmetric F-matroid is an equivalence class of restricted G–P functions, where $\varphi \sim c\varphi$ for $c \in F^\times$.

Theorem (K. 2024)

$\{$*Antisymmetric F-matroids*$\} \overset{\text{1-1}}{\longleftrightarrow} \{$*Antisymmetric F-circuit sets*$\}$

Antisymmetric matroids with coefficients in tracts F generalize

(1) matroids with coefficients in tracts (Baker and Bowler 2019) and

(2) points in the projective space satisfying the restricted Grassmann-Plücker relations (equivalently, Lagrangian subspaces) if $F = K$ is a field.

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Furthermore, they are compatible with several other concepts such as

(3) the Lagrangian orthogonal matroids (Coxeter matroids of type D) with coefficients in tracts (Jin and Kim 2023) if $-1 = 1$ in F,

(4) the symplectic Dressian and isotropic tropical linear spaces (Rincón 2012 & Balla and Olarte 2023) if $F = \mathbb{T}$ is the tropical hyperfield, and

(5) oriented gaussoids (Boege et al. 2019) if $F = \mathbb{S}$ is the sign hyperfield.

Thank you!