Generalized Chvátal-Gomory closures for integer programs with bounds on variables

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Abstract

Integer programming problems that arise in practice often involve decision variables with one or two sided bounds. In this paper, we consider a generalization of Chvátal-Gomory inequalities obtained by strengthening Chvátal-Gomory inequalities using the bounds on the variables. We prove that the closure of a rational polyhedron obtained after applying the generalized Chvátal-Gomory inequalities is also a rational polyhedron. This generalizes a result of Dunkel and Schulz on 0-1 problems to the case when some of the variables have upper or lower bounds or both while the rest of them are unbounded.

1 Introduction

Chvátal-Gomory cutting planes [4, 15] (or *CG cuts* for short) form an important class of cutting planes for integer programming problems. Besides being useful in practice, with separation routines for many subclasses of CG cuts implemented in commercial MIP solvers, there is a significant body of literature on theoretical properties of CG cuts, especially on the notions of "closure" and "rank".

Let $\alpha x \leq \beta$ be a valid inequality for a polyhedron $P \subseteq \mathbb{R}^n$ with $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{R} \setminus \mathbb{Z}$. The inequality $\alpha x \leq \lfloor \beta \rfloor$ is called a *CG cut for P derived from the inequality* $\alpha x \leq \beta$ and is valid for all integer points x' satisfying $\alpha x' \leq \beta$ (and therefore, for $P \cap \mathbb{Z}^n$). Here, $\lfloor \beta \rfloor$ stands for the largest integer less than or equal to β. Therefore,

$$\lfloor \beta \rfloor \geq \max\{\alpha x : x \in \mathbb{Z}^n, \alpha x \leq \beta\}, \tag{1}$$

with equality when the coefficients of α are relatively prime. The *Chvátal closure* of a polyhedron P [4] is the set of points in P that satisfy all possible CG cuts for P. Schrijver [20] proved that the Chvátal closure of a rational polyhedron is again a rational polyhedron, thus showing that there are only a finite number of nonredundant CG cuts for a rational polyhedron. Additional results on the polyhedrality of the Chvátal closure were given by Dunkel and Schulz [11] for nonrational polytopes, and Dadush, Dey, and Vielma [6] for compact convex sets.

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Dunkel and Schulz [10] proposed a generalization of CG cuts for 0-1 integer programs. Let $\alpha x \le \beta$ be a valid inequality for a polytope $P \subseteq [0,1]^n$. Optimal solutions of the maximization problem in (1) may not be contained in $\{0,1\}^n$, the set of possible integral points in P. Therefore, if $\beta' = \max\{\alpha x : x \in \{0,1\}^n, \alpha x \le \beta\}$ (assuming the maximum exists), then $\beta' \le \lfloor \beta \rfloor$ and $\alpha x \le \beta'$ is at least as strong as the CG cut $\alpha x \le \lfloor \beta \rfloor$. Moreover, $\alpha x \le \beta'$ is a valid inequality for $P \cap \{0,1\}^n$. Dunkel and Schulz showed that the set of all points in a polytope $P \subseteq [0,1]^n$ that satisfy all cuts of the type above define a rational polytope. These cuts are clearly valid for the 0-1 knapsack set $\{x \in \{0,1\}^n : \alpha x \le \beta\}$; valid inequalities for such knapsack sets are used to solve practical problem instances in Crowder, Johnson, and Padberg [3] and an associated closure operation is defined by Fischetti and Lodi [13].

Pokutta [19] generalized the Dunkel-Schulz definition by considering arbitrary subsets of $\mathbb{Z}^n$ and studied bounds on the rank of resulting cuts for certain families of polytopes. Let $S \subseteq \mathbb{Z}^n$, let P be a polyhedron, and let $\alpha x \le \beta$ be a valid inequality for P. Assume further that S has a point satisfying $\alpha x \le \beta$. We call the inequality $\alpha x \le \beta'$ where $\beta' = \max\{\alpha x : x \in S, \alpha x \le \beta\}$, an *$S$-Chvátal-Gomory cut* (or *S-CG cut,* for short) for P and this inequality is valid for $P \cap S$. Let

$$\lfloor \beta \rfloor_{S,\alpha} = \max\{\alpha x : x \in S, \alpha x \le \beta\}.$$

We view $\lfloor \beta \rfloor_{S,\alpha}$ as a generalization of the operator $\lfloor \beta \rfloor$ (Pokutta uses the notation $[\![\alpha, \beta]\!]_S$ instead). We then represent the above S-CG cut as $\alpha x \le \lfloor \beta \rfloor_{S,\alpha}$. If $\alpha x \le \beta$ is valid for P, but $\mathrm{conv}(S)$ is a rational polyhedron and does not contain a point satisfying this inequality, then $P \cap S$ is empty. In this case, we say that $\mathbf{0}x \le -1$ is an S-CG cut for P derived from $\alpha x \le \beta$. In a similar manner, we define

$$\lceil \beta \rceil_{S,\alpha} = \min\{\alpha x : x \in S, \alpha x \ge \beta\},$$

assuming S has a point satisfying $\alpha x \ge \beta$. Then we say that $\alpha x \ge \lceil \beta \rceil_{S,\alpha}$ is the S-CG cut obtained from $\alpha x \ge \beta$. We define the *S-CG closure* of a polyhedron P to be the set of all points in P that satisfy all S-CG cuts for P, and we denote this set by P_S (Pokutta uses the notation $GCG(P)_S$ to refer to P_S). We focus only on rational polyhedra in this paper.

When $S = \mathbb{Z}^n$, the family of S-CG cuts for P is the same as the set of CG cuts for P of the form $\alpha x \le \lfloor \beta \rfloor$ where c is a vector of coprime integers and $\alpha x \le \beta$ is valid for P; in this case, the hyperplane $\alpha x = \delta$ is moved (by reducing δ from the starting value of β) till it first hits an integer point. In the case of an S-CG cut where $S \ne \mathbb{Z}^n$, the hyperplane $\alpha x = \delta$ is moved till it first hits a point in S. These new inequalities can also be viewed as cutting planes from "wide split disjunctions", introduced recently by Bonami, Lodi, Tramontani, and Wiese [2], where the cut coincides with one side of the disjunction (or the associated inequality).

In this paper, we consider some natural choices of S not considered earlier, for example, S is a finite set or $S = \mathbb{Z}^n_+$. Polyhedrality results were given for $S = \{0,1\}^n$ by Dunkel and Schulz [10] and for $S = \mathbb{Z}^n$ by Schrijver [20]. We prove that the S-CG closure is a polyhedron when S is finite, generalizing the result of Dunkel and Schulz above. Any integer program with a bounded linear programming relaxation can be strengthened by the S-CG closure for some finite S. Moreover, our result for finite S allows S to have "holes" ($S \ne \mathrm{conv}(S) \cap \mathbb{Z}^n$) and is thus relevant to some recent work by Bonami et al. [2] on the so-called Lazy Bureaucrat Problem (defined by Furini, Ljubić, and

Sinnl [14]) and Vielma's [21] *embedding* approach to formulating disjunctive programs in a higher-dimensional space (also studied by Huchette [16]).

The case $S = \mathbb{Z}^n_+$ is also highly relevant in practice as many practical integer programs involve nonnegative integer variables. The results in this paper imply that the S-CG closure of a rational polyhedron is a polyhedron when $S = \mathbb{Z}^n_+$. This is the most difficult case considered in this paper, and the proof reduces to proving polyhedrality when P is a rational packing or covering polyhedron contained in $\mathbb{R}^n_+$. Recently, Pashkovich, Poirrier, and Pulyassary [18] and Zhu, Del Pia, and Linderoth [22] studied aggregation cuts for packing polyhedra and proved that the associated *aggregation closure* is a polyhedron (Pashkovich et al. also proved polyhedrality of the aggregation closure in the case of covering polyhedra). An aggregation cut for a packing polyhedron P is a valid inequality for the unbounded knapsack set $\{x \in \mathbb{Z}^n_+ : \alpha x \leq \beta\}$ where $\alpha x \leq \beta$ is a valid inequality for P, and the aggregation closure is the set of points in P that satisfy all aggregation cuts. In this case, the inequality $\alpha x \leq \lfloor \beta \rfloor_{\mathbb{Z}^n_+, \alpha}$ is a special case of an aggregation cut, and therefore, the aggregation closure is a subset of P_S. A similar connection holds between aggregation cuts and $\mathbb{Z}^n_+$-CG cuts for covering polyhedra. However, the proof in [18] that the aggregation closure is a polyhedron does not imply the polyhedrality of P_S: all facets of the aggregation closure are aggregation cuts, whereas we show that some facets of P_S are not S-CG cuts but are limits of such cuts.

We combine results for the cases $S = \mathbb{Z}^n_+$, S is finite, and $S = \mathbb{Z}^n$ in our main result:

Theorem 1.1. *Let $T \subseteq \mathbb{Z}^{n_1}$ be finite, $\ell \in \mathbb{Z}^{n_3}, u \in \mathbb{Z}^{n_4}$, and let S be*

$$S = \left\{(x, y, w^1, w^2) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3} \times \mathbb{Z}^{n_4} :\ x \in T,\ w^1 \geq \ell,\ w^2 \leq u \right\}.$$

If $P \subseteq \operatorname{conv}(S)$ is a rational polyhedron, then the S-CG closure of P is a rational polyhedron.

In Section 2, we formally define the S-CG closure of a rational polyhedron and give some of its basic properties. In Section 2.3, we prove that the S-CG closure is polyhedral for every finite $S \subseteq \mathbb{Z}^n$. In Section 3, we show that the S-CG closure of a rational polyhedron is also a rational polyhedron when $S = T \times \mathbb{Z}^{n_2}$ where $T \subseteq \mathbb{Z}^{n_1}$ is a finite set. In Section 4, we prove Theorem 1.1 by reducing the general case to the case when $S \subseteq \mathbb{Z}^n_+$. We conclude in Section 5 with some remarks on the separation problem for S-CG cuts.

2 Preliminaries

We start with a formal definition of the S-CG closure of a rational polyhedron. Let $S \subseteq \mathbb{Z}^n$, and let

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} \tag{2}$$

be a rational polyhedron, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Let $P_I = \operatorname{conv}(P \cap \mathbb{Z}^n)$ denote the *integer hull* of P. Let

$$\Pi^*_P = \{(\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{R}\ :\ \exists \lambda \in \mathbb{R}^m_+ \text{ s.t. } \alpha = \lambda A,\ \beta \geq \lambda b\} \tag{3}$$

be the set of all vectors that define valid inequalities for P with integral left-hand-side coefficients. As P is rational, these inequalities define P. Let Π_P be the subset of Π^*_P corresponding to supporting valid inequalities only:

$$\Pi_P = \{(\alpha, \beta) \in \Pi^*_P\ :\ \beta = \max\{\alpha x : x \in P\}\}. \tag{4}$$

Though Π_P^* is a polyhedral mixed-integer set, Π_P is the union of a finite number of polyhedral mixed-integer sets. We defined the S-CG closure of P as the set of points in P that satisfy every S-CG cut for P. Throughout the paper, we assume that $\operatorname{conv}(S)$ is a rational polyhedron. Then if $P \cap \operatorname{conv}(S)$ is empty, there is a vector $(\alpha, \beta) \in \Pi_P^*$ such that the inequality $\alpha x \le \beta$ strictly separates $\operatorname{conv}(S)$ from P:

$$x \in P \Rightarrow \alpha x \le \beta \quad \text{and} \quad x \in \operatorname{conv}(S) \Rightarrow \alpha x > \beta.$$

Then, by definition, $\mathbf{0}x \le -1$ is an S-CG cut for P, and $P_S = \emptyset$.

On the other hand, if $P \cap \operatorname{conv}(S) \neq \emptyset$, then $\lfloor \beta \rfloor_{S,\alpha} = \max\{\alpha x : x \in S, \alpha x \le \beta\}$ is well-defined for all $(\alpha, \beta) \in \Pi_P^*$. Then the S-CG closure of P can be written as

$$P_S = \bigcap_{(\alpha,\beta)\in\Pi_P^*} \{x \in \mathbb{R}^n : \ \alpha x \le \lfloor \beta \rfloor_{S,\alpha}\}\,. \tag{5}$$

Note that P_S can be empty even when $P \cap \operatorname{conv}(S) \neq \emptyset$, for example, when $S = \mathbb{Z}^n$ and P is a polyhedron whose Chvátal closure is empty. It is straightforward to see that the closure operation (5) has the following properties, observed first by Pokutta [19].

Remark 2.1. *Let $S \subseteq \mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then*

(1) $P_I \subseteq P_S \subseteq P$,
(2) if $S \subseteq T$, for some $T \subseteq \mathbb{Z}^n$, then $P_S \subseteq P_T$,
(3) if $Q \supseteq P$ is a rational polyhedron, then $Q_S \supseteq P_S$.

For any $\Gamma \subseteq \Pi_P^*$, we consider a relaxation of P_S defined by S-CG cuts obtained from Γ as follows:

$$P_{S,\Gamma} = \bigcap_{(\alpha,\beta)\in\Gamma} \{x \in \mathbb{R}^n : \ \alpha x \le \lfloor \beta \rfloor_{S,\alpha}\}\,.$$

Remark 2.2. *Let $S \subseteq \mathbb{Z}^n$, let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, and let $\Gamma \subseteq \Pi_P^*$. Then*

(1) if $\Gamma \subseteq \Omega \subseteq \Pi_P^*$, then $P_S \subseteq P_{S,\Omega} \subseteq P_{S,\Gamma}$,
(2) if $\Gamma = \bigcup_{i=1}^k \Gamma_i$, then $P_{S,\Gamma} = \bigcap_{i=1}^k P_{S,\Gamma_i}$.

Therefore, if $\Gamma = \bigcup_{i=1}^k \Gamma_i$ and P_{S,Γ_i} is a rational polyhedron for each $i \in \{1, \ldots, k\}$, then $P_{S,\Gamma}$ is a rational polyhedron. For any Γ satisfying $\Pi_P \subseteq \Gamma \subseteq \Pi_P^*$, we have

$$P_{S,\Pi_P} = P_{S,\Gamma} = P_{S,\Pi_P^*}. \tag{6}$$

It is easy to see that the first set above is equal to the third set as an S-CG cut for P is dominated by an S-CG cut for P arising from a valid supporting inequality for P. As the second set is trivially contained in the third set and contains the first set, the remaining equality relations follow.

2.1 Examples

We next present two simple examples to highlight some differences between regular CG cuts and S-CG cuts. The first example highlights the strength of S-CG cuts.

Example 2.3. *Consider a rational polyhedron $P \subseteq \mathbb{R}^2$ such that the inequality $3x+5y \geq 3.4$ is valid. Clearly, the associated CG cut $3x+5y \geq 4$ is valid for $P \cap \mathbb{Z}^n$. Notice that the CG cut is tight at point $(3,-1)$. Now, consider $S = \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 3\right\}$, and note that $(3,-1) \notin S$. In fact, the S-CG cut $3x+5y \geq 5$, obtained from $3x+5y \geq 3.4$, is valid for $P \cap S$ and is tight at the point $(0,1) \in S$. See Figure 1 for an illustration.* $\square$

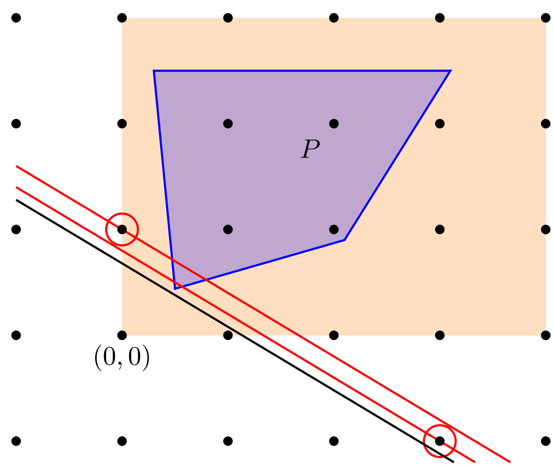

Figure 1: Illustration of an S-CG cut

The next example highlights the fact that the S-CG closure can have facet-defining inequalities that are not S-CG cuts. In contrast, it is known that all facets of the Chvátal closure of a rational polyhedron are defined by CG cuts [20]. In the following example, a sequence of S-CG cuts converge to a facet-defining inequality that is not an S-CG cut itself.

Example 2.4. *Let $S = \{0,1\}^4$, and let P be the convex hull of the following six points in $[0,1]^4$.*

$$P = \operatorname{conv}\left\{(1/2,0,0,0), (1,0,0,0), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,1)\right\}.$$

Observe that $2x_1+x_2+x_3+x_4 \geq 1$ is a valid inequality for P and is tight at the vertex $(1/2,0,0,0)$. As the point $(0,1,0,0) \in S$ satisfies $2x_1+x_2+x_3+x_4 = 1$, one cannot obtain

$$2x_1+x_2+x_3+x_4 \geq 2$$

as an S-CG cut for P. However, we claim that this inequality is valid for the S-CG closure of P. Note that for any $0 < \delta \leq 1/2$, the inequality $2x_1+(1-\delta)x_2+(1-\delta)x_3+(1-\delta)x_4 \geq 1$ is valid for P as it is satisfied by all its vertices. Moreover, any point $x^* \in S$ that satisfies this inequality either has $x_1^* = 1$ or $x_2^*+x_3^*+x_4^* \geq 2$. Therefore, the smallest value of $2x_1+(1-\delta)x_2+(1-\delta)x_3+(1-\delta)x_4$ at such points in S is exactly $2-2\delta$. This implies that for any rational $0 < \delta \leq \frac{1}{2}$,

$$2x_1+(1-\delta)x_2+(1-\delta)x_3+(1-\delta)x_4 \geq 2-2\delta$$

is an S-CG cut (after scaling by a positive integer M such that $M\delta$ is integral). Taking the limit of this inequality as $\delta \to 0$, we can infer that $2x_1+x_2+x_3+x_4 \geq 2$ is valid for P_S. As this inequality is facet-defining for P_I, it is also facet-defining for $P_S \supseteq P_I$. $\square$

We next illustrate this fact in Figure 2, where $S = \{s_1, s_2, s_3\} \subseteq \mathbb{Z}^2$ and $P \subseteq \mathbb{R}^2$ is the blue (larger) triangle. The S-CG closure has a facet-defining inequality (indicated by the thick line passing through s_2) that is not an S-CG cut. The supporting hyperplane for P (which is parallel to this inequality – depicted by the thick line passing through s_3) also touches the point $s_3 \in S$.

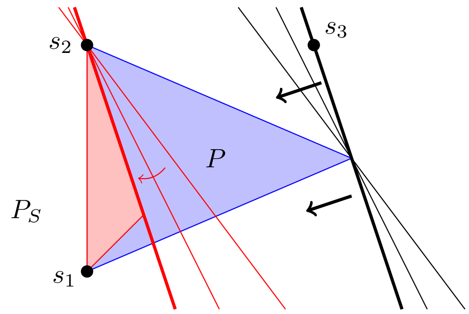

Figure 2: Some facets are not defined by S-CG cuts

2.2 The polar lemma

We next show an important property of closures of polyhedra with respect to an infinite family of valid inequalities. The following lemma will be useful, and it is related to a result of Dunkel and Schulz [10, Lemma 2.4].

Lemma 2.5 (Polar lemma). *Let $P \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^n \times \mathbb{R}$ be rational polyhedra. Assume $H \cap (\mathbb{Z}^n \times \mathbb{Z})$ is nonempty and is contained in $\operatorname{rec}(H)$, the recession cone of H. Then*

$$\bigcap_{(\alpha,\beta)\in H\cap(\mathbb{Z}^n\times\mathbb{Z})} \{x \in P:\ \alpha x \le \beta\} = \bigcap_{(\alpha,\beta)\in \operatorname{rec}(H)} \{x \in P:\ \alpha x \le \beta\}. \tag{7}$$

Moreover, both sets are rational polyhedra.

Proof. For ease of notation, let Π denote $H \cap (\mathbb{Z}^n \times \mathbb{Z})$. By Meyer's Theorem [17], as Π is nonempty, $\operatorname{conv}(\Pi)$ is a rational polyhedron and has the same recession cone as H, namely $\operatorname{rec}(H)$. Let P_1 denote the set on the left-hand-side of equation (7), and let P_2 denote the right-hand-side set. As $\Pi \subseteq \operatorname{rec}(H)$, P_2 is a subset of P_1. We will show, by contradiction, that for any $(\alpha, \beta) \in \operatorname{rec}(H)$, $\alpha x \le \beta$ is valid for P_1, thereby proving that $P_1 \subseteq P_2$. Assume this is false. Then there exist $(\alpha, \beta) \in \operatorname{rec}(H)$ and $\bar{x} \in P_1$ such that $\alpha\bar{x} > \beta$. Consider an arbitrary $(\alpha^0, \beta^0) \in \Pi$; then $\alpha^0\bar{x} \le \beta^0$ as $\bar{x} \in P_1$. Therefore, we can choose a positive μ such that $\mu(\alpha\bar{x} - \beta) > \beta^0 - \alpha^0\bar{x}$. So, we have

$$(\alpha^0 + \mu\alpha)\bar{x} > \beta^0 + \mu\beta. \tag{8}$$

On the other hand, since $(\alpha^0, \beta^0) \in \Pi \subseteq \operatorname{conv}(\Pi)$ and $(\alpha, \beta) \in \operatorname{rec}(H) = \operatorname{rec}(\operatorname{conv}(\Pi))$, it follows that $(\alpha^0, \beta^0) + \mu(\alpha, \beta) \in \operatorname{conv}(\Pi)$. Every vector of Π defines a valid inequality for P_1, and – by convexity – so does every vector of $\operatorname{conv}(\Pi)$. This implies that $(\alpha^0 + \mu\alpha)\bar{x} \le \beta^0 + \mu\beta$, a contradiction to (8). Therefore, $P_1 = P_2$.

To complete the proof, we show that P_2 is a rational polyhedron. As H is a rational polyhedron, $\operatorname{rec}(H)$ is a rational polyhedral cone, and therefore, there exist $(\alpha^1, \beta^1), \ldots, (\alpha^r, \beta^r) \in \operatorname{rec}(H) \cap (\mathbb{Q}^n \times \mathbb{Q})$ such that any

$(\alpha, \beta) \in \text{rec}(H)$ can be written as a conic combination of these vectors. Therefore, P_2 is equal to $\{x \in P : \alpha^i x \leq \beta^i, i = 1, \ldots, r\}$, so P_2 is a rational polyhedron, as required. □

Using Lemma 2.5, we can prove that the S-CG closure of a rational polyhedron $P \subseteq \mathbb{R}^n$ is a rational polyhedron by constructing a rational polyhedron $H \subseteq \mathbb{R}^n \times \mathbb{R}$ such that $H \subseteq \text{rec}(H)$ and the S-CG closure is equal to $\bigcap_{(\alpha,\beta) \in H \cap (\mathbb{Z}^n \times \mathbb{Z})} \{x \in P : \ \alpha x \leq \beta\}$. We note that the idea of constructing a polyhedron H such that its integer points correspond to CG cuts for a polyhedron P is well-known; see Bockmayr and Eisenbrand [1].

2.3 S-CG closure when S is finite

Dunkel and Schulz [10] proved the following result:

Theorem 2.6 (Dunkel and Schulz [10])**.** *Let $S = \{0, 1\}^n$, and let $P \subseteq [0, 1]^n$ be a rational polytope. Then P_S is a rational polytope.*

We extend this result to the case when S is a finite subset of $\mathbb{Z}^n$ and P is a rational polyhedron not necessarily contained in $\text{conv}(S)$.

Theorem 2.7. *Let S be a finite subset of $\mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Let $H \subseteq \mathbb{R}^n \times \mathbb{R}$ be a rational polyhedron that is contained in its recession cone $\text{rec}(H)$, and let $\Gamma = \Pi_P \cap H$. Then $P_{S,\Gamma}$ is a rational polyhedron. In particular, P_S is a rational polyhedron.*

Proof. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. As S is finite, it has a finite number of ordered partitions with cardinality two, and let $\mathcal{F}$ be the family of all such partitions. For any $(L, G) \in \mathcal{F}$, we define

$$H_{(L,G)} = \left\{ \begin{array}{c} (\alpha, \delta) \\ \in \mathbb{R}^n \times \mathbb{R} \end{array} : \begin{array}{c} \exists(\beta, \lambda) \\ \in \mathbb{R} \times \mathbb{R}^m_+ \end{array} \text{ s.t. } \begin{array}{rcll} (\alpha, \beta) & = & (\lambda A, \lambda b), & \\ \alpha z & \leq & \delta & \forall z \in L, \\ \alpha z & \geq & \beta + \frac{1}{\kappa} & \forall z \in G, \\ \delta & \leq & \beta, & \\ (\alpha, \beta) & \in & H & \end{array} \right\}$$

where $\kappa > 0$ is the least common multiple of nonzero subdeterminants of A. We define $P_{(L,G)}$ as

$$P_{(L,G)} = \bigcap_{(\alpha,\delta) \in H_{(L,G)} \cap (\mathbb{Z}^n \times \mathbb{Z})} \{x \in P : \ \alpha x \leq \delta\}. \tag{9}$$

For some $(L, G) \in \mathcal{F}$, the set $H_{(L,G)}$ may be empty, in which case $P_{(L,G)} = P$.

By definition, the polyhedron $H_{(L,G)}$ is the projection of a polyhedral set V – defined on the variables $\alpha, \delta, \beta, \lambda$ – onto the space of variables α, δ. Using the fact that the recession cone of $H_{(L,G)}$ is equal to the projection of $\text{rec}(V)$ onto the the space of variables α, δ, we obtain

$$\text{rec}(H_{(L,G)}) = \left\{ \begin{array}{c} (\alpha, \delta) \\ \in \mathbb{R}^n \times \mathbb{R} \end{array} : \begin{array}{c} \exists(\beta, \lambda) \\ \in \mathbb{R} \times \mathbb{R}^m_+ \end{array} \text{ s.t. } \begin{array}{rcll} (\alpha, \beta) & = & (\lambda A, \lambda b), & \\ \alpha z & \leq & \delta & \forall z \in L, \\ \alpha z & \geq & \beta & \forall z \in G, \\ \delta & \leq & \beta, & \\ (\alpha, \beta) & \in & \text{rec}(H) & \end{array} \right\}.$$

Since $1/\kappa > 0$ and $H \subseteq \text{rec}(H)$, it follows that

$$H_{(L,G)} \subseteq \text{rec}(H_{(L,G)}) \quad \Rightarrow \quad H_{(L,G)} \cap (\mathbb{Z}^n \times \mathbb{Z}) \subseteq \text{rec}(H_{(L,G)}).$$

Then Lemma 2.5, along with equation (9), implies that $P_{(L,G)}$ is a rational polyhedron.

We will next prove that

$$P_{S,\Gamma} = \bigcap_{(L,G)\in\mathcal{F}} P_{(L,G)}. \tag{10}$$

As $\mathcal{F}$ is a finite set and $P_{(L,G)}$ is a rational polyhedron for any $(L, G) \in \mathcal{F}$, the theorem will follow.

To show that $P_{S,\Gamma}$ contains the right-hand-side of (10), we will show that all valid inequalities for $P_{S,\Gamma}$ are valid for some $P_{(L,G)}$. By (6), it suffices to consider valid inequalities for $P_{S,\Gamma}$ that have the form $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$ for some $(\alpha, \beta) \in \Pi_P \cap H$. Consider one such inequality. Then there exists some $\lambda \in \mathbb{R}^m_+$ such that $(\alpha, \beta) = (\lambda A, \lambda b)$ and $\beta = \max\{\alpha x : x \in P\}$. Assume that $\alpha x \leq \beta$ partitions S into L and G as follows:

$$L = \{x \in S : \ \alpha x \leq \beta\} \quad \text{and} \quad G = \{x \in S : \ \alpha x > \beta\}. \tag{11}$$

We will show that $(\alpha, \lfloor \beta \rfloor_{S,\alpha}) \in \mathbb{Z}^n \times \mathbb{Z}$ is contained in $H_{(L,G)}$. Let $\delta = \lfloor \beta \rfloor_{S,\alpha} \leq \beta$, and notice that α, β, δ trivially satisfy the first, second, and fourth set of inequalities defining $H_{(L,G)}$. If G is empty, then the third set of inequalities defining $H_{(L,G)}$ is trivially satisfied. So, we may assume that G is nonempty. As $\beta = \max\{\alpha x : x \in P\}$, the maximum is attained at a point $\bar{x}$ such that its components are integral multiples of $1/\kappa$. Then β is an integral multiple of $1/\kappa$ as α is integral. Therefore, if $\alpha \bar{z} > \beta$ for some integral $\bar{z}$, then $\alpha \bar{z} \geq \beta + 1/\kappa$. Consequently, $(\alpha, \lfloor \beta \rfloor_{S,\alpha}) \in H_{(L,G)}$, as desired. Therefore, $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$ is valid for $P_{(L,G)}$. We have thus shown that $P_{S,\Gamma}$ contains the right-hand-side of (10).

To show the reverse containment, consider an arbitrary $(L, G) \in \mathcal{F}$ such that $H_{(L,G)} \neq \emptyset$, and let $(\alpha, \delta) \in H_{(L,G)} \cap (\mathbb{Z}^n \times \mathbb{Z})$. Then there exist some β and λ such that $\alpha, \beta, \delta, \lambda$ satisfy the constraints defining $H_{(L,G)}$. Therefore, $\alpha x \leq \beta$ is valid for P, $\alpha z > \beta$ for all $z \in G$, and $\alpha z \leq \delta \leq \beta$ for all $z \in L$. Furthermore, $\lfloor \beta \rfloor_{S,\alpha} \leq \delta$ as all points in S that satisfy $\alpha x \leq \beta$ also satisfy $\alpha x \leq \delta$. Therefore, $\alpha x \leq \delta$ is valid for $P_{S,\Gamma}$ as it is dominated by the S-CG cut $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$. We have thus shown that $P_{S,\Gamma}$ is contained in the right-hand-side of (10), and therefore, the equality in (10) holds.

If we let $H = \mathbb{R}^n \times \mathbb{R}$, then $P_S = P_{S,\Gamma}$, and therefore, P_S is a rational polyhedron. $\square$

As a direct corollary of Theorem 2.7, we obtain the following:

Corollary 2.8. *Let $B = \{x \in \mathbb{R}^n : \ell \leq x \leq u\}$ for some $\ell, u \in \mathbb{Z}^n$ such that $\ell \leq u$. Let $P \subseteq B$ be a rational polytope, and let $S = B \cap \mathbb{Z}^n$. Then P_S is a rational polytope.*

It is possible that $H_{(L,G)} \cap (\mathbb{Z}^n \times \mathbb{Z})$, defined in the proof of Theorem 2.7, is strictly contained in $\text{rec}(H_{(L,G)})$. Therefore, for some $\alpha, \beta, \delta, \lambda$ that satisfy the constraints describing $\text{rec}(H_{(L,G)})$, we might have a point $z \in G$ that satisfies $\alpha z = \beta$. In this case, $\lfloor \beta \rfloor_{S,\alpha} = \beta > \delta$, and therefore, the inequality $\alpha x \leq \delta$ cannot be obtained as an S-CG cut from $\alpha x \leq \beta$. In Example 2.4, the limiting inequality that is valid for the S-CG closure but is not an S-CG cut precisely falls into this category.

3 S-CG closure when S is a cylinder

In Section 2.3, we showed that P_S is a rational polyhedron when S is a finite subset of $\mathbb{Z}^n$ and P is a rational polyhedron. In this section, we consider the case where

$$S = T \times \mathbb{Z}^l \text{ for some finite } T \subseteq \mathbb{Z}^n, \tag{12}$$

$$P = \Big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^l : Ax + Cy \leq b\Big\}, \tag{13}$$

and the matrices A, C, b are integral and have m rows and $n, l, 1$ columns, respectively. In this case, we will prove that P_S is a rational polyhedron. For P defined in (13), the set Π_P – defined in (4) – can be written as

$$\begin{aligned}\Pi_P = \{(\alpha, \gamma, \beta) \in \mathbb{Z}^n \times \mathbb{Z}^l \times \mathbb{R} \ : \ \exists \lambda \in \mathbb{R}^m_+ \text{ s.t. } (\alpha, \gamma, \beta) &= (\lambda A, \lambda C, \lambda b), \\ \beta &= \max\{\alpha x + \gamma y : (x, y) \in P\}\}.\end{aligned} \tag{14}$$

Let $\mathbf{0}$ be the vector of all zeros of appropriate dimension, and let

$$\Pi_0 = \{(\alpha, \gamma, \beta) \in \Pi_P : \gamma = \mathbf{0}\}. \tag{15}$$

By Remark 2.2, $P_S = P_{S,\Pi_0} \cap P_{S,\Pi_P \setminus \Pi_0}$. To prove that P_S is a rational polyhedron, we will first argue that P_{S,Π_0} is a rational polyhedron. This result follows from the lemma below, which will also be used in Section 4.

Lemma 3.1 (**Projection lemma**). *Let P be defined as in (13). Let*

$$S = T \times \mathbb{Z}^l \textit{ for some } T \subseteq \mathbb{Z}^n.$$

Let $\Gamma \subseteq \Pi_0$, and let $\Omega = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : (\alpha, \mathbf{0}, \beta) \in \Gamma\}$. If $Q = \operatorname{proj}_x(P)$, then $\Omega \subseteq \Pi_Q$ and

$$P_{S,\Gamma} = P \cap \left(Q_{T,\Omega} \times \mathbb{R}^l\right).$$

Proof. We first argue that $\Omega \subseteq \Pi_Q$. For any $(\alpha, \beta) \in \Omega$, we have $(\alpha, \mathbf{0}, \beta) \in \Gamma$, implying in turn that

$$\beta = \max\{\alpha x : (x, y) \in P\} = \max\{\alpha x : x \in \operatorname{proj}_x(P)\} = \max\{\alpha x : x \in Q\}.$$

Therefore, $(\alpha, \beta) \in \Pi_Q$, and thus $\Omega \subseteq \Pi_Q$.

Next we argue that $Q_{T,\Omega} = \operatorname{proj}_x(P_{S,\Gamma})$. For any $(\alpha, \beta) \in \Omega$ (i.e., $(\alpha, \mathbf{0}, \beta) \in \Gamma$), we have

$$\lfloor \beta \rfloor_{T,\alpha} = \max\{\alpha x : x \in T, \alpha x \leq \beta\} = \max\{\alpha x : (x, y) \in S, \alpha x \leq \beta\} = \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}.$$

Let $(\bar{x}, \bar{y}) \in P_{S,\Gamma}$. Then for any $(\alpha, \beta) \in \Omega$, we have $\alpha \bar{x} \leq \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}$ and thus $\alpha \bar{x} \leq \lfloor \beta \rfloor_{T,\alpha}$, implying in turn that $\bar{x} \in Q_{T,\Omega}$. Conversely, let $x \in Q_{T,\Omega}$. As $x \in Q$, there exists $y \in \mathbb{R}^l$ such that $(x, y) \in P$. Then for any $(\alpha, \mathbf{0}, \beta) \in \Gamma$, we have $\alpha x \leq \lfloor \beta \rfloor_{T,\alpha}$ and thus $\alpha x \leq \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}$, which in turn implies that $(x, y) \in P_{S,\Gamma}$. Therefore, $Q_{T,\Omega} = \operatorname{proj}_x(P_{S,\Gamma})$, and it follows that

$$P_{S,\Gamma} \subseteq P \cap (Q_{T,\Omega} \times \mathbb{R}^l).$$

Suppose for a contradiction that $P_{S,\Gamma} \neq P \cap (Q_{T,\Omega} \times \mathbb{R}^l)$. Then there exists a point $(\bar{x}, \bar{y}) \in P$ such that $\bar{x} \in Q_{T,\Omega}$ and $(\bar{x}, \bar{y}) \notin P_{S,\Gamma}$. Since $(\bar{x}, \bar{y}) \in P \setminus P_{S,\Gamma}$, there must exist some $(\alpha, \mathbf{0}, \beta) \in \Gamma$ such that $\alpha \bar{x} > \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}$, and therefore, $\alpha \bar{x} > \lfloor \beta \rfloor_{T,\alpha}$. This is a contradiction as $\bar{x} \in Q_{T,\Omega}$. So, $P_{S,\Gamma} = P \cap (Q_{T,\Omega} \times \mathbb{R}^l)$, as required. $\square$

Notice that $T \subseteq \mathbb{Z}^n$ in Lemma 3.1 does not need to be finite. As a consequence of Lemma 3.1, we obtain the following lemma:

Lemma 3.2. *Let S and P be defined as in (12)–(13), and let Π_0 be defined as in (15). Then P_{S,Π_0} is a rational polyhedron.*

Proof. Let $\Omega = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : (\alpha, \mathbf{0}, \beta) \in \Pi_0\}$, and let $Q = \mathrm{proj}_x(P)$. Then it follows that $\Omega = \Pi_Q$, implying in turn that $Q_{T,\Omega} = Q_T$. So, by Theorem 2.7, $Q_{T,\Omega}$ is a rational polyhedron. Moreover, by Lemma 3.1, $P_{S,\Pi_0} = P \cap \left(Q_{T,\Omega} \times \mathbb{R}^l\right)$. Therefore, P_{S,Π_0} is a rational polyhedron. □

Given two cutting planes for P, we say that the first *dominates* the second if the points in P satisfying the first also satisfy the second inequality. By this definition, any cut for P dominates, and is dominated by, itself. It is well-known that the Chvátal closure of P is described by $\lambda Ax + \lambda Cy \leq \lfloor \lambda b \rfloor$ for $\lambda \in \mathbb{R}^m_+$ such that $(\lambda A, \lambda C) \in \mathbb{Z}^n \times \mathbb{Z}^l$ and $\mathbf{0} \leq \lambda < \mathbf{1}$ [20]. In fact, every CG-cut for a rational polyhedron is dominated by another CG cut obtained via bounded multipliers. The next result for S-CG cuts is analogous to this result. We define the following constant K that depends on P and T:

$$K = \max\left\{\mathbf{1}^\top |b - Ax| : x \in T\right\}, \tag{16}$$

where $|b - Ax|$ denotes the vector whose entries are the absolute values of the entries of $b - Ax$. Given a vector γ, let $\gcd(\gamma)$ denote the greatest common divisor of the entries of γ.

Lemma 3.3. *Let S, T, P, and Π_P be defined as in (12)–(14). Then for any $(\alpha, \gamma, \beta) \in \Pi_P$, there exists $(\alpha', \gamma', \beta') \in \Pi_P$ that satisfies the following:*

(1) the S-CG cut derived from $(\alpha', \gamma', \beta')$ dominates the S-CG cut derived from (α, γ, β),
(2) either $\gamma' = \mathbf{0}$, or there exists $\mu \in \mathbb{R}^m$ with $\mathbf{0} \leq \mu < \gcd(\gamma')\mathbf{1}$ such that (a) $(\alpha', \gamma', \beta') = (\mu A, \mu C, \mu b)$ and (b) $|\beta' - \alpha' x| \leq \gcd(\gamma')K$ for all $x \in T$.

Proof. Let $(\alpha, \gamma, \beta) \in \Pi_P$. Then $(\alpha, \gamma, \beta) = (\lambda A, \lambda C, \lambda b)$ for some $\lambda \in \mathbb{R}^m_+$, and α, γ are integral vectors. If $\gamma = \mathbf{0}$, then the S-CG cut derived from $(\alpha, \gamma, \beta) = (\alpha, \mathbf{0}, \beta)$ dominates itself. So, we assume that $\gamma \neq \mathbf{0}$. Let g denote $\gcd(\gamma)$. If $\lambda_i < g$ for $i = 1, \ldots, m$, then $(\alpha', \gamma', \beta') = (\alpha, \gamma, \beta)$ is the desired vector as $|\beta - \alpha x| \leq gK$, and therefore, we may assume that this is not the case.

Let $\delta, \mu \in \mathbb{R}^m$ be defined by $\delta_i = g\lfloor \lambda_i / g \rfloor$ and $\mu = \lambda - \delta$. Since $\mu_i \equiv \lambda_i \pmod{g}$ for $i = 1, \ldots, m$, we have $\mathbf{0} \leq \mu < g\mathbf{1}$. Let $(\alpha', \gamma', \beta') = (\mu A, \mu C, \mu b)$. Since $\mu \leq \lambda$ and $\beta = \max\{\alpha x + \gamma y : (x, y) \in P\}$, it follows that $\beta' = \max\{\alpha' x + \gamma' y : (x, y) \in P\}$. So, $(\alpha', \gamma', \beta') \in \Pi_P$.

To show that $\alpha' x + \gamma' y \leq \lfloor \beta' \rfloor_{S,(\alpha',\gamma')}$ dominates the inequality $\alpha x + \gamma y \leq \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$, we will argue that

$$\delta b + \lfloor \beta' \rfloor_{S,(\alpha',\gamma')} \leq \lfloor \beta \rfloor_{S,(\alpha,\gamma)}. \tag{17}$$

By definition, there exists $(\bar{x}, \bar{y}) \in S$ such that $\alpha' \bar{x} + \gamma' \bar{y} = \lfloor \beta' \rfloor_{S,(\alpha',\gamma')}$, which implies that

$$\delta b + \lfloor \beta' \rfloor_{S,(\alpha',\gamma')} = \delta b + (\alpha - \delta A)\bar{x} + (\gamma - \delta C)\bar{y} = \alpha \bar{x} + \gamma \bar{y} + \left(\delta b - \delta A \bar{x} - \delta C \bar{y}\right). \tag{18}$$

As the components of the vector δ are multiples of g and $A, C, b, \bar{x}, \bar{y}$ are all integral, the expression

$$\frac{1}{g}(\delta b - \delta A\bar{x} - \delta C\bar{y}) \tag{19}$$

is an integer. Since $\frac{1}{g}\gamma$ is an integral vector with $\gcd\left(\frac{1}{g}\gamma\right) = 1$, there exists $\hat{y} \in \mathbb{Z}^l$ such that $\frac{1}{g}\gamma\hat{y}$ is equal to the integer in (19). Making this substitution in (18), we obtain $\delta b + \lfloor\beta'\rfloor_{S,(\alpha',\gamma')} = \alpha\bar{x} + \gamma(\bar{y}+\hat{y})$. As $\delta b + \lfloor\beta'\rfloor_{S,(\alpha',\gamma')} \leq \delta b + \beta' = \beta$, it follows that $\alpha\bar{x} + \gamma(\bar{y}+\hat{y}) \leq \lfloor\beta\rfloor_{S,(\alpha,\gamma)}$ as $(\bar{x}, \bar{y}+\hat{y}) \in S$. Therefore, the inequality (17) holds.

If $\gamma' = \mathbf{0}$, the proof is complete. If $\gamma' \neq \mathbf{0}$, then we note that all components of γ' are multiples of g as $\gamma' = \gamma - \delta$ and all components of γ and δ are multiples of g. Therefore, $\gcd(\gamma') = g' = kg$ for some positive integer k. Moreover, for $i = 1, \ldots, m$, we have $0 \leq \mu_i < g'$ as $0 \leq \mu_i < g$, so (a) holds. To see that (b) also holds, note that $\beta' - \alpha' x = \mu b - \mu A x = \mu(b - Ax)$ for all $x \in T$. As A and b are fixed, T is a finite set of integers, and $\mathbf{0} \leq \mu < g'\mathbf{1}$, the result follows with K defined in (16). $\square$

Using Lemma 3.3, we can prove the following theorem:

Theorem 3.4. *Let $S = T \times \mathbb{Z}^l$ for some finite $T \subseteq \mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^{n+l}$ be a rational polyhedron. Then P_S is a rational polyhedron.*

Proof. We may assume that $P_S \neq \emptyset$; otherwise P_S is trivially polyhedral. As $\mathrm{conv}(S)$ is a rational polyhedron, if $P \cap \mathrm{conv}(S) = \emptyset$, then we have $P_S = \emptyset$. Therefore, we assume that $P \cap \mathrm{conv}(S)$ is nonempty. Let Π_P and Π_0 be defined as in (14)–(15). Remark 2.2 implies that $P_S = P_{S,\Pi_0} \cap P_{S,\Pi_P \setminus \Pi_0}$, and Lemma 3.2 implies that P_{S,Π_0} is a rational polyhedron.

Let $\Theta = \{\lambda C \in \mathbb{Z}^l : \mathbf{0} \leq \lambda \leq \mathbf{1}\} \setminus \{\mathbf{0}\}$. Let $t = |T|$, $T = \{x^1, \ldots, x^t\}$, and $I = \{1, \ldots, t\}$. Let K be defined as in (16). Given $\mu \in \Theta$ and $\rho \in [-K, K]^t \cap \mathbb{Z}^t$, we define $H_{(\mu,\rho)}$ as follows:

$$H_{(\mu,\rho)} = \left\{ \begin{array}{c} (\alpha,\gamma,\delta) \\ \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R} \end{array} : \exists \begin{pmatrix} \beta \in \mathbb{R} \\ \lambda \in \mathbb{R}^m_+ \\ g \in \mathbb{Z} \end{pmatrix} \text{ s.t. } \begin{array}{rcll} (\alpha,\gamma,\beta) & = & (\lambda A, \lambda C, \lambda b), & \\ \alpha x^i + g(\rho_i + 1) & \geq & \beta + \frac{1}{\kappa} & \forall i \in I, \\ \alpha x^i + g\rho_i & \leq & \delta & \forall i \in I, \\ \lambda & \leq & g\mathbf{1}, & \\ \delta & \leq & \beta, & \\ \gamma & = & g\mu, & \\ g & \geq & 1 & \end{array} \right\}$$

where $\kappa > 0$ is the least common multiple of nonzero subdeterminants of (A, C). For all $\mu \in \Theta$ and $\rho \in [-K, K]^t \cap \mathbb{Z}^t$, let

$$P_{(\mu,\rho)} = \bigcap_{(\alpha,\gamma,\delta) \in H_{(\mu,\rho)} \cap (\mathbb{Z}^n \times \mathbb{Z}^l \times \mathbb{Z})} \{(x,y) \in P : \alpha x + \gamma y \leq \delta\}.$$

If $H_{(\mu,\rho)} \cap (\mathbb{Z}^n \times \mathbb{Z}^l \times \mathbb{Z})$ is nonempty, then it is contained in $\mathrm{rec}(H_{(\mu,\rho)})$ and Lemma 2.5 implies that $P_{(\mu,\rho)}$ is a rational polyhedron. When $H_{(\mu,\rho)} \cap (\mathbb{Z}^n \times \mathbb{Z}^l \times \mathbb{Z}) = \emptyset$, we have $P_{(\mu,\rho)} = P$. We will prove that

$$P_{S,\Pi_P \setminus \Pi_0} = \bigcap_{(\mu \in \Theta, \rho \in [-K,K]^t \cap \mathbb{Z}^t)} P_{(\mu,\rho)}, \tag{20}$$

thereby proving the theorem.

To show that $P_{S,\Pi_P\setminus\Pi_0}$ contains the right-hand-side of (20), we will show that for all $(\alpha,\gamma,\beta)\in\Pi_P\setminus\Pi_0$, the vector $(\alpha,\gamma,\lfloor\beta\rfloor_{S,(\alpha,\gamma)})\in H_{(\mu,\rho)}\cap(\mathbb{Z}^n\times\mathbb{Z}^l\times\mathbb{Z})$ for some $\mu\in\Theta$ and $\rho\in[-K,K]^t\cap\mathbb{Z}^t$. To this end, let $(\alpha,\gamma,\beta)\in\Pi_P\setminus\Pi_0$. Then $\gamma\neq\mathbf{0}$ and $(\alpha,\gamma,\beta)=(\lambda A,\lambda C,\lambda b)$ for some $\lambda\in\mathbb{R}^m_+$. Moreover, by Lemma 3.3, we may assume that $\gcd(\gamma)=g$ for some positive integer g with $\mathbf{0}\leq\lambda<g\mathbf{1}$. As $\gamma/g=(\lambda/g)C$ is an integral vector and $\mathbf{0}\leq\lambda/g<\mathbf{1}$, we see that $\gamma=g\mu$ for some $\mu\in\Theta$. By our choice of U in (16), for each $i\in I$, there exists an integer $\rho_i\in[-K,K]$ such that

$$g\rho_i\leq\beta-\alpha x^i<g(\rho_i+1). \tag{21}$$

As $\beta=\max\{\alpha x+\gamma y:(x,y)\in P\}$ is finite, the maximum is attained at a rational point $(\bar{x},\bar{y})$ that has the denominators of its components equal to a subdeterminant of (A,C). Therefore, β is an integer multiple of $\frac{1}{\kappa}$. Hence, $\beta\leq\alpha x^i+g(\rho_i+1)-\frac{1}{\kappa}$ for all $i\in I$. Let ρ denote the vector whose entries are ρ_i $(i\in I)$. As the components of $\mu=\frac{1}{g}\gamma$ are relatively prime, we can find a vector $y^i\in\mathbb{Z}^l$ such that $\mu y^i=\rho_i$ for all $i\in I$. So, $\gamma y^i=g\rho_i$, and it follows from (21) that

$$\alpha x^i+\gamma y^i=\alpha x^i+g\rho_i\leq\beta.$$

Since $(x^i,y^i)\in S$, we have that $\alpha x^i+g\rho_i\leq\lfloor\beta\rfloor_{S,(\alpha,\gamma)}$. Therefore, $(\alpha,\gamma,\lfloor\beta\rfloor_{S,(\alpha,\gamma)})\in H_{(\mu,\rho)}$, as required.

We next show that if $H_{(\mu,\rho)}\cap(\mathbb{Z}^n\times\mathbb{Z}^l\times\mathbb{Z})\neq\emptyset$ for some $\mu\in\Theta$ and $\rho\in[-K,K]^t\cap\mathbb{Z}^t$, then $P_{S,\Pi_P\setminus\Pi_0}\subseteq P_{(\mu,\rho)}$. Let $(\alpha,\gamma,\delta)\in H_{(\mu,\rho)}$. Then there exists some $\beta\geq\delta$ such that the inequality $\alpha x+\gamma y\leq\beta$ is valid for P and $\delta\geq\max\{\alpha x^i+g\rho_i:i\in I\}$. As $P\cap\mathrm{conv}(S)$ is nonempty, $\lfloor\beta\rfloor_{S,(\alpha,\gamma)}$ is well-defined. So, for some $j\in I$ and some $y^*\in\mathbb{Z}^l$ we have

$$\alpha x^j+\gamma y^*=\lfloor\beta\rfloor_{S,(\alpha,\gamma)}\leq\beta<\alpha x^j+g(\rho_j+1).$$

If $\alpha x^j+g\rho_j<\lfloor\beta\rfloor_{S,(\alpha,\gamma)}$, then the previous inequality implies that

$$g\rho_j<\gamma y^*<g(\rho_j+1).$$

This is not possible as $\gamma=g\mu$ and γy^* is a multiple of g. Therefore, we have $\alpha x^j+g\rho_j\geq\lfloor\beta\rfloor_{S,(\alpha,\gamma)}$, implying that $\lfloor\beta\rfloor_{S,(\alpha,\gamma)}\leq\delta$. Therefore, $\alpha x+\gamma y\leq\delta$ is valid for P_S, as required. □

As a directly corollary of Theorem 3.4, we obtain the following result:

Corollary 3.5. *Let $T=\{x\in\mathbb{R}^n:u\leq x\leq v\}$ for some $u\leq v\in\mathbb{Z}^n$, and let $S=(T\cap\mathbb{Z}^n)\times\mathbb{Z}^l$. Let $P\subseteq\mathbb{R}^n\times\mathbb{R}^l$ be a rational polyhedron. Then P_S is a rational polyhedron.*

4 Integer points with bounds on components

In this section, we consider the set

$$S=\left\{(x,y,w^1,w^2)\in\mathbb{Z}^{n_1}\times\mathbb{Z}^{n_2}\times\mathbb{Z}^{n_3}\times\mathbb{Z}^{n_4}:\ x\in T,\ w^1\geq\ell,\ w^2\leq u\right\} \tag{22}$$

where $T \subseteq \mathbb{Z}^{n_1}$ is finite, $\ell \in \mathbb{Z}^{n_3}$ and $u \in \mathbb{Z}^{n_4}$. We will show that the S-CG closure of a rational polyhedron is again a rational polyhedron. To simplify the proof, we will first argue that we may focus on the setting where S is of the form

$$S = T \times \mathbb{Z}^{n_2} \times \mathbb{Z}_+^{n_3} \quad \text{for some finite } \; T \subseteq \mathbb{Z}_+^{n_1}. \tag{23}$$

Remember that a unimodular transformation is a mapping τ which maps $x \in \mathbb{R}^n$ to $Ux + v \in \mathbb{R}^n$ for some unimodular matrix $U \in \mathbb{Z}^{n\times n}$ and some integral vector $v \in \mathbb{Z}^n$. Note that the inverse mapping $\tau^{-1}(x) = U^{-1}x - U^{-1}v$ is also a unimodular transformation. For $\Pi \subseteq \Pi_P$, we abuse our notation and define $\tau(\Pi)$ as $\{(\alpha U^{-1}, \beta + \alpha U^{-1}v) : (\alpha,\beta) \in \Pi\} \subseteq \Pi_{\tau(P)}$.

Lemma 4.1 (Unimodular mapping lemma). *Let $S \subseteq \mathbb{Z}^n$, and let $P \subseteq \operatorname{conv}(S)$ be a rational polyhedron. Then $\tau(P) \subseteq \operatorname{conv}(\tau(S))$, and for any $\Pi \subseteq \Pi_P$, $\tau(P_{S,\Pi}) = \tau(P)_{\tau(S),\tau(\Pi)}$. In particular, $\tau(P_S) = \tau(P)_{\tau(S)}$.*

Proof. It is clear that $\tau(\operatorname{conv}(S)) = \operatorname{conv}(\tau(S))$, so we have $\tau(P) \subseteq \operatorname{conv}(\tau(S))$. For any $(\alpha,\beta) \in \mathbb{Z}^n \times \mathbb{R}$, $\tau\left(\{x \in \mathbb{R}^n : \; \alpha x \leq \beta\}\right) = \left\{z \in \mathbb{R}^n : \; \alpha\tau^{-1}(z) \leq \beta\right\}$, which implies that $\alpha x \leq \beta$ is a valid and supporting inequality for P if and only if $\alpha U^{-1} z \leq \beta + \alpha U^{-1} v$ is a valid and supporting inequality for $\tau(P)$. Moreover,

$$\tau\left(\{x \in \mathbb{R}^n : \; \lfloor\beta\rfloor_{S,\alpha} < \alpha x \leq \beta\}\right) = \left\{z \in \mathbb{R}^n : \; \lfloor\beta\rfloor_{S,\alpha} + \alpha U^{-1}v < \alpha U^{-1}z \leq \beta + \alpha U^{-1}v\right\}.$$

This implies that $\lfloor\beta + \alpha U^{-1}v\rfloor_{\tau(S),\alpha U^{-1}} = \lfloor\beta\rfloor_{S,\alpha} + \alpha U^{-1}v$. As a result,

$$\tau\left(\{x \in \mathbb{R}^n : \; \alpha x \leq \lfloor\beta\rfloor_{S,\alpha}\}\right) = \left\{z \in \mathbb{R}^n : \; \alpha U^{-1}z \leq \lfloor\beta + \alpha U^{-1}v\rfloor_{\tau(S),\alpha U^{-1}}\right\}.$$

Therefore, we get $\tau(P_{S,\Pi}) = \tau(P)_{\tau(S),\tau(\Pi)}$. In particular, when $\Pi = \Pi_P$, we have $\tau(P_S) = \tau(P)_{\tau(S)}$. ☐

Essentially, we can argue that there is a unimodular transformation mapping a set of the form (22) to a set of the form (23).

Lemma 4.2. *Let $S \subseteq \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3} \times \mathbb{Z}^{n_4}$ be of the form (22). Then there exists a unimodular transformation τ such that $\tau(S)$ is of the form (23).*

Proof. As T is finite, $T \subseteq \{x \in \mathbb{R}^{n_1} : p \leq x \leq q\}$ for some $p, q \in \mathbb{Z}^{n_1}$. Let τ be the unimodular transformation defined as follows: for each $(x, y, w^1, w^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \mathbb{R}^{n_4}$,

$$\tau\left((x, y, w^1, w^2)\right) = \left(x - p, \; y, \; w^1 - \ell, \; u - w^2\right).$$

Let $S' = T' \times \mathbb{Z}^{n_2} \times \mathbb{Z}_+^{n_3} \times \mathbb{Z}_+^{n_4}$ where $T' = \{x - p : x \in T\}$. Then $S' = \tau(S)$. Notice that T' is contained in $[\mathbf{0}, q - p]$. Therefore, S' is of the form (23) and τ is the desired unimodular transformation. ☐

Given S of the form (23), let S_0 be defined as

$$S_0 = T \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}. \tag{24}$$

Notice that S_0 is obtained from S after relaxing the nonnegativity restriction on the third set of variables. Since $S \subseteq S_0$, we have $P_S \subseteq P_{S_0}$ by Remark 2.1. Henceforth, we use N_1, N_2, N_3 to denote $\{1,\ldots,n_1\}$, $\{1,\ldots,n_2\}$, $\{1,\ldots,n_3\}$, respectively. Next we prove the following two lemmas:

Lemma 4.3. *Let S and S_0 be sets of the form (23) and (24), respectively. If $P \subseteq \operatorname{conv}(S)$ is a rational polyhedron, then $P_S = P_{S_0} \cap P_{S,\Pi^0}$ where*

$$\Pi^0 = \left\{(\alpha, \beta) \in \Pi_P : \ \alpha = (\alpha^1, \mathbf{0}, \alpha^3)\right\}. \tag{25}$$

Proof. To prove the claim, we will argue that if the S-CG cut derived from $(\alpha, \beta) \in \Pi_P$ is violated by a point in P_{S_0}, then $(\alpha, \beta) \in \Pi^0$. To this end, take a pair $(\alpha, \beta) \in \Pi_P$, where $\alpha = (\alpha^1, \alpha^2, \alpha^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$. If $\lfloor \beta \rfloor_{S,\alpha} = \lfloor \beta \rfloor_{S_0,\alpha}$, then the associated S-CG cut is the same as the associated S_0-CG cut, implying that any S-CG cut violated by a point in P_{S_0} must have $\lfloor \beta \rfloor_{S,\alpha} < \lfloor \beta \rfloor_{S_0,\alpha}$. This means that while S_0 contains a point $z = (z^1, z^2, z^3)$ such that $\alpha z = \lfloor \beta \rfloor_{S_0,\alpha}$, there is no such point in S.

Suppose for a contradiction that $\alpha^2 \neq \mathbf{0}$. Then $\alpha_i^2 \neq 0$ for some $i \in N_2$. Let $r = (r^1, r^2, r^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$ where

$$r^1 = \mathbf{0}, \qquad r^2 = -\frac{|\alpha_i^2|}{\alpha_i^2} \left(\sum_{j \in N_3} \alpha_j^3 \right) e_2^i, \qquad r^3 = |\alpha_i^2| \sum_{j \in N_3} e_3^j,$$

and e_2^i denotes the i^{th} unit vector in $\mathbb{R}^{n_2}$ and e_3^j denotes the j^{th} unit vector in $\mathbb{R}^{n_3}$. As $r^3 > \mathbf{0}$, there exists a sufficiently large integer M such that $\alpha^3 z^3 + M r^3 \geq \mathbf{0}$, and therefore, $z + Mr \in S$. Moreover, it can be readily checked that $\alpha r = 0$ and that $\alpha(z + Mr) = \alpha z$, implying in turn that $\lfloor \beta \rfloor_{S,\alpha} = \lfloor \beta \rfloor_{S_0,\alpha}$, a contradiction to our assumption that $\lfloor \beta \rfloor_{S,\alpha} < \lfloor \beta \rfloor_{S_0,\alpha}$. Therefore, it follows that $\alpha^2 = \mathbf{0}$. ☐

Lemma 4.4. *Let S be a set of the form (23), and let $P \subseteq \operatorname{conv}(S)$ be a rational polyhedron. Assume the following holds:*

> *For every $S' = T \times \mathbb{Z}_+^{n_3}$ for some finite $T \subseteq \mathbb{Z}_+^{n_1}$ and for every rational polyhedron $Q \subseteq \operatorname{conv}(S')$, both Q_{S',Ω_Q^+} and Q_{S',Ω_Q^-} are rational polyhedra where*
>
> $$\Omega_Q^+ = \left\{(\alpha, \beta) \in \Pi_Q : \ \alpha = (\alpha^1, \alpha^3), \ \alpha^3 \geq \mathbf{0}\right\} \ \textit{ and } \ \Omega_Q^- = \left\{(\alpha, \beta) \in \Pi_Q : \ \alpha = (\alpha^1, \alpha^3), \ \alpha^3 \leq \mathbf{0}\right\}. \tag{26}$$

Then P_S is a rational polyhedron.

Proof. By our choice of S, we have $S = T \times \mathbb{Z}^{n_2} \times \mathbb{Z}_+^{n_3}$ for some finite $T \subseteq \mathbb{Z}_+^{n_1}$. Let Π^0 be defined as in (25). Then, by Theorem 3.4 and Lemma 4.3, it suffices to show that P_{S,Π^0} is a rational polyhedron. Let $S' = T \times \mathbb{Z}_+^{n_3}$ and Q be the projection of S and the projection of P obtained after projecting out the second set of coordinates, respectively. By Lemma 3.1, P_{S,Π^0} is a rational polyhedron if and only if $Q_{S'}$ is a rational polyhedron.

Let $S_0' = T \times \mathbb{Z}^{n_3}$. We first show that $Q_{S'} = Q_{S_0'} \cap Q_{S',\Omega_Q^+} \cap Q_{S',\Omega_Q^-}$ where Ω_Q^+ and Ω_Q^- are defined as in (26). It is sufficient to argue that if the S'-CG cut derived from $(\alpha, \beta) \in \Pi_Q$ is violated by a point in $Q_{S_0'}$, then $(\alpha, \beta) \in \Omega_Q^+ \cup \Omega_Q^-$. To this end, take $(\alpha, \beta) \in \Pi_Q$, where $\alpha = (\alpha^1, \alpha^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_3}$, and assume that $\lfloor \beta \rfloor_{S',\alpha} < \lfloor \beta \rfloor_{S_0',\alpha}$. So, S_0' contains a point $z = (z^1, z^3)$ such that $\alpha z = \lfloor \beta \rfloor_{S_0',\alpha}$.

Suppose for a contradiction that there are distinct $i, j \in N_3$ such that $\alpha_i^3 > 0$ and $\alpha_j^3 < 0$. Let $J^+ = \{i \in N_3 :$

$\alpha_i^3 \geq 0\}$ and $J^- = \{j \in N_3 : \alpha_j^3 < 0\}$. As before, we construct a vector $r = (r^1, r^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_3}$ where

$$r^1 = \mathbf{0}, \qquad r^3 = \left(\sum_{i \in J^+} \alpha_i^3\right) \sum_{j \in J^-} e_3^j + \left(-\sum_{j \in J^-} \alpha_j^3\right) \sum_{i \in J^+} e_3^i.$$

As $r^3 > \mathbf{0}$, there exists an integer M such that $\alpha^3 z^3 + M r^3 \geq \mathbf{0}$, and therefore, $z + Mr \in S'$. Moreover, note that $\alpha r = 0$, and therefore, $\alpha(z + Mr) = \alpha z$, which implies that $\lfloor \beta \rfloor_{S',\alpha} = \lfloor \beta \rfloor_{S'_0,\alpha}$, a contradiction. Therefore, it follows that $\alpha^3 \geq \mathbf{0}$ or $\alpha^3 \leq \mathbf{0}$ holds, so $(\alpha, \beta) \in \Omega_Q^+ \cup \Omega_Q^-$. This in turn implies that $Q_{S'} = Q_{S'_0} \cap Q_{S',\Omega_Q^+} \cap Q_{S',\Omega_Q^-}$.

Notice that, by the assumption, both Q_{S',Ω_Q^+} and Q_{S',Ω_Q^-} are rational polyhedra. Since $Q_{S'_0}$ is a rational polyhedron by Theorem 3.4 and $Q_{S'} = Q_{S'_0} \cap Q_{S',\Omega_Q^+} \cap Q_{S',\Omega_Q^-}$, it follows that $Q_{S'}$ is a rational polyhedron. Therefore, P_S is a rational polyhedron, as required. $\square$

In Figure 3, we give an example where $\lfloor \beta \rfloor_{S,\alpha} = \lfloor \beta \rfloor_{S_0,\alpha}$, because α has both positive and negative coefficients. Here, $S = \mathbb{Z}_+^2$ and $S_0 = \mathbb{Z}^2$.

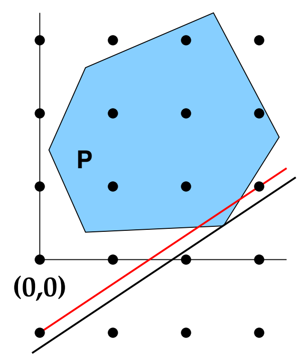

Figure 3: A situation where an S-CG cut is not strictly stronger than an S_0-CG cut

For a rational polyhedron P, we define Π_P^+ and Π_P^- as follows:

$$\Pi_P^+ = \{(\alpha, \beta) \in \Pi_P : \ \alpha \geq \mathbf{0}\}, \tag{27}$$

$$\Pi_P^- = \{(\alpha, \beta) \in \Pi_P : \ \alpha \leq \mathbf{0}\}. \tag{28}$$

When it is clear from the context, we will drop the subscript P from Π_P^+, Π_P^- and use Π^+, Π^- instead. Finally, we observe that one only needs to study the following narrow case to prove the main result:

Proposition 4.5. *Let S be a set of the form (23), and let $P \subseteq \operatorname{conv}(S)$ be a rational polyhedron. Assume the following holds:*

> *For every $S' = T \times \mathbb{Z}_+^{n_3}$ where $T \subseteq \mathbb{Z}_+^{n_1}$ is finite and for every rational polyhedron $Q \subseteq \operatorname{conv}(S')$, both Q_{S',Π_Q^+} and Q_{S',Π_Q^-} are rational polyhedra.*

Then P_S is a rational polyhedron.

Proof. Let $S' = T \times \mathbb{Z}_+^{n_3}$ where $T \subseteq \mathbb{Z}^{n_1}$ is finite, and let $Q \subseteq \operatorname{conv}(S')$ be a rational polyhedron. Let Ω_Q^+ and Ω_Q^- be defined as in (26). By Lemma 4.4, it suffices to show that Q_{S',Ω_Q^+} and Q_{S',Ω_Q^-} are rational polyhedra. To show that, we first partition the sets Ω_Q^+ and Ω_Q^- based on the sign pattern of the components of α^1:

$$\Omega_Q^+(J) = \left\{(\alpha, \beta) \in \Omega_Q^+ : \ \alpha_j^1 \geq 0 \ \forall j \in J, \quad \alpha_j^1 \leq 0 \ \forall j \in N_1 \setminus J\right\},$$

$$\Omega_Q^-(J) = \left\{(\alpha, \beta) \in \Omega_Q^- : \ \alpha_j^1 \leq 0 \ \forall j \in J, \quad \alpha_j^1 \geq 0 \ \forall j \in N_1 \setminus J\right\},$$

for all $J \subseteq N_1$. Clearly $\Omega_Q^+ = \cup_{J \subseteq N_1} \Omega_Q^+(J)$ and $\Omega_Q^- = \cup_{J \subseteq N_1} \Omega_Q^-(J)$. Then it follows from Remark 2.2 that $Q_{S',\Omega_Q^+} = \cap_{J \subseteq N_1} Q_{S',\Omega_Q^+(J)}$ and $Q_{S',\Omega_Q^-} = \cap_{J \subseteq N_1} Q_{S',\Omega_Q^-(J)}$. Hence, it suffices to prove that $Q_{S',\Omega_Q^+(J)}$ and $Q_{S',\Omega_Q^-(J)}$ are rational polyhedra for all $J \subseteq N_1$.

Let $J \subseteq N_1$, and let $u \in \mathbb{Z}_+^{n_1}$ be such that $T \subseteq [\mathbf{0}, u]$. Consider the unimodular transformation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ that maps $x \in \mathbb{R}^n$ to $z = \tau(x) \in \mathbb{R}^n$ where

$$z_i = \begin{cases} u_i - x_i, & \text{if } i \in N_1 \setminus J \\ x_i, & \text{otherwise.} \end{cases}$$

Then $\tau(S') = \tau(T) \times \mathbb{Z}_+^{n_3}$ for some $\tau(T) \subseteq [\mathbf{0}, u] \cap \mathbb{Z}^{n_1}$. Moreover, $\tau(\Omega_Q^+(J)) = \Pi_{\tau(Q)}^+$ and $\tau(\Omega_Q^-(J)) = \Pi_{\tau(Q)}^-$. By the assumption, both $\tau(Q)_{\tau(S'),\Pi_{\tau(Q)}^+}$ and $\tau(Q)_{\tau(S'),\Pi_{\tau(Q)}^-}$ are rational polyhedra. Then Lemma 4.1 implies that $Q_{S',\Omega_Q^+(J)}$ and $Q_{S',\Omega_Q^-(J)}$ are rational polyhedra. Therefore, Q_{S',Ω_Q^+} and Q_{S',Ω_Q^-} are rational polyhedra, as required. $\square$

In Figure 4, we show how to make all coefficients of a valid inequality $\alpha x \geq \beta$ for P nonnegative by applying the unimodular transformation in Proposition 4.5. Here, $S = \{0, 1, 2, 3\} \times \mathbb{Z}_+$, $P \subseteq \operatorname{conv}(S)$, and the unimodular transformation is given by $\tau(x_1, x_2) = (3 - x_1, x_2)$.

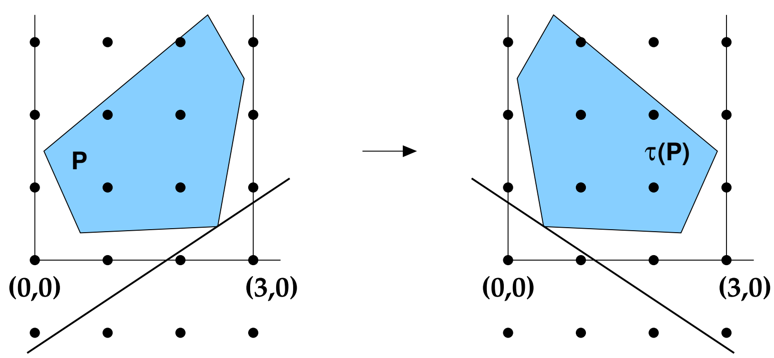

Figure 4: Transforming valid inequalities to make all coefficients nonnegative

4.1 Covering polyhedra

In this section, we consider covering polyhedra of the form

$$P^{\uparrow} = \left\{x \in \mathbb{R}^n : Ax \geq b\right\} \subseteq \mathbb{R}_+^n, \tag{29}$$

where $A \in \mathbb{Z}_+^{m\times n}$ and $b \in \mathbb{Z}_+^m$. Since $P^\uparrow \subseteq \mathbb{R}_+^n$, $P^\uparrow$ is pointed, which implies that $m \geq 1$. We assume that every extreme point of $P^\uparrow$ is contained in $\operatorname{conv}(S)$ where

$$S = T \times \mathbb{Z}_+^{n_2}, \quad T \subseteq \mathbb{Z}_+^{n_1} \text{ finite}, \quad \text{and } \ n = n_1 + n_2. \tag{30}$$

With this assumption, we will prove that ${P^\uparrow}_S$ is a rational polyhedron. Notice that, as $P^\uparrow$ may have a ray in $\mathbb{R}_+^{n_1} \times \{\mathbf{0}\}$, $P^\uparrow$ is not necessarily contained in $\operatorname{conv}(S)$. We also remark that $m \geq 1$, because $P^\uparrow$ is pointed. Note also that every valid inequality for $P^\uparrow$ is of the form

$$\alpha x \geq \beta \ \text{ where } \ \alpha \geq \mathbf{0}, \ \beta \geq 0. \tag{31}$$

Since we assumed that $P^\uparrow \subseteq \mathbb{R}_+^n$ and every extreme point of $P^\uparrow$ is in $\operatorname{conv}(S)$, we have $\min\{\alpha x : x \in P^\uparrow\} \geq \min\{\alpha x : x \in S\}$ for every $\alpha \in \mathbb{Z}_+^n$. As we will be dealing with inequalities of the greater or equal to form in this section, we will abuse notation and define $\Pi_{P^\uparrow}$ as follows:

$$\Pi_{P^\uparrow} = \big\{(\alpha,\beta) \in \mathbb{Z}^n \times \mathbb{R} \ : \ (\alpha,\beta) = (\lambda A, \lambda b) \text{ for some } \lambda \in \mathbb{R}_+^m, \ \beta = \min\{\alpha x : x \in P^\uparrow\}\big\}. \tag{32}$$

Given $(\alpha,\beta) \in \Pi_{P^\uparrow}$, the S-CG cut obtained from $\alpha x \geq \beta$ is $\alpha x \geq \lceil \beta \rceil_{S,\alpha}$.

We define the *support* of a vector $v \in \mathbb{R}^n$ to be the set $W \subseteq \{1,\ldots,n\}$ such that $v_i \neq 0$ if and only if $i \in W$, and we denote this by supp(v). For any set $I \subseteq \{1,\ldots,n\}$, we let supp$(v,I) = $ supp$(v) \cap I$ and refer to this set as the *support of* v *on* I. Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}$. For $j \in$ supp(α), the intercept of the hyperplane $\{x \in \mathbb{R}^n : \alpha x = \beta\}$ on the nonnegative axis $\{x \in \mathbb{R}_+^n : x_i = 0 \text{ for all } i \neq j\}$ equals β/α_j (and for convenience is referred to simply as an "intercept"). We define $I_2 = \{n_1+1,\ldots,n\}$.

The next result implies that if all nondominated S-CG cuts for $P^\uparrow$ have bounded intercepts (in the components corresponding to the support of the cut on I_2), then ${P^\uparrow}_S$ is a rational polyhedron.

Lemma 4.6. *Let M^* be a positive integer, and let*

$$\Pi = \{(\alpha,\beta) \in \Pi_{P^\uparrow} : \ \beta/\alpha_j \leq M^* \ \textit{ for all } j \in \textit{supp}(\alpha, I_2)\}\,. \tag{33}$$

Then ${P^\uparrow}_{S,\Pi}$ is a rational polyhedron.

Proof. Let $S^* = T \times \{0,\ldots,M^*\}^{n_2}$. Then S^* is a finite subset of S, and by Remark 2.1, ${P^\uparrow}_{S^*,\Pi} \subseteq {P^\uparrow}_{S,\Pi}$. We will next show that ${P^\uparrow}_{S^*,\Pi} = {P^\uparrow}_{S,\Pi}$.

Let $(\alpha,\beta) \in \Pi$. Then $\alpha x \geq \beta$ is valid for $P^\uparrow$, $\alpha \geq \mathbf{0}$, $\beta \geq 0$, and $0 \leq \beta/\alpha_j \leq M^*$ for every $j \in I_2$ such that $\alpha_j > 0$. It is sufficient to show that $\lceil \beta \rceil_{S^*,\alpha} = \lceil \beta \rceil_{S,\alpha}$. Let $z^* = (z^1, z^2) \in S = T \times \mathbb{Z}_+^{n_2}$ be such that

$$\alpha z^* = \lceil \beta \rceil_{S,\alpha} = \min\{\alpha x : x \in S, \alpha x \geq \beta\}. \tag{34}$$

If $z^* \in S^*$, then $\alpha z^* = \lceil \beta \rceil_{S^*,\alpha}$, and therefore, $\lceil \beta \rceil_{S^*,\alpha} = \lceil \beta \rceil_{S,\alpha}$. Thus, we may assume that $z^* \notin S^*$. Then for some $j \in I_2$, we have $z_j^* > M^*$. Let $\bar{z} \in S^*$ be obtained from z^* by reducing all such components to M^*. Note that if $\alpha_j > 0$ for any one of these components, then $\alpha \bar{z} \geq \beta$ as $\alpha_j M^* \geq \beta$. If, on the other hand, they are all zero, then $\alpha z^* = \alpha \bar{z}$ and $\alpha \bar{z} \geq \beta$ still holds. Consequently, in both cases, we have $\alpha z^* \geq \alpha \bar{z} \geq \beta$. Therefore, by (34), we have $\alpha z^* = \alpha \bar{z}$, implying in turn that $\lceil \beta \rceil_{S^*,\alpha} = \lceil \beta \rceil_{S,\alpha}$ and ${P^\uparrow}_{S^*,\Pi} = {P^\uparrow}_{S,\Pi}$, as desired.

To complete the proof, we will next argue that $P^{\uparrow}{}_{S^*,\Pi}$ is a rational polyhedron. Note that we can write

$$\Pi = \bigcup_{I \subseteq I_2} \Pi(I)$$

where

$$\Pi(I) = \{(\alpha, \beta) \in \Pi : \text{ supp}(\alpha, I_2) = I\}.$$

Therefore, $\Pi(I) = \Pi_{P^{\uparrow}} \cap H(I)$ where

$$H(I) = \left\{(\alpha, \beta) \in \mathbb{R}^{n+1} : \; M^* \alpha_j \geq \beta \text{ and } \alpha_j \geq 1 \;\; \forall j \in I, \quad \alpha_j = 0 \;\; \forall j \in I_2 \setminus I\right\}.$$

Notice that $H(I) \subseteq \text{rec}(H(I))$, and therefore, Theorem 2.7 implies that $P^{\uparrow}{}_{S^*,\Pi(I)}$ is a rational polyhedron. As $P^{\uparrow}{}_{S^*,\Pi} = \cap_{I \subseteq I_2} P^{\uparrow}{}_{S^*,\Pi(I)}$, the proof is complete. $\square$

We will next give a series of results which will show that all nondominated S-CG cuts for $P^{\uparrow}$ have "bounded" intercepts, in the sense that these inequalities belong to Π defined in (33). So, in the end, we will argue that $P^{\uparrow}{}_S = P^{\uparrow}{}_{S,\Pi}$.

Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$. For $j \in \{1, \ldots, n\}$, let $(\lambda A)_j$ denote the j^{th} component of λA, and consider the hyperplane $\{x \in \mathbb{R}^n : \lambda A x = \lambda b\}$. Notice that if each row a_i of A has the same support as λA, then the intercept on the positive x_j axis must lie between $\min_i\{b_i/a_{ij}\}$ and $\max_i\{b_i/a_{ij}\}$ for any j in supp(λA). In other words, all intercepts are trivially bounded by a function of A and b. Therefore, the difficult case for us is when not all rows of A have the same support. In that case, $a_{ij} = 0$ for some i, and therefore, $\max_i\{b_i/a_{ij}\}$ is unbounded and the intercept on the positive x_j axis can be arbitrarily large.

Definition 4.7. *Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. The tilting ratio of λ with respect to A is defined as*

$$r(\lambda, A) = \frac{\lambda_1}{\lambda_{t(\lambda,A)}} \tag{35}$$

where $t(\lambda, A)$ denotes the smallest index $j \in \{1, \ldots, m\}$ such that the support of $\sum_{i=1}^{j} \lambda_i a_i$ on I_2 is the same as the support of λA. In other words, $t(\lambda, A) = \min\{j \in \{1, \ldots, m\} : \bigcup_{i=1}^{j} \mathit{supp}(a_i, I_2) = \mathit{supp}\,(\lambda A, I_2)\}$. In particular, $\lambda_1, \ldots, \lambda_{t(\lambda,A)} > 0$ and $r(\lambda, A) > 0$.

We will later show (in Theorem 4.11) that for any $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, if $r(\lambda, A)$ is bounded above by a constant that depends only on A and b, then the intercepts of $\{x \in \mathbb{R}^n : \lambda A x = \lambda b\}$ corresponding to I_2 are also bounded above by a constant that depends only on A and b. We next focus on bounding $r(\lambda, A)$ for $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ defining a nondominated S-CG cut for $P^{\uparrow}$, with the bounding constants (that depend only on A and b, not on the cut) defined below.

Definition 4.8. *Let $B = \max\{b_i : i \in \{1, \ldots, m\}\}$ and $D = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$. We define $M_1 = 2\,(mB + 2D)$ and $M = \prod_{i=1}^{m-1} M_i$ where*

$$M_i = \left(2mB \prod_{j=1}^{i-1} M_j\right)^{i-1} M_1 \;\; \textit{for } i = 2, \ldots, m-1. \tag{36}$$

In particular, $M = 1$ if $m = 1$ and $M \geq M_1 \geq 4$ if $m \geq 2$. Furthermore, $(M_i/M_1)^{1/(i-1)} \geq 4$, and therefore, $(M_1/M_i)^{1/(i-1)} \leq 1/4$ for all $i \geq 2$.

We will show in the the following technical lemma that if $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ has tilting ratio $r(\lambda, A) > M$, then there exists a $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ that defines an S-CG cut dominating the one defined by λ, but with $\|\mu\|_1 \leq \|\lambda\|_1 - 1$. We will need the following well-known result of Dirichlet.

Theorem 4.9 (Simultaneous Diophantine Approximation Theorem [9]). *Let k be a positive integer. Given any real numbers $r_1, \ldots, r_k$ and $0 < \varepsilon < 1$, there exist integers $p_1, \ldots, p_k$ and q such that $\left| r_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}$ for $i = 1, \ldots, k$ and $1 \leq q \leq \left(\frac{1}{\varepsilon}\right)^k$.*

We are ready to prove the following technical lemma:

Lemma 4.10. *Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be such that $(\lambda A, \lambda b) \in \Pi_{P\uparrow}$. If $r(\lambda, A) > M$, then there exists $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ that satisfies the following: (i) $\|\mu\|_1 \leq \|\lambda\|_1 - 1$, (ii) $(\mu A, \mu b) \in \Pi_{P\uparrow}$, and (iii) $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$.*

Proof. After relabeling the rows of $Ax \geq b$, we may assume that $\lambda_1 \geq \cdots \geq \lambda_m$. Let t stand for $t(\lambda, A)$. If $t = 1$, we have $r(\lambda, A) = 1 \leq M$, a contradiction to our assumption. This implies that $t \geq 2$, so we have $m \geq 2$ as well. Let Δ be defined as

$$\Delta = \min \left\{ (\lambda A)_j : j \in \operatorname{supp}(\lambda A, I_2) \right\}, \tag{37}$$

and let

$$k = \operatorname{argmin} \left\{ (\lambda A)_j : \; j \in \operatorname{supp}(\lambda A, I_2) \setminus \bigcup_{i=1}^{t-1} \operatorname{supp}(a_i, I_2) \right\}. \tag{38}$$

By the definition of t, it follows that $\operatorname{supp}(\lambda A, I_2) \setminus \bigcup_{i=1}^{t-1} \operatorname{supp}(a_i, I_2)$ is not empty, and therefore, k is a well-defined index. Moreover, by (37) and (38),

$$\Delta \leq (\lambda A)_k = \sum_{i=t}^{m} \lambda_i a_{ik} \leq \lambda_t \sum_{i=t}^{m} a_{ik} \leq D\lambda_t. \tag{39}$$

Notice that as

$$r(\lambda, A) = \frac{\lambda_1}{\lambda_t} = \frac{\lambda_1}{\lambda_2} \times \cdots \times \frac{\lambda_{t-1}}{\lambda_t} > M \geq M_1 \times \cdots \times M_{t-1},$$

there exists $\ell \in \{1, \ldots, t-1\}$ such that

$$\lambda_i/\lambda_{i+1} \leq M_i \text{ for all } i \in \{1, \ldots, \ell - 1\} \quad \text{and} \quad \lambda_\ell/\lambda_{\ell+1} > M_\ell. \tag{40}$$

We now construct the vector $\mu \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. We consider the case $\ell \geq 2$ first. It follows from the Simultaneous Diophantine Approximation Theorem (with $k = \ell - 1$ and $r_i = \lambda_i/\lambda_\ell$ for $i \in \{1, \ldots, \ell - 1\}$) that there exist positive integers $p_1, \ldots, p_\ell$ satisfying

$$\left| \frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell} \right| < \frac{\varepsilon}{p_\ell}, \; i \in \{1, \ldots, \ell\} \quad \text{and} \quad p_\ell \leq \left(\frac{1}{\varepsilon}\right)^{\ell-1} \tag{41}$$

where $\varepsilon = (M_1/M_\ell)^{1/(\ell-1)}$. Moreover, for all $i \in \{1,\ldots,\ell-1\}$ we can assume that $p_i \geq p_{i+1} \geq p_\ell$, as $\lambda_i \geq \lambda_{i+1}$. If $p_i < p_{i+1}$ for some $i \in \{1,\ldots,\ell-1\}$, then increasing p_i to p_{i+1} can only reduce $|\lambda_i/\lambda_\ell - p_i/p_\ell|$. Now we define $\mu_1,\ldots,\mu_m$ as follows:

$$\mu_i = \begin{cases} \lambda_i - p_i\Delta & \text{for } i \in \{1,\ldots,\ell\}, \\ \lambda_i & \text{otherwise.} \end{cases} \tag{42}$$

If, on the other hand, $\ell = 1$, we define μ as in (42) with $p_1 = 1$. We divide the rest of the proof into several parts to improve readability.

Claim 1. *$\mu \geq \mathbf{0}$ and supp(μ) = supp(λ).*

Proof of Claim. If $\ell = 1$, then $\mu_1 = \lambda_1 - \Delta$ and $\mu_i = \lambda_i$ for $i \geq 2$. As $\lambda_1 > M_1\lambda_2$, it follows that $\mu_1 = \lambda_1 - \Delta > M_1\lambda_2 - \Delta$, so by (39), $\mu_1 > \lambda_2(M_1 - D)$. This in turn implies that $\mu_1 > \lambda_2$ as $M_1 - D \geq 1$ by Definition 4.8.

Now consider the case $\ell \geq 2$. Notice that

$$p_\ell \leq \frac{M_\ell}{M_1} \quad \text{and} \quad \lambda_i > \frac{p_i}{2p_\ell}\lambda_\ell,\ i \in \{1,\ldots,\ell\} \tag{43}$$

where the first inequality follows from (41) and the second one follows from the fact that $\varepsilon \leq \frac{1}{2}$, $|\lambda_i/\lambda_\ell - p_i/p_\ell| < \varepsilon/p_\ell \leq 1/(2p_\ell)$, and the fact that $p_i \geq p_\ell \geq 1$ for all $i \leq \ell$.

We will first show that $\mu \geq \mathbf{0}$. Clearly for $i \geq \ell+1$, we have $\mu_i = \lambda_i \geq 0$. We next show that $\mu_1,\ldots,\mu_\ell \geq \mu_{\ell+1}$. Let $i \in \{1,\ldots,\ell\}$. By definition, we have

$$\lambda_\ell > M_\ell\lambda_{\ell+1} \geq M_1 p_\ell \lambda_{\ell+1} \quad \Rightarrow \quad \lambda_\ell/p_\ell > M_1\lambda_{\ell+1}.$$

As $\lambda_i > \frac{p_i}{2p_\ell}\lambda_\ell$, we can conclude that

$$\lambda_i > p_i M_1 \lambda_{\ell+1}/2 \quad \text{and} \quad \mu_i = \lambda_i - p_i\Delta > p_i(\frac{1}{2}M_1\lambda_{\ell+1} - \Delta).$$

But as $\Delta \leq D\lambda_t \leq D\lambda_{\ell+1}$, we can conclude that

$$\mu_i > p_i(\frac{1}{2}M_1 - D)\lambda_{\ell+1}.$$

Since $M_1/2 - D \geq 1$ by Definition 4.8 and $p_i \geq 1$, the inequality above implies that $\mu_i \geq \lambda_{\ell+1} = \mu_{\ell+1} > 0$ for all $i \leq \ell$, as required. Therefore, $\mu \geq \mathbf{0}$. Moreover, we have shown that $\mu_i > 0$ if and only if $\lambda_i > 0$, for $i = 1,\ldots,m$, implying supp(μ) = supp(λ). □

By Claim 1, supp(μA) = supp(λA), which implies that $t(\mu, A) = t(\lambda, A)$. Moreover, since $\mu \geq \mathbf{0}$, it follows that $\|\mu\|_1 \leq \|\lambda\|_1 - 1$. So, claim (i) is satisfied whether.

Claim 2. *$\mu b = \min\left\{\mu A x : x \in P^{\uparrow}\right\}$, and therefore, $(\mu A, \mu b) \in \Pi_{P^{\uparrow}}$.*

Proof of Claim. Remember that $\lambda b = \min\left\{\lambda A x : x \in P^{\uparrow}\right\}$ and $P^{\uparrow} = \{x \in \mathbb{R}^n : Ax \geq b\}$. Let $x^* \in P^{\uparrow}$ be such that $\lambda A^* x = \lambda b$. By complementary slackness, if $\lambda_i > 0$ for an $i \in \{1,\ldots,m\}$, then $a_i x^* = b_i$. As $\lambda \geq \mu \geq \mathbf{0}$, if $\mu_i > 0$ then $a_i x^* = b_i$ also holds. Therefore, $\mu A x^* = \mu b = \min\left\{\mu A x : x \in P^{\uparrow}\right\}$. □

Claim 3. *Let $Q = \{x \in \mathbb{R}^n_+ : \mu b \le \mu A x \le \mu b + \Delta\}$. There is no point $x \in Q$ that satisfies*

$$\sum_{i=1}^{\ell} p_i a_i x \ge 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{44}$$

Proof of Claim. Suppose for a contradiction that there exists $\tilde{x} \in Q$ satisfying (44). Recall that for the index k defined in (38) the inequality $(\mu A)_k > 0$ holds. Let e^k denote the k^{th} unit vector and

$$v = \frac{\mu b}{(\mu A)_k} e^k$$

denote the intercept of the hyperplane defined by $\mu A x = \mu b$ on the nonnegative axis corresponding to x_k. Note that $\mu A v = \mu b$ and $v \in Q$. In addition, for the index ℓ defined in (40)

$$\sum_{i=1}^{\ell} p_i a_i v = 0 \tag{45}$$

since $k \notin \bigcup_{i=1}^{t-1} \operatorname{supp}(a_i, I_2)$ and $a_i e^k = 0$ for $i \le t-1$. As $\tilde{x} \in Q$ satisfies (44) and $v \in Q$ satisfies (45), we can take a convex combination of these points to get a point $\bar{x} \in Q$ such that

$$\sum_{i=1}^{\ell} p_i a_i \bar{x} = 1 + \sum_{i=1}^{\ell} p_i b_i \quad \Rightarrow \quad \sum_{i=1}^{\ell} p_i (a_i \bar{x} - b_i) = 1. \tag{46}$$

As $\mu A \bar{x} \le \mu b + \Delta$, we have

$$\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) \le - \sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + \Delta. \tag{47}$$

Note that for all i, we have $|\lambda_i/\lambda_\ell - p_i/p_\ell| < \varepsilon/p_\ell$, and therefore, we can define $\varepsilon_i \in [-\varepsilon, \varepsilon]$ such that

$$\frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell} = \frac{\varepsilon_i}{p_\ell} \quad \Rightarrow \quad \lambda_i = \frac{\lambda_\ell}{p_\ell}(p_i + \varepsilon_i) = \frac{\lambda_\ell}{p_\ell} p_i + \frac{\lambda_\ell}{p_\ell} \varepsilon_i.$$

Therefore, using the fact that $\mu_i = \lambda_i - p_i \Delta$ for $i \le \ell$ we can rewrite the left hand side of (47):

$$\sum_{i=1}^{\ell} (\lambda_i - p_i \Delta)(a_i \bar{x} - b_i) = \sum_{i=1}^{\ell} [\frac{\lambda_\ell}{p_\ell} p_i + \frac{\lambda_\ell}{p_\ell} \varepsilon_i - p_i \Delta](a_i \bar{x} - b_i) = (\frac{\lambda_\ell}{p_\ell} - \Delta) + \frac{\lambda_\ell}{p_\ell} \sum_{i=1}^{\ell} \varepsilon_i (a_i \bar{x} - b_i) \tag{48}$$

where the second equality follows from (46). Therefore, we can rewrite (47) as:

$$\begin{aligned} \frac{\lambda_\ell}{p_\ell} \left(1 + \sum_{i=1}^{\ell} \varepsilon_i (a_i \bar{x} - b_i) \right) \quad &\le \quad - \sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + 2\Delta \\ &\le \quad \sum_{j=\ell+1}^{m} \mu_j b_j + 2\Delta \le \lambda_{\ell+1}(mB + 2D) \; = \; \frac{1}{2} \lambda_{\ell+1} M_1 \end{aligned} \tag{49}$$

where the second inequality in (49) follows the assumption that $a_j \ge 0$ and $\bar{x} \ge \mathbf{0}$, the third inequality follows from the fact that $\mu_i = \lambda_i \le \lambda_{\ell+1}$ for $i = \ell+1, \ldots, m$ by (42) and that $b_j \le B$ by Definition 4.8. The last equality simply follows from the definition of M_1.

We will obtain a lower bound on the first term in (49). As $a_i \bar{x} \ge 0$, $b_i \ge 0$, and $\varepsilon_i \in [-\varepsilon, \varepsilon]$, we have

$$\sum_{i=1}^{\ell} \varepsilon_i \left(a_i \bar{x} - b_i \right) = \sum_{i=1}^{\ell} \varepsilon_i a_i \bar{x} - \sum_{i=1}^{\ell} \varepsilon_i b_i \ge -\varepsilon \sum_{i=1}^{\ell} (a_i \bar{x} + b_i). \tag{50}$$

As $p_i \geq p_\ell$ for $i \in \{1, \dots, \ell\}$ and $b_i \leq B$, we have

$$\sum_{i=1}^{\ell} a_i \bar{x} \;\leq\; \sum_{i=1}^{\ell} \frac{p_i}{p_\ell} a_i \bar{x} \;=\; \frac{1}{p_\ell}\left(1 + \sum_{i=1}^{\ell} p_i b_i\right) \;\leq\; \frac{1}{p_\ell} + B \sum_{i=1}^{\ell} \frac{p_i}{p_\ell} \tag{51}$$

where the equality above follows from (46). Moreover,

$$\sum_{i=1}^{\ell} \frac{p_i}{p_\ell} \;\leq\; 1 + \sum_{i=1}^{\ell-1} \left(\frac{\lambda_i}{\lambda_\ell} + \frac{\varepsilon}{p_\ell}\right) \;=\; 1 + (\ell - 1)\frac{\varepsilon}{p_\ell} + \sum_{i=1}^{\ell-1} \frac{\lambda_i}{\lambda_\ell} \;\leq\; 1 + (\ell - 1)\frac{\varepsilon}{p_\ell} + \sum_{i=1}^{\ell-1} \prod_{j=i}^{\ell-1} M_j \tag{52}$$

where the first inequality follows from $p_i/p_\ell \leq \lambda_i/\lambda_\ell + \varepsilon/p_\ell$ for $i \leq \ell - 1$ by (41) and the second inequality follows from the fact that $\lambda_i/\lambda_\ell = \prod_{j=i}^{\ell-1} \lambda_j/\lambda_{j+1}$ and that $\lambda_j/\lambda_{j+1} \leq M_j$ for $j \leq \ell - 1$. Putting (51), (52) and $\sum_{i=1}^{\ell} b_i \leq mB$ together, we obtain the following inequality:

$$\sum_{i=1}^{\ell} (a_i \bar{x} + b_i) \leq B \left(m + \frac{1}{B p_\ell} + 1 + (\ell - 1)\frac{\varepsilon}{p_\ell} + \sum_{i=1}^{\ell-1} \prod_{j=i}^{\ell-1} M_j \right).$$

The term $\sum_{i=1}^{\ell-1} \prod_{j=i}^{\ell-1} M_j$ can be bounded above by $(\ell - 1) \prod_{j=1}^{\ell-1} M_j$. Moreover, it is not difficult to see that

$$m + \frac{1}{B p_\ell} + 1 + (\ell - 1)\frac{\varepsilon}{p_\ell} \leq \prod_{j=1}^{\ell-1} M_j.$$

Therefore,

$$\sum_{i=1}^{\ell} (a_i \bar{x} + b_i) \leq B m \prod_{j=1}^{\ell-1} M_j$$

It follows from (36) and (41) that $Bm \prod_{j=1}^{\ell-1} M_j = \frac{1}{2\varepsilon}$, implying in turn that

$$-\varepsilon \sum_{i=1}^{\ell} (a_i \bar{x} + b_i) \geq -\frac{1}{2}.$$

By (50), it follows that $\sum_{i=1}^{\ell} \varepsilon_i (a_i \bar{x} - b_i) \geq -1/2$. Then the left hand side of (49) is lower bounded by $\lambda_\ell / 2p_\ell$, so we obtain $\lambda_\ell \leq p_\ell \lambda_{\ell+1} M_1$ from (49), implying in turn that $M_\ell < p_\ell M_1$ as we assumed that $\lambda_\ell > M_\ell \lambda_{\ell+1}$ (40). However, this contradicts the first inequality in (43). $\square$

Claim 4. *$\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$.*

Proof of Claim. We will first show that

$$\mu b \leq \lceil \mu b \rceil_{S,\mu A} \leq \mu b + \Delta \tag{53}$$

holds. Set $(\alpha, \beta) = (\mu A, \mu b)$. By Claim 2, we have that $\beta = \min\{\alpha x : x \in P^{\uparrow}\}$. We have $\beta \geq \min\{\alpha x : x \in S\}$ because $P^{\uparrow} \subseteq \mathbb{R}^n_+$ and every extreme point of $P^{\uparrow}$ is contained in $\operatorname{conv}(S)$. If $\beta = \min\{\alpha x : x \in S\}$, then $\beta = \lceil \beta \rceil_{S,\alpha}$. Thus we may assume that $\beta > \min\{\alpha x : x \in S\}$, so there exists $z' \in S$ such that $\beta > \alpha z'$.

Remember that by (37), $\Delta = \min\{(\lambda A)_j : j \in \operatorname{supp}(\lambda A, I_2)\}$, and let j be such that $(\lambda A)_j = \Delta$. As $\operatorname{supp}(\lambda A, I_2) = \operatorname{supp}(\mu A, I_2)$, we have $\alpha_j > 0$ and $\kappa = (\beta - \alpha z')/\alpha_j > 0$. Therefore, $z'' = z' + \lceil \kappa \rceil e^j \in S$. Observe that

$$\beta \;=\; \alpha z' + (\beta - \alpha z') \;=\; \alpha\left(z' + \kappa e^j\right) \;\leq\; \alpha\left(z' + \lceil \kappa \rceil e^j\right) \;=\; \beta + \alpha_j(\lceil \kappa \rceil - \kappa) \;\leq\; \beta + \alpha_j.$$

As $\lambda \geq \mu$, we have $\Delta \geq \alpha_j$ implying $\beta \leq \alpha z'' \leq \beta + \Delta$ and (53) hold as desired.

Let $z \in S$ be such that $\mu A z = \lceil \mu b \rceil_{S,\mu A}$. As z is integral, Claim 3 implies that

$$\sum_{i=1}^{\ell} p_i a_i z < 1 + \sum_{i=1}^{\ell} p_i b_i \quad \Rightarrow \quad \sum_{i=1}^{\ell} p_i a_i z = \sum_{i=1}^{\ell} p_i b_i - f$$

for some integer $f \in [0, \sum_{i=1}^{\ell} p_i b_i]$. Consider $z + f e^j \in S$ and observe that

$$\lambda A \left(z + f e^j\right) = \lambda A z + f\,(\lambda A)_j = \left(\mu A + \Delta \sum_{i=1}^{\ell} p_i a_i\right) z + \Delta \sum_{i=1}^{\ell} p_i (b_i - a_i z) = \lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i.$$

Since $\lceil \mu b \rceil_{S,\mu A} \geq \mu b$, we must have

$$\lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \geq \mu b + \Delta \sum_{i=1}^{\ell} p_i b_i = \lambda b.$$

Then $\lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \geq \lceil \lambda b \rceil_{S,\lambda A}$. So, the inequality $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$ is dominated by $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$, as the former is implied by the latter and a nonnegative combination of the inequalities in $Ax \geq b$, as required. □

□

We remark that the proof of Claim 4 is the only part where we use the assumption that every extreme point of $P^{\uparrow}$ is contained in $\mathrm{conv}(S)$.

Theorem 4.11. *Let $P^{\uparrow}$ and S be defined as in (29) and (30), respectively. Let*

$$\Pi = \left\{(\alpha, \beta) \in \Pi_{P^{\uparrow}} : \quad \beta/\alpha_j \leq M^* \text{ for all } j \in \mathit{supp}(\alpha, I_2)\right\}$$

where $M^* = mBM$. Then ${P^{\uparrow}}_S = {P^{\uparrow}}_{S,\Pi}$, and in particular, ${P^{\uparrow}}_S$ is a rational polyhedron.

Proof. By Remark 2.2, we have ${P^{\uparrow}}_S \subseteq {P^{\uparrow}}_{S,\Pi}$ as $\Pi \subseteq \Pi_{P^{\uparrow}}$. To show that they are equal, we will argue that for each $(\alpha, \beta) \in \Pi_{P^{\uparrow}}$, there is an $(\alpha', \beta') \in \Pi$ such that the S-CG cut derived from (α', β') dominates the S-CG cut derived from (α, β) on $P^{\uparrow}$.

Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be such that $(\lambda A, \lambda b) \in \Pi_{P^{\uparrow}}$, and set $(\alpha, \beta) = (\lambda A, \lambda b)$. If $\beta/\alpha_j \leq M^*$ for all $j \in \mathrm{supp}(\alpha, I_2)$, then $(\alpha, \beta) \in \Pi$ as desired. Otherwise, consider an arbitrary $j \in \mathrm{supp}(\alpha, I_2)$ such that $\beta/\alpha_j > M^*$. Let t stand for $t(\lambda, A)$ and note that

$$M^* \;<\; \frac{\beta}{\alpha_j} \;=\; \frac{\sum_{i=1}^{m} \lambda_i b_i}{\sum_{i=1}^{m} \lambda_i a_{ij}} \;\leq\; \frac{\lambda_1 \sum_{i=1}^{m} b_i}{\lambda_t \sum_{i=1}^{t} a_{ij}} \;=\; r(\lambda, A) \frac{\sum_{i=1}^{m} b_i}{\sum_{i=1}^{t} a_{ij}} \;\leq\; mB\, r(\lambda, A)$$

where the last inequality follows from the fact that $b_i \leq B$ for all $i \in \{1, \ldots, m\}$, and the fact that $\sum_{i=1}^{t} a_{ij} \geq 1$ as $\bigcup_{i=1}^{t} \mathrm{supp}\,(a_i, I_2) = \mathrm{supp}\,(\lambda A, I_2)$.

As $M^* = mBM$, we have $r(\lambda, A) > M$. Then, by Lemma 4.10, there exists a $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that $\|\mu\|_1 \leq \|\lambda\|_1 - 1$ and the S-CG cut generated by μ dominates the S-CG cut generated by λ for $P^{\uparrow}$. If necessary, we can repeat this argument and construct a sequence of vectors $\mu^1, \mu^2, \ldots,$ with decreasing norms such that each vector in the sequence defines an S-CG cut that dominates the previous one. Therefore, after at most $\|\lambda\|_1$ iterations, we must obtain a vector $\hat{\mu}$ such that $r(\hat{\mu}, A) \leq M$ and $(\hat{\mu}A, \hat{\mu}b) \in \Pi$. As $(\hat{\mu}A, \hat{\mu}b) \in \Pi$ and the S-CG cut generated by $\hat{\mu}$ dominates the S-CG cut generated by λ for $P^{\uparrow}$, we conclude that ${P^{\uparrow}}_S = {P^{\uparrow}}_{S,\Pi}$. Moreover, as ${P^{\uparrow}}_{S,\Pi}$ is a rational polyhedron by Lemma 4.6, it follows that ${P^{\uparrow}}_S$ is a rational polyhedron, as desired. □

4.2 Packing polyhedra

In this section, we consider packing polyhedra of the form

$$P^{\downarrow} = \Big\{x \in \mathbb{R}^n : Ax \leq b\Big\}, \tag{54}$$

where $A \in \mathbb{Z}_+^{m \times n}$ and $b \in \mathbb{Z}_+^m$. We will prove that ${P^{\downarrow}}_S$ is a rational polyhedron where S is of the form (30). If ${P^{\downarrow}}_S = \emptyset$, then ${P^{\downarrow}}_S$ is trivially a rational polyhedron. So, for the rest of this section, we will assume that ${P^{\downarrow}}_S$ is nonempty. Unlike the covering polyhedra considered in Section 4.1, $P^{\downarrow}$ is not necessarily pointed. Moreover, we do not assume that every extreme point of $P^{\downarrow}$ is contained in $\operatorname{conv}(S)$. We may assume that $m \geq 1$. If otherwise, $P^{\downarrow} = \mathbb{R}^n$, and therefore, ${P^{\downarrow}}_S = \mathbb{R}^n$ is trivially a rational polyhedron. Notice that every valid inequality for $P^{\downarrow}$ is of the form $\alpha x \leq \beta$ where α and β are nonnegative. Recall that $\Pi_{P^{\downarrow}}$ is defined as

$$\Pi_{P^{\downarrow}} = \big\{(\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{R} \ : \ (\alpha, \beta) = (\lambda A, \lambda b) \text{ for some } \lambda \in \mathbb{R}_+^m, \ \beta = \max\{\alpha x : x \in P^{\downarrow}\}\big\}. \tag{55}$$

As before, we use $I_2 = \{n_1 + 1, \ldots, n\}$ for convenience. We first consider cuts with bounded intercepts.

Lemma 4.12. *Let M^* be a positive integer, and let*

$$\Pi = \{(\alpha, \beta) \in \Pi_{P^{\downarrow}} : \ \beta/\alpha_j \leq M^* \ \textit{for all } j \in \textit{supp}(\alpha, I_2)\}. \tag{56}$$

Then ${P^{\downarrow}}_{S,\Pi}$ is a rational polyhedron.

Proof. The proof is very similar to that of Lemma 4.6. Let $S^* = T \times \{0, \ldots, M^*\}^{n_2}$. Then S^* is a finite subset of S, and by Remark 2.1, ${P^{\downarrow}}_{S^*,\Pi} \subseteq {P^{\downarrow}}_{S,\Pi}$. We first show that ${P^{\downarrow}}_{S^*,\Pi} = {P^{\downarrow}}_{S,\Pi}$.

Let $(\alpha, \beta) \in \Pi$. Then $\alpha x \leq \beta$ is valid for $P^{\downarrow}$, $\alpha \geq \mathbf{0}$, $\beta \geq 0$, and $0 \leq \beta/\alpha_j \leq M^*$ for every $j \in I_2$ such that $\alpha_j > 0$. Notice that there exists $z^* = (z^1, z^2) \in S = T \times \mathbb{Z}_+^{n_2}$ such that

$$\alpha z^* = \lfloor \beta \rfloor_{S,\alpha} = \max\{\alpha x : \ x \in S, \ \alpha x \leq \beta\}, \tag{57}$$

for otherwise, ${P^{\downarrow}}_{S,\Pi}$ is empty. Let $j \in I_2$. If $\alpha_j > 0$, then $\beta \leq M^* \alpha_j$, implying in turn that $z_j^* \leq M^*$. If $\alpha_j = 0$, then we may assume that $z_j^* = 0$. Therefore, we may assume that $z^* \in S^*$, so it follows that $\lfloor \beta \rfloor_{S^*,\alpha} = \lfloor \beta \rfloor_{S,\alpha}$. This implies that ${P^{\downarrow}}_{S^*,\Pi} = {P^{\downarrow}}_{S,\Pi}$, as desired.

To complete the proof, we next show that ${P^{\downarrow}}_{S^*,\Pi}$ is a rational polyhedron. We first write $\Pi = \cup_{I \subseteq I_2} \Pi(I)$ where $\Pi(I) = \{(\alpha, \beta) \in \Pi : \ \operatorname{supp}(\alpha, I_2) = I\}$. Therefore, $\Pi(I) = \Pi_{P^{\downarrow}} \cap H(I)$ where

$$H(I) = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : \ M^* \alpha_j \geq \beta \ \text{ and } \ \alpha_j \geq 1 \ \forall j \in I, \quad \alpha_j = 0 \ \forall j \in N_2 \setminus I\}.$$

As $H(I) \subseteq \operatorname{rec}(H(I))$, it follows from Theorem 2.7 that ${P^{\downarrow}}_{S^*,\Pi(I)}$ is a rational polyhedron. So, as ${P^{\downarrow}}_{S^*,\Pi} = \cap_{I \subseteq I_2} {P^{\downarrow}}_{S^*,\Pi(I)}$, ${P^{\downarrow}}_{S^*,\Pi}$ is a rational polyhedron, implying in turn that ${P^{\downarrow}}_{S,\Pi}$ is a rational polyhedron. □

The proof of Lemma 4.13 is basically the same as that of Lemma 4.10. Given $\lambda \in \mathbb{R}^m_+$, as in Definition 4.7, we can define the tilting ratio of λ with respect to A, and we denote it by $r(\lambda, A)$. Let B, D, M_i for $i \in \{1, \ldots, m-1\}$, and M be defined as in Definition 4.8.

Lemma 4.13. *Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be such that $(\lambda A, \lambda b) \in \Pi_{P^\downarrow}$. If $r(\lambda, A) > M$, then there exists $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ that satisfies the following: (i) $\|\mu\|_1 \leq \|\lambda\|_1 - 1$, (ii) $(\mu A, \mu b) \in \Pi_{P^\downarrow}$, and (iii) $\mu A x \leq \lfloor \mu b \rfloor_{S,\mu A}$ dominates $\lambda A x \leq \lfloor \lambda b \rfloor_{S,\lambda A}$.*

Proof. After relabeling the rows of $Ax \leq b$, we may assume that $\lambda_1 \geq \cdots \geq \lambda_m$. Let $t(\lambda, A)$ be defined as in Definition 4.7, and let t stand for $t(\lambda, A)$. If $t = 1$, we have $r(\lambda, A) = 1 \leq M$, a contradiction to our assumption. So, $t \geq 2$. Let Δ and k be defined as in (37) and (38). As $\operatorname{supp}(\lambda A, I_2) \setminus \bigcup_{i=1}^{t-1} \operatorname{supp}(a_i, I_2)$ is not empty, k is a well-defined index. Moreover, as $r(\lambda, A) > M_1 \times \cdots \times M_{m-1}$, there exists some $\ell \in \{1, \ldots, t-1\}$ such that (40) is satisfied. By the Simultaneous Diophantine Approximation theorem (with $k = \ell - 1$ and $r_i = \lambda_i / \lambda_\ell$ for $i \in \{1, \ldots, k\}$), there exist positive integers $p_1, \ldots, p_\ell$ that satisfy (41).

As in the proof of Lemma 4.10, we now construct $\mu \in \mathbb{R}^m$ as follows:

$$\mu_i = \begin{cases} \lambda_i - p_i \Delta & \text{for } i = 1, \ldots, \ell, \\ \lambda_i & \text{otherwise} \end{cases} \tag{58}$$

When $\ell = 1$, we let $p_1 = 1$, and let μ be defined by (58). Using the same arguments as in the proof of Lemma 4.10, we can show that $\mu \geq \mathbf{0}$, $\operatorname{supp}(\mu) = \operatorname{supp}(\lambda)$ and $\mu b = \max\left\{\mu A x : x \in P^\downarrow\right\}$, and therefore, $(\mu A, \mu b) \in \Pi_{P^\downarrow}$. Consequently, we get that $\|\mu\|_1 \leq \|\lambda\|_1 - 1$ is satisfied.

We next define $Q = \{x \in \mathbb{R}^n_+ : \mu b - \Delta \leq \mu A x \leq \mu b\}$ and show that there is no point $x \in Q$ that satisfies

$$\sum_{i=1}^{\ell} p_i a_i x \geq 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{59}$$

Suppose for a contradiction that there exists $\tilde{x} \in Q$ satisfying (59). (Note that Q here is defined differently than the one defined in Claim 3 of Lemma 4.10.) Taking a convex combination of $\tilde{x}$ with the point $v \in Q$ defined in the proof of Lemma 4.10, we can construct $\bar{x} \in Q$ such that $\sum_{i=1}^{\ell} p_i a_i \bar{x} = 1 + \sum_{i=1}^{\ell} p_i b_i$. As $\bar{x} \in Q$, we have $\mu A \bar{x} \leq \mu b$, and this inequality can be rewritten as $\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) \leq -\sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j)$. As $\Delta > 0$, it follows that

$$\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) \leq -\sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + \Delta. \tag{60}$$

Note that inequality (60) is the same as (47). The same argument used for proving Claim 3 of Lemma 4.10 can be repeated, and we obtain the desired contradiction.

Finally, to show that $\lambda A x \leq \lfloor \lambda b \rfloor_{S,\lambda A}$ is implied by $\mu A x \leq \lfloor \mu b \rfloor_{S,\mu A}$ and the inequalities in $Ax \leq b$, we first show that

$$\mu b - \Delta \leq \lfloor \mu b \rfloor_{S,\mu A} \leq \mu b \tag{61}$$

holds. Let α, β denote $\mu A, \mu b$, respectively. There exists $z \in S$ such that $\alpha z = \lfloor \beta \rfloor_{S,\alpha}$. Recall that by (37), $\Delta = \min\{(\lambda A)_j : j \in \operatorname{supp}(\lambda A, I_2)\}$, and let j be such that $(\lambda A)_j = \Delta$. Note that $z + e^j \in S$ and that

$\alpha(z + e^j) = \alpha z + \alpha_j$. As $\alpha z = \lfloor \beta \rfloor_{S,\alpha}$, it follows that $\alpha(z + e^j) = \lfloor \beta \rfloor_{S,\alpha} + \alpha_j > \lfloor \beta \rfloor_{S,\alpha}$. That means $\alpha(z + e^j) > \beta$. So, we obtain $\lfloor \beta \rfloor_{S,\alpha} + \alpha_j > \beta$. Since $\lambda \geq \mu$, we have $\Delta \geq \alpha_j$, so it follows that $\lfloor \beta \rfloor_{S,\alpha} \geq \beta - \alpha_j \geq \beta - \Delta$, as required.

There exists $z \in S$ such that $\mu A z = \lfloor \mu b \rfloor_{S,\mu A}$, and (61) implies that $\mu b - \Delta \leq \mu A z \leq \mu b$. Since we have shown that there is no point $x \in Q$ satisfying (59), it follows that $\sum_{i=1}^{\ell} p_i a_i z = \sum_{i=1}^{\ell} p_i b_i - f$ for some integer $f \in \left[0, \sum_{i=1}^{\ell} p_i b_i\right]$, as z is integral. It can be observed that $\lambda A\left(z + f e^j\right) = \lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i$. Since $\lfloor \mu b \rfloor_{S,\mu A} \leq \mu b$, we must have $\lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \leq \mu b + \Delta \sum_{i=1}^{\ell} p_i b_i = \lambda b$. Then $\lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \leq \lfloor \lambda b \rfloor_{S,\lambda A}$. So, the inequality $\lambda A x \leq \lfloor \lambda b \rfloor_{S,\lambda A}$ is dominated by $\mu A x \leq \lfloor \mu b \rfloor_{S,\mu A}$, as the former is implied by the latter and a nonnegative combination of the inequalities in $Ax \leq b$, as required. $\square$

Theorem 4.14. *Let $P^{\downarrow}$ and S be defined as in (54) and (30), respectively. Let*

$$\Pi = \left\{(\alpha, \beta) \in \Pi_{P^{\downarrow}} : \quad \beta/\alpha_j \leq M^* \text{ for all } j \in \mathit{supp}(\alpha, I_2)\right\}$$

where $M^* = mBM$. Then $P^{\downarrow}{}_S = P^{\downarrow}{}_{S,\Pi}$, and in particular, $P^{\downarrow}{}_S$ is a rational polyhedron.

Proof. Recall that $P^{\downarrow}{}_S = P^{\downarrow}{}_{S,\Pi_{P^{\downarrow}}}$ by (6). As $\Pi \subseteq \Pi_{P^{\downarrow}}$, Remark 2.2 implies that $P^{\downarrow}{}_{S,\Pi_{P^{\downarrow}}} \subseteq P^{\downarrow}{}_{S,\Pi}$. To show that $P^{\downarrow}{}_{S,\Pi_{P^{\downarrow}}} = P^{\downarrow}{}_{S,\Pi}$, we will argue that for each $(\alpha, \beta) \in \Pi_{P^{\downarrow}}$, there is an $(\alpha', \beta') \in \Pi$ such that the S-CG cut derived from (α', β') dominates the S-CG cut derived from (α, β) on $P^{\downarrow}$.

Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be such that $(\lambda A, \lambda b) \in \Pi_{P^{\downarrow}}$ and let $\alpha = \lambda A$, and $\beta = \lambda b$. If $\beta/\alpha_j \leq M^*$ for all $j \in \text{supp}(\alpha, I_2)$, then $(\alpha, \beta) \in \Pi$ as desired. Otherwise, consider an arbitrary $j \in \text{supp}(\alpha, I_2)$ such that $\beta/\alpha_j > M^*$. As argued in the proof of Theorem 4.11, it can be shown that $M^* < mBr(\lambda, A)$. As $M^* = mBM$, we have $r(\lambda, A) > M$. So, by Lemma 4.13, there exists a $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that (i) $\|\mu\|_1 \leq \|\lambda\|_1 - 1$, (ii) $(\mu A, \mu b) \in \Pi_{P^{\downarrow}}$, and (iii) $\mu A x \leq \lfloor \mu b \rfloor_{S,\mu A}$ dominates $\lambda A x \leq \lfloor \lambda b \rfloor_{S,\lambda A}$. As we argued in the proof of Theorem 4.11, after repeating this process for at most $\|\lambda\|_1$ iterations, we may assume that $r(\mu, A) \leq M$ and $(\mu A, \mu b) \in \Pi$. Since the S-CG cut generated by μ dominates the S-CG cut generated by λ for $P^{\downarrow}$, it follows that $P^{\downarrow}{}_S = P^{\downarrow}{}_{S,\Pi}$. Since $P^{\downarrow}{}_{S,\Pi}$ is a rational polyhedron by Lemma 4.12, it follows that $P^{\downarrow}{}_S$ is a rational polyhedron, as required. $\square$

4.3 The main result

We are now ready to prove the main result of this paper.

Theorem 1.1. *Let $T \subseteq \mathbb{Z}^{n_1}$ be finite, $\ell \in \mathbb{Z}^{n_3}, u \in \mathbb{Z}^{n_4}$, and let S be*

$$S = \left\{(x, y, w^1, w^2) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3} \times \mathbb{Z}^{n_4} : \ x \in T, \ w^1 \geq \ell, \ w^2 \leq u \right\}.$$

If $P \subseteq \text{conv}(S)$ is a rational polyhedron, then the S-CG closure of P is a rational polyhedron.

Proof. By Lemma 4.2 and Proposition 4.5, it is sufficient to show that for every $S = T \times \mathbb{Z}^{n_2}_+$ where $T \subseteq \mathbb{Z}^{n_1}_+$ is finite and for every rational polyhedron $Q \subseteq \text{conv}(S)$, Q_{S,Π^+_Q} and Q_{S,Π^-_Q} are rational polyhedra where Π^+_Q and

Π_Q^- are defined as in (27)–(28). To this end, take a set $S = T \times \mathbb{Z}_+^{n_2}$ for some finite $T \subseteq \mathbb{Z}_+^{n_1}$ and a rational polyhedron $Q \subseteq \operatorname{conv}(S)$. We abbreviate Π_Q^+ and Π_Q^- by Π^+ and Π^-, respectively. Let $Q^\uparrow$ and $Q^\downarrow$ be defined as follows:

$$Q^\uparrow = Q + \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \quad \text{and} \quad Q^\downarrow = Q - \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2}.$$

Let $n = n_1 + n_2$. Since $Q \subseteq \operatorname{conv}(S)$ and $\operatorname{conv}(S) \subseteq \mathbb{R}_+^n$, there exist some matrices A, b, C, d of appropriate dimension whose entries are nonnegative integers such that $Q^\uparrow = \{x \in \mathbb{R}^n : Ax \geq b\}$ and $Q^\downarrow = \{x \in \mathbb{R}^n : Cx \leq d\}$. Moreover, $Q^\uparrow$ is pointed and its extreme points of $Q^\uparrow$ are contained in $\operatorname{conv}(S)$.

We first claim that ${Q^\uparrow}_S \cap Q = Q_{S,\Pi^-}$. We will show that $\Pi^- = \Gamma$ where

$$\Gamma = \big\{(-\alpha, -\beta) \in \mathbb{Z}^n \times \mathbb{R} \;:\; (\alpha, \beta) = (\lambda A, \lambda b) \text{ for some } \lambda \in \mathbb{R}_+^m,\ \beta = \min\{\alpha x : x \in Q^\uparrow\}\big\}.$$

Let $(-\alpha, -\beta) \in \Gamma$. Then $\alpha x \geq \beta$ is a valid inequality for $Q^\uparrow$. Since the entries of A are nonnegative, it follows that $\alpha \geq \mathbf{0}$, implying in turn that $\min\{\alpha x : x \in Q^\uparrow\} = \min\{\alpha x : x \in Q\}$. Then $-\beta = \max\{-\alpha x : x \in Q\}$, so $(-\alpha, -\beta) \in \Pi^-$. Conversely, take $(-\alpha, -\beta) \in \Pi^-$. Then $-\beta = \max\{-\alpha x : x \in Q\}$, so $\beta = \min\{\alpha x : x \in Q\}$. As $\alpha \geq \mathbf{0}$, it follows that $\min\{\alpha x : x \in Q\} = \min\{\alpha x : x \in Q^\uparrow\}$, so $(-\alpha, -\beta) \in \Gamma$. Therefore, as $\Pi^- = \Gamma$, it follows that $Q_{S,\Pi^-} = \{x \in Q : \ \alpha x \geq \lceil \beta \rceil_{S,\alpha} \ \forall (-\alpha, -\beta) \in \Gamma\} = Q \cap {Q^\uparrow}_S$.

Similarly, we claim that ${Q^\downarrow}_S \cap Q = Q_{S,\Pi^+}$. We will show that $\Pi_{Q^\downarrow} = \Pi^+$. Let $(\alpha, \beta) \in \Pi_{Q^\downarrow}$. Then $\alpha x \leq \beta$ is a valid inequality for $Q^\downarrow$. Since the entries of C are nonnegative, it follows that $\alpha \geq \mathbf{0}$, implying in turn that $\max\{\alpha x : x \in Q^\downarrow\} = \max\{\alpha x : x \in Q\}$. So, it follows that $(\alpha, \beta) \in \Pi^+$. Conversely, take $(\alpha, \beta) \in \Pi^+$. Then, as $\alpha \geq \mathbf{0}$ and $\beta = \max\{\alpha x : x \in Q\}$, we have $\beta = \max\{\alpha x : x \in Q^\downarrow\}$. In turn, we get $(\alpha, \beta) \in \Pi_{Q^\downarrow}$. Therefore, as $\Pi_{Q^\downarrow} = \Pi^+$, it follows that $Q_{S,\Pi^+} = \big\{x \in Q : \ \alpha x \leq \lfloor \beta \rfloor_{S,\alpha} \ \forall (\alpha, \beta) \in \Pi_{Q^\downarrow}\big\} = Q \cap {Q^\downarrow}_S$.

By Theorems 4.11 and 4.14, both ${Q^\uparrow}_S$ and ${Q^\downarrow}_S$ are rational polyhedra. In turn, both Q_{S,Π_Q^-} and Q_{S,Π_Q^+} are rational polyhedra. Therefore, P_S is a rational polyhedron, as required. $\square$

5 Concluding remarks

In this paper, we proved that the closure of a rational polyhedron obtained after applying S-Chvátal-Gomory inequalities is also a rational polyhedron when S is the set of integer points that satisfy arbitrary bound constraints. Note that in our setting, classical Chvátal-Gomory inequalities can be seen as S-Chvátal-Gomory inequalities where S contains all integer points.

Our result generalizes an earlier result of Dunkel and Schulz who studied the same question when S is the set of all vertices of the $\{0,1\}$ cube. Their proof is already more difficult than the proof of the same result for the classical CG inequalities. Our proof builds on proof techniques for the cases $S = \mathbb{Z}^n$ and $S = \{0,1\}^n$, but is significantly more difficult. One source of difficulty is the fact that not every facet of the S-CG closure is defined by an S-CG cut but instead some facet-defining inequality could be the limiting inequality obtained from an infinite sequence of S-CG cuts, as seen in Example 2.4 and Figure 2. In contrast, all facets of many other closures such as the Chvátal closure [20], the split closure [5], the t-branch split closure [7], and the lattice closure [8] are in fact defined by the cuts from the corresponding family. Related to this fact, there is no finite set of S-CG cuts that imply the rest.

One question we have not answered is whether or not the S-CG closure of polyhedra is still polyhedral for more general S. As we discussed in Section 1, S-CG cuts can also be considered as a special case of *wide split cuts* [2]. In the same way that S-CG cuts generalize CG cuts, one can generalize split cuts to define S*-split cuts* and study the associated closure. A natural question then is whether or not such closures of rational polyhedra are polyhedral.

It is known that the separation problem for CG cuts is NP-hard [12], although it is easy to certify the validity of a CG cut. The separation problem for S-CG cuts for a given polyhedron $P \subseteq \mathbb{R}^n$ is clearly also NP-hard, as S can be chosen to be $\mathbb{Z}^n$. Furthermore, even establishing the validity of an S-CG cut is NP-hard for certain choices of S. For example, when $S = \mathbb{Z}^n_+$, establishing validity is equivalent to solving an unbounded knapsack problem. In [2], computational methods for separating wide split cuts were studied. A natural question is whether one can devise effective methods to separate S-CG cuts for different choices of S.

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