On the Rational Polytopes with Chvátal Rank 1

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December 2016, revised May 2018

Abstract

We study the following problem: given a rational polytope with Chvátal rank 1, does it contain an integer point? Boyd and Pulleyblank observed that this problem is in the complexity class NP $\cap$ co-NP, an indication that it is probably not NP-complete. It is open whether there is a polynomial time algorithm to solve the problem, and we give several special classes where this is indeed the case. We show that any compact convex set whose Chvátal closure is empty has an integer width of at most n, and we give an example showing that this bound is tight within an additive constant of 1. This determines the time complexity of a Lenstra-type algorithm. However, the promise that a polytope has Chvátal rank 1 seems hard to verify. We prove that deciding emptiness of the Chvátal closure of a rational polytope given by its linear description is NP-complete, even when the polytope is contained in the unit hypercube or is a rational simplex and it does not contain any integer point.

1 Introduction

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, and let P_I denote its integer hull, namely $P_I := \mathrm{conv}(P \cap \mathbb{Z}^n)$, the convex hull of the integer points in P. If an inequality $cx \le d$ with $c \in \mathbb{Z}^n$ is valid for P, then $cx \le \lfloor d \rfloor$ is valid for all the integer solutions contained in P, and thus for P_I. We call $cx \le \lfloor d \rfloor$ the *Chvátal inequality* of P obtained from $cx \le d$. Chvátal [6] introduced the following beautiful notion of closure, which is obtained by applying all possible Chvátal inequalities.

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \le \lfloor \max\{cx : \ x \in P\} \rfloor\}$$

It follows from the definition that $P_I \subseteq P' \subseteq P$, and we call P' the *Chvátal closure* of P. Although P' is defined as the intersection of infinitely many half-spaces, P' turns out to be a rational polytope when P is a rational polytope [6]. Schrijver [35] later extended this result to rational polyhedra. We can recursively apply the operation of taking the Chvátal closure. The set obtained after k recursive applications of the closure operation to a polyhedron P is called the kth Chvátal closure of P. We say that a Chvátal inequality of the $(k-1)$th Chvátal closure of P is a rank-k Chvátal inequality of P. In fact, there exists a finite integer k such that the kth Chvátal closure of a rational polyhedron P coincides with the integer hull of P [6, 35], and the Chvátal rank of P is defined as

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the smallest such k. In this paper, we study rational polyhedra that have Chvátal rank 1, meaning that the integer hull can be obtained by applying all the (rank-1) Chvátal inequalities.

The problem of deciding whether a rational polyhedron given by its linear description contains an integer point is NP-complete [3]. What if we assume that the Chvátal rank of the input polyhedron is 1? In fact, the main motivation of this paper is the following question: for a rational polyhedron with Chvátal rank 1 given by its linear description, can the integer feasibility problem be solved in polynomial time? Boyd and Pulleyblank [4] observed that this problem belongs to the complexity class NP $\cap$ co-NP. This is an indication that this problem is not NP-complete (unless NP = co-NP). It is open whether there is a polynomial time algorithm to solve the integer feasibility problem over a rational polyhedron with Chvátal rank 1 given by its linear description. There are some examples of problems which we can indeed solve in polynomial time. The fractional matching polytope of a graph, defined by the degree constraints and the nonnegativity, has Chvátal rank 1, and we know that the blossom algorithm by Edmonds [18] solves the matching problem in polynomial time. Another example is the stable set problem of t-perfect graphs. As the matching problem, the fractional stable set polytope of a t-perfect graph, defined by the edge constraints and the nonnegativity, has Chvátal rank 1 and there is a polynomial time algorithm to find a maximum weight stable set of a t-perfect graph in polynomial time [20, 22]. In Section 2, we motivate the question by considering other special cases.

For the general case, we consider a Lenstra-type algorithm for the integer feasibility problem. Lenstra's algorithm [31] relies on the fact that any lattice-free compact convex set has bounded integer width. In Section 3, we prove that any compact convex set in $\mathbb{R}^n$ whose Chvátal closure is empty has an integer width of at most n. We extend this result to unbounded closed convex sets that can be represented as the Minkowski sum of a compact convex set and a convex cone, under a rationality assumption on the cone. We also give an example showing that this bound is tight within an additive constant of 1. We remark that the upper bound on the width implies the existence of a deterministic $2^{O(n)}n^n$ Lenstra-type algorithm for the integer feasibility problem over a given rational polyhedron with Chvátal rank 1. On the other hand, the lower bound indicates that we cannot improve this time complexity if we use a Lenstra-type procedure.

Although the integer width of a closed convex set whose Chvátal closure is empty is well-understood, it seems very difficult to cleverly use the Chvátal rank 1 condition imposed on the input polytope. In Section 4, we prove that deciding whether the Chvátal closure of a rational polytope given by its linear description is NP-hard, even when its integer hull is empty and the input polytope is contained in the unit hypercube or is a simplex, and this resolves an open question raised by Cornuéjols and Li [10, 11]. This hardness result has some nice corollaries. In particular, it implies that it is NP-hard to optimize over the Chvátal closure of a rational polytope contained in the unit hypercube given by its linear description, which answers a question of Letchford, Pokutta and Schulz [32].

2 Integer programming over polytopes with Chvátal rank 1

In this section, we introduce the problem of deciding whether a rational polyhedron P contains an integer point under the promise that P has Chvátal rank 1, which is the main motivation of this paper. This promise on the input P very likely modifies the computational complexity of the integer feasibility problem. A result of Boyd and Pulleyblank ([4], Theorem 5.4) implies the following theorem.

Theorem 1 ([4]). *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polyhedron with Chvátal rank 1. The problem of deciding whether P contains an integer point is in the complexity class NP $\cap$ co-NP.*

The problems in NP $\cap$ co-NP are probably not NP-complete (since otherwise NP = co-NP), so we have the following question:

Open question 1. Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polyhedron with Chvátal rank 1. Can we decide whether P contains an integer point in time polynomial in the encoding size of P?

However, it does not seem straightforward to use the Chvátal rank 1 condition. In fact, it is NP-hard to certify that the Chvátal rank of a rational polytope given by its linear description is 1, even under some assumptions on the input polytope. We show this in Section 4. We also note that the Chvátal rank of a polyhedron is not directly related to its geometry. In particular, the Chvátal rank is not invariant under translation. The following example illustrates that the Chvátal rank of a polyhedron may vary significantly under translation.

Example 2. Let $Q_1 := \{x \in [0,1]^n : \sum_{j=1}^n v_j(1-x_j) + (1-v_j)x_j \ge \frac{1}{2}\ \forall v \in \{0,1\}^n\}$. Notice that Q_1 contains no integer point. Chvátal, Cook, and Hartmann ([7], Lemma7.2) proved that the Chvátal rank of Q_1 is exactly n. Now, let us translate Q_1 so that its center point is at the origin, and we denote by Q_2 the resulting polytope. Since $Q_2 \subseteq [-\frac{1}{2}, \frac{1}{2}]^n$, the only integer point contained in Q_2 is the origin. We can obtain both $x_i \ge 0$ and $x_i \le 0$ as Chvátal inequalities for Q_2 for all $i \in [n]$. Hence, the Chvátal rank of Q_2 is exactly 1.

The difficulty in understanding the Chvátal rank 1 condition is an indication that Open question 1 might not be easy to answer in general. Next, we present several special cases of the question, which seem easier to tackle and still remain interesting, for motivation.

2.1 Satisfiability problem with Chvátal rank 1

The satisfiability problem is NP-complete (see [21]), and it can be formulated as a binary integer program. Given a formula in conjunctive normal form with m clauses that consist of literals $x_1, \cdots, x_n$ and their negations, the problem of finding a satisfying assignment $x \in \{0,1\}^n$ can be equivalently formulated as the 0,1 feasibility problem over a polytope. Given a clause $\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} \neg x_j$ for some disjoint subsets I, J of $[n]$, we make a linear inequality $\sum_{i \in I} x_i + \sum_{j \in J}(1-x_j) \ge 1$. Notice that an assignment $x \in \{0,1\}^n$ satisfies all the clauses if and only if it satisfies all the corresponding inequalities. Inequalities of the form

$$\sum_{i \in I} x_i + \sum_{j \in J}(1-x_j) \ge 1 \qquad I, J \subseteq [n],\ I \cap J = \emptyset$$

are called *generalized set covering inequalities.* Then, the satisfiability problem of a given formula is equivalent to the integer feasibility problem of a polytope defined by generalized set covering inequalities and the bounds $0 \le x \le 1$. We call such a polytope a *SAT polytope.*

Open question 2. Given a SAT polytope P whose Chvátal rank is 1, can we decide in polynomial time whether P contains an integer point?

The k-satisfiability problem is a variant of the satisfiability problem where each clause in a given formula has at most k literals. It remains NP-complete for $k \ge 3$ (see [21]). On the other hand, there is a simple polynomial algorithm for the case of $k = 2$. We consider a formula whose SAT polytope has Chvátal rank 1 and each of whose clauses contains at least 3 literals. We remark that such a formula always has a satisfying assignment.

Remark 3. *Let P be a SAT polytope such that each generalized set covering inequality in its description has at least 3 variables. If P has Chvátal rank 1, then P always contains an integer point.*

Proof. Observe that setting any variable to 0 or 1, and all other $n-1$ variables to 1/2 satisfies all the constraints of P (because every generalized set covering inequality involves at least three variables). In other words, the middle point of each facet of the hypercube $[0,1]^n$ is contained in P. A result of Chvátal, Cook and Hartmann ([7], Lemma 7.2) implies that the Chvátal closure of P contains the middle point $(\frac{1}{2}, \cdots, \frac{1}{2})$ of the hypercube, so the Chvátal closure of P is always nonempty. Because the Chvátal rank of P is 1, P contains an integer point. $\square$

A natural question is whether one can actually find an integer point in polynomial time, under the assumptions of Remark 3. This is open. The following example provides a positive answer when each generalized set covering inequality contains n variables.

Example 4. Take an integer $n \geq 3$. Given $\overline{S} \subseteq \{0,1\}^n$, we construct a SAT polytope as follows:

$$P = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} ((1-v_i)x_i + v_i(1-x_i)) \geq 1, \ \forall v \in \overline{S} \right\}$$

Notice that $P \cap \{0,1\}^n = \{0,1\}^n \setminus \overline{S}$. Theorems 3 and 4 in [9] imply that P has Chvátal rank 1 if and only if $G[\overline{S}]$, the induced subgraph of G by $\overline{S}$ where G denotes the skeleton graph of the hypercube $[0,1]^n$, has max degree 2. It is easy to find a 0,1 point contained in P. First, check whether $\mathbf{0} \in P$. If not, then $\mathbf{0} \in \overline{S}$ and at least $n-2$ points among $e^1, \ldots, e^n$ (the unit vectors) are contained in P since the degree of $\mathbf{0}$ in $G[\overline{S}]$ is at most 2.

The gap between Open question 2 and Remark 3 is on the SAT formulas involving both clauses with 2 literals and clauses with at least 3 literals. SAT polytopes whose generalized set covering inequalities have at most 2 variables are well understood by Gerards and Schrijver [22]. They gave a characterization of the Chvátal closure in such a case, and they provided a polynomial algorithm to separate over it. Furthermore, we remark that the Chvátal rank of a SAT polytope in that case is always 1 whenever it contains no integer point. However, the Chvátal closure of a SAT polytope that includes both generalized set covering inequalities with 2 variables and 3 variables has not been studied.

2.2 When a few Chvátal cuts are sufficient

In this section, we consider another special case of Open question 1, where we assume that the integer hull of a given polyhedron can be obtained by adding a constant number of (rank-1) Chvátal inequalities.

Open question 3. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, and assume that the integer hull of P can be obtained by adding at most k (rank-1) Chvátal inequalities of P to the description of P, for some constant k. Can we solve the integer feasibility problem of P in polynomial time?

In fact, Open question 3 is open even when $k = 1$. We will show in Section 4.4 that verifying the promise that the integer hull of a given rational polytope is obtained after adding one Chvátal inequality is NP-hard. Thus, Open question 3 might be difficult to answer as well.

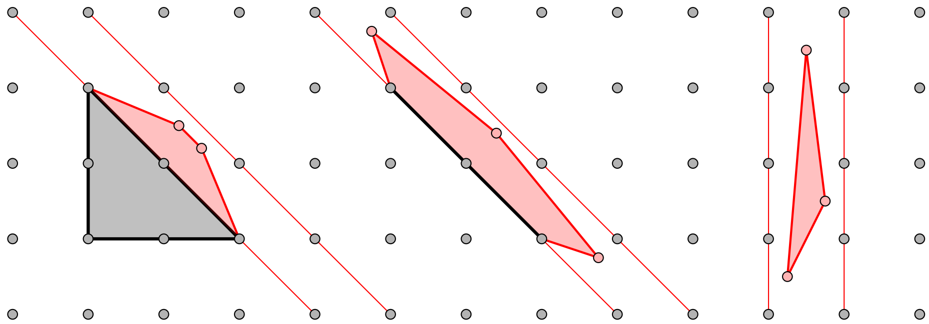

Figure 1: When one Chvátal inequality is sufficient in $\mathbb{R}^2$

Remark 5. *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polytope such that adding one Chvátal inequality to the description of P gives its integer hull. Then there exists an algorithm for the integer feasibility problem over P which runs in time bounded by $m^n n^3 poly(L)$ where m and L denote the number of constraints in P and the encoding size of P, respectively.*

Proof. This is easy to show because a fractional vertex of P should be removed by the Chvátal inequality. Therefore P contains an integer point if and only if an extreme point of P is integral. In this case, a trivial algorithm solves the integer feasibility problem: check all the vertices of P and conclude that $P_I \neq \emptyset$ if there exists an integral vertex or $P_I = \emptyset$ otherwise. Since there are $O(m^n)$ extreme points of P and the time complexity of the Gaussian elimination method is bounded by $n^3\text{poly}(L)$, the algorithm runs in time bounded by $m^n n^3 \text{poly}(L)$. □

In fact, Proposition 24 will show the existence of a $2^{O(n)}\text{poly}(L)$ time algorithm for the case of $k = 1$.

In the following, we consider a special case of Open question 3, where the input is a rational simplex. A polytope $P \subseteq \mathbb{R}^n$ is called a *simplex* of dimension ℓ for some $\ell \le n$ if it is the convex hull of $\ell + 1$ affinely independent points. One can show that the integer feasibility problem over a rational simplex is NP-complete by the following polynomial reduction of the knapsack problem to it [37]: consider positive integers $a_1, \cdots, a_n, b$. Let $v^i := \frac{b}{a_i} e^i$ where e^i denotes the ith unit vector for $i \in [n]$. Let $v^{n+1} := \frac{b-\frac{1}{2}}{n}(\frac{1}{a_1}, \cdots, \frac{1}{a_n})$. Let $\text{conv}\{v^1, \cdots, v^{n+1}\}$ denote the convex hull of $v^1, \cdots, v^{n+1}$. Note that $av^{n+1} = b - \frac{1}{2}$ and $av^i = b$ for $i \in [n]$. Then, $\text{conv}\{v^1, \cdots, v^{n+1}\} \cap \mathbb{Z}^n = \{x \in \mathbb{Z}^n : ax = b,\ x \ge 0\}$. However, if we further assume that the integer hull of a rational simplex can be obtained by adding a constant number of (rank-1) Chvátal inequalities, then we can solve the integer feasibility problem over the simplex in polynomial time.

Proposition 6. *Let k be a positive integer. Given a rational simplex $P \subseteq \mathbb{R}^n$ such that its integer hull can be obtained from P by adding at most k (rank-1) Chvátal inequalities, and a vector $w \in \mathbb{Q}^n$, there is an algorithm to optimize wx over P_I in time $n^{O(k)}poly(k, L)$, where L is the encoding size of P and w.*

Proof. Suppose that the dimension of P is ℓ for some $\ell \le n$. Let $P = \{x \in \mathbb{R}^n : Ax = b,\ Cx \le d\}$ be a minimal linear system defining P such that $Cx \le d$ define the facets of P. We denote by $Ex \le f$ the set of k Chvátal inequalities of P such that $P_I = \{x \in \mathbb{R}^n : Ax = b,\ Cx \le d,\ Ex \le f\}$. So the inequalities in $Ex \le f + \epsilon\mathbf{1}$ are valid for P, where $\epsilon \in (0,1)$ and $\mathbf{1}$ denotes the vector of all ones, and $P \subseteq S$, where $S := \{x \in \mathbb{R}^n : Ax = b,\ Ex \le f + \epsilon\mathbf{1}\}$.

We first argue that we may assume that P is full-dimensional. If not, we can find in polynomial time an unimodular matrix U such that $AU = (D, 0)$ is a Hermite normal form of A. If $D^{-1}b$ is not integral, we can just conclude that P does not contain an integer point. Thus, we may assume that $D^{-1}b$ is integral. Let U_1 and U_2 denote the two submatrices of U which consist of the first $n-\ell$ columns of U and the last ℓ columns of U, respectively. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be an unimodular transformation defined by $u(x) = U^{-1}x$. Consider the images of P, P_I, and S under u:

$$\begin{aligned}
u(P) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} :\ y_1 = D^{-1}b,\ CU_2y_2 \le d - CU_1D^{-1}b\right\},\\
u(P_I) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} :\ y_1 = D^{-1}b,\ CU_2y_2 \le d - CU_1D^{-1}b,\ EU_2y_2 \le f - EU_1D^{-1}b\right\},\\
u(S) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} :\ y_1 = D^{-1}b,\ EU_2y_2 \le f + \epsilon\mathbf{1} - EU_1D^{-1}b\right\}.
\end{aligned}$$

Note that $u(P)$ is an ℓ-dimensional simplex in $\mathbb{R}^n$, so $Q := \left\{y_2 \in \mathbb{R}^\ell : CU_2y_2 \le d - CU_1D^{-1}b\right\}$ is an ℓ-dimensional simplex in $\mathbb{R}^\ell$. Furthermore, $u(P_I)$ is integral. Since $D^{-1}b$ is integral, $\{y_2 \in \mathbb{R}^\ell : CU_2y_2 \le d - CU_1D^{-1}b,\ EU_2y_2 \le f - EU_1D^{-1}b\}$ is integral and thus $Q \cap \{y_2 \in \mathbb{R}^\ell : EU_2y_2 \le f - EU_1D^{-1}b\}$ is integral. We claim that the inequalities in the system $EU_2y_2 \le f - EU_1D^{-1}b$ are Chvátal inequalities of Q. In fact, we know that $u(P) \subseteq u(S)$, so $Q \subseteq \left\{y_2 \in \mathbb{R}^\ell : EU_2y_2 \le f + \epsilon\mathbf{1} - EU_1D^{-1}b\right\}$. That means the inequalities in $EU_2y_2 \le f+\epsilon\mathbf{1}-EU_1D^{-1}b$ are all valid for Q, so those in the system $EU_2y_2 \le f - EU_1D^{-1}b$ are Chvátal inequalities of Q. Now, we have obtained a full-dimensional rational simplex Q in $\mathbb{R}^\ell$ such that its integer hull Q_I can be described by adding at most k Chvátal inequalities.

Q has $\ell+1$ inequalities in its description, so Q_I can be described by $\ell+k+1$ linear inequalities. When $\ell \le k$, the dimension of Q is fixed and we can optimize a linear function over Q_I in polynomial time by Lenstra's algorithm [31]. Thus, we may assume that $\ell > k$. Suppose that Q_I is not empty. Then let $z \in \mathbb{Z}^\ell$ be an extreme point of Q_I. So there are ℓ linearly independent inequalities in the description of Q_I that are active at z. This means that at least $\ell - k$ inequalities among the $\ell + 1$ inequalities in the original description of Q are active at z. Thus, z belongs to a k-dimensional face of Q. Hence, if no k-dimensional face of Q contains an integer point, Q_I is empty. Since k is fixed, we can optimize a linear function over the integer hull of each k-dimensional face of Q. Notice that there are exactly $\binom{\ell+1}{k+1}$ k-dimensional faces of Q. Therefore, we can optimize a linear function over Q_I in $\ell^{O(k)}\text{poly}(L)$ time. Since we can compute the Hermite normal form of A in time polynomial in the encoding size of P and $\ell \le n$, the result follows, as required. □

The only property of a simplex in $\mathbb{R}^n$ used in the proof of Proposition 6 is that the number of its facets is at most $n+1$. The result should generalize to the case where a rational polytope $P \subseteq \mathbb{R}^n$ has $n+t$ facets, where t is a constant, and the integer hull of P is obtained by adding k (rank-1) Chvátal inequalities.

2.3 Rounded polytopes

A full-dimensional polytope $P \subseteq \mathbb{R}^n$ is *rounded with factor* $\ell > 1$ if $B_2^n(a, r) \subseteq P \subseteq B_2^n(a, \ell r)$, where $B_2^n(p, q)$ denotes an Euclidean ball $\{x \in \mathbb{R}^n : \|x - p\|_2 \le q\}$ centered at p with radius q. We first remark the following:

Remark 7. *Let $\ell > 1$ be a constant, and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rounded polytope with factor ℓ. We can decide whether P contains an integer point and find one if there exists any in $\ell^{O(n)}$poly(L) time, where L is the encoding size of P.*

Proof. One can find an Euclidean ball $B_2^n(c, R) \subseteq P$ of the largest radius by solving a linear program whose encoding size is bounded above by poly(L) (see Section 4.3 in [5]). If R is at least $\frac{\sqrt{n}}{2}$, an integer point that is nearest to c is contained in the ball, so we can obtain an integer point in P by rounding c. If that is not the case, we consider two Euclidean balls $B_2^n(a, r)$ and $B_2^n(a, \ell r)$ for some $a \in P$ and $0 < r < \frac{\sqrt{n}}{2}$ such that $B_2^n(a, r) \subseteq P \subseteq B_2^n(a, \ell r)$. As $c \in P$, the distance between a and c is at most ℓr, and therefore, $B_2^n(c, 2\ell r)$ contains $B_2^n(a, \ell r)$ by the triangle inequality. So, P is also contained in $B_2^n(c, 2\ell r)$. As $2\ell r < \ell\sqrt{n}$, we can enumerate all the $\ell^{O(n)}$ integer points in $B_2^n(c, 2\ell r)$ and check whether at least one of them belongs to P. $\square$

Now, we further assume that the integer hull of P can be obtained by adding one Chvátal inequality, which is another special case of Open question 3.

Proposition 8. *Let $\ell > 1$ be a constant, and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rounded polytope with factor ℓ. If the integer hull of P can be obtained by adding one Chvátal inequality to the description of P, then we can decide whether P contains an integer point in $n^{O(\ell)}$poly(L) time, where L is the encoding size of P.*

To prove this, we use the notion of *integer width* of a convex set, which will also be used in Section 3.

Definition 9. *Let $K \subseteq \mathbb{R}^n$ be a convex set and $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. The integer width of K along d is*

$$w(K, d) := \lfloor \sup\{dx : \ x \in K\} \rfloor - \lceil \inf\{dx : \ x \in K\} \rceil + 1.$$

The integer width of K, $w(K, \mathbb{Z}^n)$, is the infimum of the values $w(K, d)$ over all $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

$$w(K, \mathbb{Z}^n) := \inf_{d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} w(K, d).$$

Lemma 10. *Let $P \subseteq \mathbb{R}^n$ be a rounded polytope with factor $\ell > 1$. If there exists a direction $d \in \mathbb{Z}^n$ such that $w(P, d) \leq k$ for some nonnegative integer k, then either $\|d\|_2 \leq (k+1)\ell$ or $w(P, e^i) \leq 1$ for all $i \in [n]$.*

Proof. Since P is rounded with factor ℓ, P satisfies $B_2^n(a, r) \subseteq P \subseteq B_2^n(a, \ell r)$ for some $r > 0$ and $a \in \mathbb{R}^n$. Assume that $\|d\|_2 > (k+1)\ell$. Since $w(P, d) \leq k$, there exists $d_0 \in \mathbb{Z}$ such that $d_0 < dx < d_0 + k + 1$ for all $x \in P$. Notice that $B_2^n(a, r) \subseteq P \subseteq \{x \in \mathbb{R}^n : d_0 < dx < d_0 + k + 1\}$ and the distance between two hyperplanes $\{x \in \mathbb{R}^n : dx = d_0\}$ and $\{x \in \mathbb{R}^n : \ dx = d_0 + k + 1\}$ is exactly $(k+1)/\|d\|_2$. This implies that $2r$ is at most $(k+1)/\|d\|_2$. Hence, we get $r \leq \frac{k+1}{2\|d\|_2} < \frac{1}{2\ell}$, *i.e.*, $2\ell r < 1$. Suppose that there is some i such that $w(P, e^i) \geq 2$. Then there are two points $u, v \in P$ such that $u_i \leq b$ and $v_i \geq b + 1$ for some $b \in \mathbb{Z}$. So $\|u - v\|_2 \geq |u_i - v_i| \geq 1$. Since $B_2^n(a, \ell r)$ contains P, the distance between any two points in P is at most $2\ell r$ and thus we get $2\ell r \geq 1$. However, this contradicts the previous observation that $2\ell r < 1$. Therefore, $w(P, e^i) \leq 1$ for all $i \in [n]$. $\square$

Proof of Proposition 8. Consider the following algorithm:

(1) For each $d \in \mathbb{Z}^n$ with $\|d\|_2 \leq \ell$, compute $w(P, d)$. If $w(P, d) = 0$ for some d with $\|d\|_2 \leq \ell$, then $P_I = \emptyset$. Otherwise, go to step (2).

(2) Compute $w(P, e^i)$ for $i \in [n]$. If there exists $i \in [n]$ such that $w(P, e^i) \geq 2$, then $P_I \neq \emptyset$. If there exists $i \in [n]$ such that $w(P, e^i) = 0$, then $P_I = \emptyset$. Otherwise, go to step (3).

(3) Let $z_j := \lfloor \max\{x_j : x \in P\} \rfloor$ for $j \in [n]$. If $(z_1, \cdots, z_n) \in P$, then $P_I \neq \emptyset$. Otherwise, $P_I = \emptyset$.

Step (1) can be done in polynomial time, because there are at most $\binom{n}{\ell} 2^\ell \binom{2\ell-1}{\ell}$ integral vectors d with $\|d\|_2 \leq \ell$. By assumption, there exists a Chvátal inequality $\bar{d}x \leq \bar{d}_0$ such that $\{x \in P : \bar{d}x \leq \bar{d}_0\} = P_I$. Note that P_I is empty if and only if $w(P, \bar{d}) = 0$. Going into Step (2), we have $w(P, d) \geq 1$ for all $d \in \mathbb{Z}^n$ with $\|d\|_2 \leq \ell$ and $\|\bar{d}\|_2 > \ell$. If $w(P, e^i) \geq 2$ for some $i \in [n]$, then $w(P, \bar{d}) \geq 1$ by Lemma 10 (when $k = 0$) and thus $P_I \neq \emptyset$. If $w(P, e^i) = 0$ for some $i \in [n]$, then P_I is empty. Therefore, going into Step (3), we have $w(P, e^i) = 1$ for all $i \in [n]$, and P can have at most one integer point. z is the only possibility and we can compute z by solving n linear programs, therefore, in polynomial time. □

3 Flatness theorem for closed convex sets with empty Chvátal closure

Recall the definition of integer width of a convex set K given in Definition 9. When K is unbounded or has a large volume, there exists a direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ where $w(K, d)$ is large. On the other hand, it is possible that there is a direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $w(K, d)$ is relatively small if K does not contain any integer point. In fact, the famous flatness theorem by Khinchine [29] states that $w(K, \mathbb{Z}^n)$ for any compact convex set K containing no integer point is bounded by $f(n)$, a function that depends only on the ambient dimension n. Khinchine's flatness theorem [29] shows that $f(n) \leq (n+1)!$. A crucial component of Lenstra's algorithm [31] is to find a flat direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ of a polyhedron $P \subseteq \mathbb{R}^n$ containing no integer point. Lenstra [31] gave a polynomial algorithm to find a direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $w(K, d) \leq 2^{O(n^2)}$ for a given lattice-free compact convex set K. Then, it generates $2^{O(n^2)}$ subproblems in $\mathbb{R}^{n-1}$ by intersecting K with $2^{O(n^2)}$ parallel hyperplanes orthogonal to d. Hence, the algorithm works recursively, and the number of total steps required is $2^{O(n^3)}$.

Over the last few decades there have been huge improvements on the upper bound $f(n)$ (see [1, 2, 27, 28, 29, 34]). The current best known asymptotic upper bound is $f(n) = O(n^{4/3}\text{polylog}(n))$ given by Banaszczyk, Litvak, Pajor, and Szarek [2] and Rudelson [34]. It has been even conjectured that $f(n) = O(n)$. However, the existence of a polynomial algorithm to find a direction $d \in \mathbb{Z}^n$ such that $w(K, d) = O(n^{4/3}\text{polylog}(n))$ for a convex set K containing no integer point is not known. Dadush, Peikert and Vempala [15] and Dadush and Vempala [16] developed an algorithm to find all vectors $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $w(K, d) = w(K, \mathbb{Z}^n)$ in $2^{O(n)}\text{poly}(L)$ time and space.

In this section, we first prove that $f(n) \leq n$ if K is a compact convex set whose Chvátal closure is empty. The Chvátal closure of a closed convex set is defined similarly to that of a polyhedron [13, 14, 17]. For a closed convex set K, $\sigma_K(d) := \sup\{dx : x \in K\}$ for $d \in \mathbb{R}^n$ is its support function. It is known that any closed convex set K can be expressed as $K = \bigcap_{d \in \mathbb{R}^n} \{x \in \mathbb{R}^n : dx \leq \sigma_K(d)\}$, which is the set of solutions satisfying the system of linear inequalities given by its support function (see Theorem C.2.2.2 in [26]). Dadush, Dey, and Vielma later showed that the inequalities with integer coefficients are sufficient to describe K (Proposition 2.1 in [13]). In other words, $K = \bigcap_{d \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : dx \leq \sigma_K(d)\}$. The Chvátal closure of K is defined as what is obtained after rounding down their right hand side values.

Definition 11. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. The Chvátal closure of K is defined as*

$$K' := \bigcap_{d \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ dx \leq \lfloor \sigma_K(d) \rfloor\}.$$

By its definition, K' is contained in K and it is also clear that $K \cap \mathbb{Z}^n \subseteq K'$.

Let $K \subseteq \mathbb{R}^n$ be a convex set and $a \in \mathbb{R}^n$ be a point. We denote by $K - a := \{x - a : x \in K\}$ the translation of K by $-a$. Let ℓK for some real number ℓ be defined as $\ell K := \{\ell x : x \in K\}$.

Proposition 12. *Let $K \subseteq \mathbb{R}^n$ be a compact convex set whose Chvátal closure is empty. If $K - a \subseteq -\ell(K - a)$ for some $a \in K$ and $\ell > 0$, then the integer width of K is at most $\lceil \ell \rceil$.*

Proof. Since the Chvátal closure of K is empty, $a \in K$ should be cut off by a Chvátal inequality of K. In other words, there exists $(d, d_0) \in \mathbb{Z}^{n+1}$ such that $\max\{dx : x \in K\} < d_0$ and $da > d_0 - 1$. Then, we get $\max\{dx : x \in K - a\} = \max\{dx : x \in K\} - da < 1$, and this implies $\min\{dx : x \in -\ell(K - a)\} = -\max\{dx : x \in \ell(K - a)\} > -\ell$. We assumed that $K - a \subseteq -\ell(K - a)$, so $\min\{dx : x \in K - a\} \geq \min\{dx : x \in -\ell(K - a)\} > -\ell$. Hence, we have $\max\{dx : x \in K\} < d_0$ and $\min\{dx : x \in K\} > da - \ell > d_0 - \ell - 1$. Therefore, the integer width of K (along d) is at most $\lceil \ell \rceil$. $\square$

If $K \subseteq \mathbb{R}^n$ is a centrally symmetric compact convex set, then $K - a = -(K - a)$ for some $a \in K$. Although an asymmetric convex set K does not contain such a point $a \in K$, Süss [38] and Hammer [24] proved the following:

Theorem 13 ([24], Theorem 2, see also [38])**.** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set, then there exists $a \in K$ such that $K - a \subseteq -n(K - a)$.*

Combining Proposition 12 and Theorem 13, we obtain the following as a direct corollary.

Theorem 14. *Let $K \subseteq \mathbb{R}^n$ be a compact convex set whose Chvátal closure is empty. Then the integer width of K is at most n.*

The upper bound given by Theorem 14 turns out to be very tight as shown in the following proposition.

Proposition 15. *There exists a polytope in $\mathbb{R}^n$ such that its Chvátal closure is empty and its integer width is $n - 1$.*

Proof. Let $P_n := \{x \in \mathbb{R}^n : x \geq \frac{1}{n+1}\mathbf{1}, \ \sum_{i=1}^n x_i \leq n - 1 + \frac{n}{n+1}\}$. Figure 2 depicts P_n when $n = 2$. Then P_n is the convex hull of $(n-1)e^i + \frac{1}{n+1}\mathbf{1}$ for $i \in [n]$ and $\frac{1}{n+1}\mathbf{1}$. Since $x_i \geq 1$ is valid for P'_n for each i, $\sum_{i=1}^n x_i \geq n$ is valid for P'_n. Together with $\sum_{i=1}^n x_i \leq n - 1 + \frac{n}{n+1}$, this shows the emptiness of P'_n.

Now we show that the integer width of P_n is $n - 1$. Let $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Since the integer width of P_n along d is the same as that along $-d$, we may assume $\sum_{i=1}^n d_i \geq 0$. Notice that $\max\{dx : x \in P_n\} = (n-1)\max\{d_1, \cdots, d_n\} + \frac{1}{n+1}\sum_{i=1}^n d_i$ and $\min\{dx : x \in P_n\} = (n-1)\min\{0, d_1, \cdots, d_n\} + \frac{1}{n+1}\sum_{i=1}^n d_i$. Then the integer width of P_n along d is either $(n-1)(\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\})$ or $(n-1)(\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\}) + 1$. Clearly, $\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\}$ is at least 1. Hence, the integer width of P_n along d is at least $n - 1$. It is easy to show that the integer width of P_n along $\mathbf{1}$ is exactly $n - 1$. $\square$

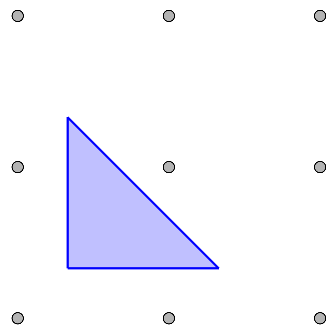

Figure 2: P_2 in $\mathbb{R}^2$

3.1 Flatness result

Can we bound the integer width of a closed convex set whose Chvátal closure is empty, even when it is unbounded? The answer is no; let us elaborate with the following example.

Example 16. Let $P := \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1\}$. P can be rewritten as $P = \{\alpha(1, \sqrt{2}) : \alpha \geq 1\}$. It is clear that P does not contain an integer point. For every $d = (d_1, d_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $d_1 + d_2\sqrt{2} \neq 0$ and thus either $\max\{dx : x \in P\}$ or $\min\{dx : x \in P\}$ is unbounded. Therefore, the integer width of P is unbounded.

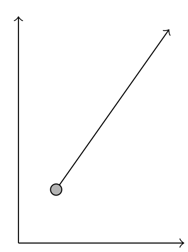

Figure 3: P in $\mathbb{R}^2$

In fact, we can prove that the Chvátal closure of P is empty. It is sufficient to show that for any $z \geq 1$, there is a Chvátal inequality that cuts off the line segment between $(1, \sqrt{2})$ and $z(1, \sqrt{2})$. By the Dirichlet approximation theorem, we can find $(d_1, d_2) \in \mathbb{Z}^2$ such that

$$\left|\sqrt{2} - \frac{d_1}{d_2}\right| < \frac{1}{2zd_2}.$$

Then, we get $|d_1 - d_2\sqrt{2}| < \frac{1}{2z}$. Since $d_1 - d_2\sqrt{2} \neq 0$, we may assume without loss of generality that $-\frac{1}{2z} < d_1 - d_2\sqrt{2} < 0$. In this case, $d_1x_1 - d_2x_2 \leq d_1 - d_2\sqrt{2}$ is a valid inequality for P. We then obtain a Chvátal inequality $d_1x_1 - d_2x_2 \leq -1$ from it, because $-1 < d_1 - d_2\sqrt{2} < 0$. Notice that $d_1z - d_2z\sqrt{2} = z(d_1 - d_2\sqrt{2})$ and $z(d_1 - d_2\sqrt{2}) > -\frac{1}{2}$, so both $(1, \sqrt{2})$ and $z(1, \sqrt{2})$ are cut off by the Chvátal inequality. In this case, we need infinitely many Chvátal inequalities to certify that the Chvátal closure of P is empty.

As explained in this example, there is no global bound on the integer width of an unbounded closed convex set whose Chvátal closure is empty. What made the integer width unbounded in the previous example was an irrational ray $(1, \sqrt{2})$ that is not contained in a proper rational linear subspace. We say that an irrational vector r is *fully irrational* if there is no proper rational linear subspace containing r. In general, we can show that

Remark 17. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If K contains a fully irrational ray $r \in \mathbb{R}^n$, then the integer width of K is unbounded.*

Proof. Let $d \in \mathbb{Z}^n \setminus \{0\}$. Notice that dr is nonzero. Otherwise, r is contained in a proper rational linear subspace $\{x \in \mathbb{R}^n : dx = 0\}$, a contradiction to the assumption. Then either $\sup\{dx : x \in K\}$ or $\inf\{dx : x \in K\}$ is unbounded, so we have that $w(K, d)$ is unbounded. Therefore, $w(K, d)$ is unbounded for each $d \in \mathbb{Z}^n \setminus \{0\}$, and the integer width of K is unbounded. $\square$

Hence, a closed convex set with bounded integer width does not contain a fully irrational ray. Let K be a closed convex set that does not contain a fully irrational ray, and consider its recession cone C, that is, the collection of all the rays contained in K. Let $\text{lin}(C)$ denote the linear hull of C, that is, the smallest linear subspace containing C. Then $\text{lin}(C)$ is a rational linear subspace. In fact, we can generalize Theorem 14 as the following:

Theorem 18. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set that can be expressed as $K = Q + C$ where Q is a compact convex set and C is a cone such that $\text{lin}(C)$ is rational. If the Chvátal closure of K is empty, then the integer width of K is at most n.*

It turns out that Theorem 18 cannot be generalized to a closed convex set K that can be expressed as $K = Q + C$ where Q is not necessarily bounded, as shown by the following example.

Example 19. Let $K := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1,\ x_3 \geq x_1^2\right\}$. The recession cone C of K is simply $\{\alpha(0, 0, 1) : \alpha \geq 0\}$, so $\text{lin}(C)$ is rational and $K = K + C$. K is contained in $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1\right\}$, and we can argue that the Chvátal closure of $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1\right\}$ is empty as we did in Example 16. That means the Chvátal closure of K is empty as well. However, the integer width of K is unbounded. Let $d = (d_1, d_2, d_3) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$. Notice that $(z, \sqrt{2}z, z^2) \in K$ for any positive integer z. As $d_1 z + d_2\sqrt{2}z \neq 0$ for any integer z, $d_1 z + d_2\sqrt{2}z + d_3 z^2 = d_3 z^2 + (d_1 + d_2\sqrt{2})z$ becomes unbounded as z goes to infinity. So, either $\sup\{dx : x \in K\}$ or $\inf\{dx : x \in K\}$ is unbounded. Therefore, the integer width of K is unbounded.

As a corollary of Theorem 18, we obtain an improved flatness theorem for rational polyhedra with Chvátal rank 1. The result will be used in developing an algorithm for solving the integer feasibility problem over the rational polyhedra with Chvátal rank 1 in the later part of this section.

Corollary 20. *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. Then, either P contains an integer point or the integer width of P is at most n.*

3.2 Proof of Theorem 18

To prove Theorem 18, we show Lemma 22 and Lemma 23 in this section. For Lemma 22, we need the following result due to Dadush, Dey, and Vielma [14].

Theorem 21 ([14], Theorem 1)**.** *If $K \subseteq \mathbb{R}^n$ is a compact convex set, then the Chvátal closure of K is a rational polytope.*

Lemma 22. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set that can be expressed as $K = Q + C$ where Q is a compact convex set and C is a cone such that $\operatorname{lin}(C)$ is rational. If the Chvátal closure of K is empty, then there exists a finite list of Chvátal inequalities such that the intersection of their corresponding half-spaces is empty.*

Proof. By Theorem 21, we may assume that K is unbounded, so C has a nontrivial ray. If $\operatorname{lin}(C)$ is a rational linear subspace, there exists a rational matrix A with full row rank such that $\operatorname{lin}(C) = \{x \in \mathbb{R}^n : Ax = 0\}$. We remark that we may assume $A = (I, 0)$ where I is the identity matrix with the same number of rows as A, which means $\operatorname{lin}(C) = \{x = (x^1, x^2) \in \mathbb{R}^{n_1+n_2} : Ix^1 + 0x^2 = x^1 = 0\}$ where $n_1 + n_2 = n$. When $A \neq (I, 0)$, we can find an unimodular matrix U such that $AU = (H, 0)$ is a Hermite normal form of A. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be an unimodular transformation defined as $u(x) = U^{-1}x$ for $x \in \mathbb{R}^n$. Notice that

$$u(K') = \bigcap_{dU \in \mathbb{Z}^n} \{y \in \mathbb{R}^n : \ dUy \le \lfloor \sup\{dUy : y \in u(K)\} \rfloor\}.$$

Hence, $u(K') = (u(K))'$. Then it is sufficient to show that there is a finite list of Chvátal inequalities of $u(K)$ whose corresponding half-spaces have empty intersection. Moreover, the recession cone of $u(K)$ is $u(C)$, and notice that $\operatorname{lin}(u(C)) = \left\{y = (y^1, y^2) \in \mathbb{R}^{n_1+n_2} : Hy^1 = 0\right\}$ and it is equal to $\left\{y = (y^1, y^2) \in \mathbb{R}^{n_1+n_2} : y^1 = 0\right\}$. Thus, we may indeed assume that $A = (I, 0)$.

We will first show that if the Chvátal closure of K is empty, then it suffices to look at the Chvátal inequalities obtained from the directions orthogonal to $\operatorname{lin}(C)$. Since $\operatorname{lin}(C)$ is a rational linear subspace, the relative interior of C contains a ray $\bar{r}$ whose components are integers. Let us consider $K + \bar{r}$, the translation of K by $\bar{r}$. Notice that $K + \bar{r} \subseteq K$. Since the Chvátal closure of K is empty, there are some Chvátal inequalities of K that remove all points in $K + \bar{r}$. Let's pick a direction $d \in \mathbb{Z}^n \setminus \{0\}$ that is not orthogonal to $\operatorname{lin}(C)$. We may assume that $\sup\{dx : x \in K\}$ has some finite value f. Otherwise, we can ignore the Chvátal inequality obtained from d. Then, $dr \le 0$ for all $r \in C$. If $d\bar{r} = 0$, then $dr = 0$ for all $r \in C$, a contradiction to the assumption that d is not orthogonal to $\operatorname{lin}(C)$. Hence, $d\bar{r} < 0$. In fact, we know that $d\bar{r} \le -1$, because both d and r have integer components. Notice that $\sup\{dx : x \in K + \bar{r}\} = f + d\bar{r}$. Since $d\bar{r} \le -1$, the Chvátal inequality $dx \le \lfloor \sup\{dx : x \in K\} \rfloor = \lfloor f \rfloor$ obtained from d does not cut off any point in $K + \bar{r}$. This implies that the points in $K + \bar{r}$ are cut off by only the Chvátal inequalities obtained from directions orthogonal to $\operatorname{lin}(C)$. So, we have

$$(K + \bar{r}) \cap \bigcap_{d \in \operatorname{lin}(C)^{\perp} \cap \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ dx \le \lfloor \sup\{dx : \ x \in K\} \rfloor\} = \emptyset,$$

where $\operatorname{lin}(C)^{\perp}$ denotes the orthogonal complement of $\operatorname{lin}(C)$. Let $\bar{x} \in K + \operatorname{lin}(C)$. Then $\bar{x} + r \in K + \bar{r}$ for some $r \in \operatorname{lin}(C)$, so there exists a direction $d \in \operatorname{lin}(C)^{\perp} \cap \mathbb{Z}^n$ such that $d(\bar{x} + r) > \lfloor \sup\{dx : x \in K\} \rfloor$. As $r \in \operatorname{lin}(C)$, we know that $dr = 0$. Then we get $d\bar{x} > \lfloor \sup\{dx : x \in K\} \rfloor$, so $\bar{x}$ is also cut off by the same Chvátal inequality. Therefore, we have that

$$(K + \operatorname{lin}(C)) \cap \bigcap_{d \in \operatorname{lin}(C)^{\perp} \cap \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ dx \le \lfloor \sup\{dx : \ x \in K\} \rfloor\} = \emptyset. \qquad (\star)$$

To complete the proof, we look at $\widetilde{K}$, that is the projection of K onto $\operatorname{lin}(C)^{\perp}$. Since $K = Q + C$, $\widetilde{K}$ is the same as the projection of Q onto $\operatorname{lin}(C)^{\perp}$. Then $\widetilde{K}$ is a compact convex set and $K + \operatorname{lin}(C)$

is the same as $\widetilde{K} + \text{lin}(C)$. Recall that $\text{lin}(C) = \{x = (x^1, x^2) \in \mathbb{R}^{n_1+n_2} : x^1 = 0\}$, so $\text{lin}(C)^\perp = \{x = (x^1, x^2) \in \mathbb{R}^{n_1+n_2} : x^2 = 0\}$. Then $\text{lin}(C)^\perp \cap \mathbb{Z}^n = \{d = (d^1, d^2) \in \mathbb{Z}^{n_1+n_2} : d^2 = 0\}$, so $dx \leq \lfloor \sup\{dx : x \in K\} \rfloor$ for $d \in \text{lin}(C)^\perp \cap \mathbb{Z}^n$ is equivalent to $d^1 x^1 \leq \lfloor \sup\{d^1 x^1 : x^1 \in \widetilde{K}\rfloor$. Then, $(\star)$ is equivalent to

$$\widetilde{K} \cap \bigcap_{d^1 \in \mathbb{Z}^{n_1}} \left\{ x^1 \in \mathbb{R}^{n_1} : \ d^1 x^1 \leq \lfloor \sup\{d^1 x^1 : \ x^1 \in \widetilde{K}\} \rfloor \right\} = \emptyset.$$

Since $\widetilde{K}$ is a compact convex set, its Chvátal closure is a rational polytope due to Theorem 21. Therefore, the Chvátal closure of $\widetilde{K}$ is described by a finite number of Chvátal inequalities. In turn, there is a finite subset $D \subseteq \mathbb{Z}^{n_1}$ such that $\bigcap_{d^1 \in D} \left\{ x^1 \in \mathbb{R}^{n_1} : d^1 x^1 \leq \lfloor \sup\{d^1 x^1 : x^1 \in \widetilde{K}\} \rfloor \right\} = \emptyset$. This implies

$$\bigcap_{d \in D \times \{\mathbf{0}\}} \{x \in \mathbb{R}^n : \ dx \leq \lfloor \sup\{dx : \ x \in K\} \rfloor\} = \emptyset,$$

so the Chvátal inequalities obtained from directions in a finite list $D \times \{\mathbf{0}\}$ are sufficient to show that the Chvátal closure of K is empty, as required. $\square$

To prove Theorem 18, we introduce the concept of a *simplicial cylinder*. Let $P \subseteq \mathbb{R}^n$ be a full-dimensional rational polyhedron. We denote by L and $L^\perp$ the lineality space of P and its orthogonal complement, respectively. We say that P is a *simplicial cylinder* if $P \cap L^\perp$ is a simplex. Observe that a simplicial cylinder $P \subseteq \mathbb{R}^n$ whose lineality space L has dimension $n - \ell$ can be described by $\ell + 1$ linear inequalities.

Let P be a rational polyhedron given by its linear description $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where each row of A has relatively prime integers and b has integer components. We call P a *thin simplicial cylinder* if it is a simplicial cylinder and $Ax \leq b - \mathbf{1}$, where $\mathbf{1}$ denotes the vector of all ones, is an infeasible system. Note that a thin simplicial cylinder is a lattice-free set, which does not contain an integer point in its interior but might include one on its boundary (see Figure 4).

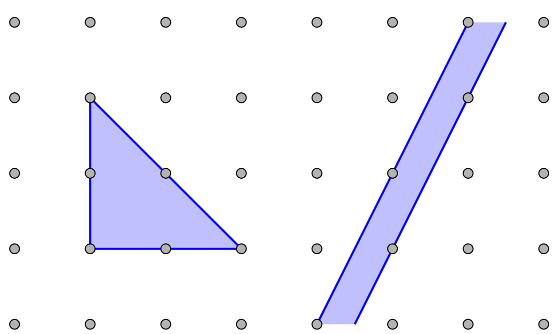

Figure 4: Thin simplicial cylinders in $\mathbb{R}^2$

Lemma 23. *Let K be a closed convex set. If there exists a finite list of Chvátal inequalities of K such that the intersection of their corresponding half-spaces is empty, K is contained in the interior of a thin simplicial cylinder.*

Proof. Helly's theorem implies that there are $\ell + 1$ Chvátal inequalities of K for some $\ell \leq n$ such that the intersection of the corresponding linear half-spaces is empty. Then, there exists a system $Ax \leq b - \epsilon\mathbf{1}$ of $\ell + 1$ linear inequalities valid for K, where (A, b) has integer entries and $0 < \epsilon < 1$, such that $Ax \leq b - \mathbf{1}$ is an infeasible system. We may assume that each row of A has relatively prime integer entries. We may also assume that the system is minimal in a sense that $\{x \in \mathbb{R}^n : a^i x \leq b_i - 1 \text{ for } i \in I\}$ is not empty for any proper subset I of $[\ell + 1]$. Now, consider the polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$. We claim that its recession cone $C := \{x : Ax \leq 0\}$ has empty interior. Otherwise, the polyhedron P contains points in the form of $x + kr$ for some $x \in P$ and some ray vector $r \in \mathbb{R}^n$ in the interior of C, where $k \in \mathbb{R}_+$. For k large enough, the points of the form are also in the polyhedron $S := \{x \in \mathbb{R}^n : Ax \leq b - \mathbf{1}\}$, which is empty, a contradiction. Therefore, the linear space $C - C$ has dimension strictly less than n. By the Minkowski-Weyl theorem, we can write the polyhedron P as $P = Q + C$ where Q is a polytope. Consider the cylinder $R := Q + C - C$. Consider all the inequalities $a^i x \leq b_i$, $i = 1, \cdots, t$, in the description of P that are valid for R. Then for $i = t + 1, \ldots, \ell + 1$, there exists $r^i \in C$ such that $a^i r^i < 0$. Consider $r = \sum_{i=t+1}^{\ell+1} r^i$. Then $a^i r \leq a^i r^i < 0$ for $i = t + 1, \ldots, \ell + 1$. We claim that the linear system $a^i x \leq b_i - 1$, $i = 1, \cdots, t$, is infeasible. If $a^i x \leq b_i - 1$, $i = 1, \cdots, t$, were feasible, then, by the same argument as given above, S would be nonempty, a contradiction. Thus $a^i x \leq b_i - 1$, $i = 1, \cdots, t$, is infeasible. By the minimality of the system, this implies $t = \ell + 1$, and therefore Q is a simplex of dimension ℓ. That means $R = P$ and P is a simplicial cylinder containing K in its interior. □

Proof of Theorem 18. Lemma 22 implies that there exists a finite list of Chvátal inequalities of K such that the intersection of their corresponding half-spaces is empty. Then, we know by Lemma 23 that there exists a thin simplicial cylinder $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ containing K in its interior. Let $\ell + 1$ be the number of rows in A for some $\ell \leq n$. We denote by $a^1, \cdots, a^{\ell+1}$ the rows of A. Notice that $P \cap L^\perp$ is an ℓ-dimensional simplex, where L and $L^\perp$ denote the lineality space of P and its orthogonal complement, respectively.

We will show that the integer width of P along some a^i is at most $\ell + 1$. Then the integer width of K is at most ℓ, because the hyperplane defined by $a^i x = b_i$ does not go through K. Suppose that the integer width of P along each a^i is at least $\ell + 2$ for the sake of contradiction. Then, the width of P along each a^i is at least $\ell + 1$. Using an affine transformation, we can transform P to $\{x \in \mathbb{R}^n : x_1, \cdots, x_\ell \geq 0,\ \sum_{i=1}^{\ell} x_i \leq 1\}$. Under the same affine transformation, we know that $\{x \in \mathbb{R}^n : Ax \leq b - \mathbf{1}\}$ is transformed to $\{x \in \mathbb{R}^n : x_i \geq \epsilon_i\ \forall\ i \in [\ell],\ \sum_{i=1}^{\ell} x_i \leq 1 - \epsilon\}$ for some $0 < \epsilon_i \leq \frac{1}{\ell+1}$ for $i \in [\ell]$ and $0 < \epsilon \leq \frac{1}{\ell+1}$. Notice that $\left(\frac{1}{\ell+1}, \ldots, \frac{1}{\ell+1}\right) \in \mathbb{R}^n$ is contained in $\{x \in \mathbb{R}^n : x_i \geq \epsilon_i\ \forall\ i \in [\ell],\ \sum_{i=1}^{\ell} x_i \leq 1 - \epsilon\}$. However, $\{x \in \mathbb{R}^n : Ax \leq b - \mathbf{1}\}$ is empty by the assumption that P is a thin simplicial cylinder, and it cannot be transformed to a nonempty set under any affine transformation. With this contradiction, we have proved that the integer width of K is at most $\ell \leq n$. □

3.3 A Lenstra-type algorithm

Recently Hildebrand and Köppe [25], Dadush, Peikert, and Vempala (see [12, 15, 16]) improved Lenstra-type algorithms for integer programming. Their algorithms are similar to Lenstra's algorithm in spirit in that a main step consists in finding a flat direction of a lattice-free convex body. In particular, Dadush, Peikert, and Vempala (see [12, 15, 16]) used a $2^{O(n)}\text{poly}(L)$ time algorithm to find a flattest direction for a convex body containing no integer point, and they proved that the time complexity of their Lenstra-type algorithm is bounded by $2^{O(n)}\,(f(n))^n\,\text{poly}(L)$, where $f(n)$

is the upper bound on the integer width of a compact convex set with no integer point. Together with the current tightest upper bound $f(n) = O(n^{4/3}\text{polylog}(n))$ [2, 34], the time complexity of the algorithm is bounded by $2^{O(n)}\left(n^{4/3}\text{polylog}(n)\right)^n \text{poly}(L)$. Corollary 20 implies that there exists a $2^{O(n)}n^n\text{poly}(L)$ time Lenstra-type algorithm for the integer feasibility problem over Chvátal rank 1 rational polyhedra. On the other hand, Proposition 15 indicates that we cannot improve this time complexity if we use a Lenstra-type procedure. Note that this does not improve the current best algorithm for integer programming. Dadush [12] provided a $2^{O(n)}n^n\text{poly}(L)$ time Kannan-type algorithm for integer programming over general convex compact sets in his doctoral dissertation, and we remark that it is the fastest algorithm for integer programming. Instead of finding one flat direction at a time, his algorithm finds many flat directions at each step, thereby reducing the number of recursive steps from $\left(n^{4/3}\text{polylog}(n)\right)^n$ to $(3n)^n$.

Based on Theorem 18 and Proposition 23, we can state the following proposition. We do not describe our algorithm in this paper, because it is similar to the earlier work done by Dadush, Peikert, and Vempala (see [12, 15, 16]). We refer the reader to Lee's dissertation [30] for details.

Proposition 24. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be rational polyhedron with Chvátal rank 1. Assume that if P contains no integer point, then P is contained in the interior of a thin simplicial cylinder defined by $\ell + 1$ inequalities for some $\ell \leq n$. Then, there exists a $2^{O(n)}\ell^n$poly(L) time Lenstra-type algorithm that decides whether P contains an integer point, where L is the encoding size of P.*

Since any rational polyhedron with empty Chvátal closure in $\mathbb{R}^n$ is always contained in the interior of a thin simplicial cylinder which is defined by at most $n + 1$ inequalities, Proposition 24 directly implies the following:

Remark 25. *There is a Lenstra-type algorithm that can decide in $2^{O(n)}n^n$poly(L) time, where L is the encoding size of P, whether a given rational polyhedron $P \subseteq \mathbb{R}^n$ with Chvátal rank 1 contains an integer point.*

Although our algorithm correctly decides whether a given rational polyhedron with Chvátal rank 1 contains an integer point, it does not find an integer point when one exists. In order to provide an algorithm that actually finds an integer point when exists, we believe that it is necessary to analyze some properties of integer feasible rational polyhedra with Chvátal rank 1, which is widely open.

4 Recognizing rational polytopes with an empty Chvátal closure is NP-hard

In Section 3, we studied closed convex sets with an empty Chvátal closure. Recently, Cornuéjols and Li [10, 11] proved that it is NP-complete to decide whether the Chvátal closure of a rational polytope is empty. In this section, we improve their result by showing that the problem remains NP-complete, even if the input polytope is contained in the unit hypercube or is a simplex. We prove this in Sections 4.1 and 4.2. This hardness result has some nice consequences. In particular, it implies that both optimizing and separating over the Chvátal closure of a rational polytope given by its linear description are NP-hard, even when the polytope is contained in the unit cube or is a simplex. This extends an earlier result of Eisenbrand [19], and we explain this in Section 4.3. Another consequence is that for any positive integer k, it is NP-hard to decide whether adding at most k (rank-1) Chvátal cuts is sufficient to describe the integer hull of a rational polytope given by its linear description, and we derive this in Section 4.4.

4.1 The case of polytopes contained in the unit hypercube

The next theorem is the main result of this section.

Theorem 26. *Let $P = \{x \in [0,1]^n : Ax \leq b\}$ be a nonempty rational polytope contained in the unit hypercube. It is NP-complete to decide whether the Chvátal closure of P is empty, even when P contains no integer point.*

Notice that when a nonempty rational polytope P contains no integer point, the Chvátal closure of P is empty if and only if the Chvátal rank of P is exactly 1. Hence, we obtain the following as a trivial corollary,

Corollary 27. *Let $P = \{x \in [0,1]^n : Ax \leq b\}$ be a rational polytope contained in the unit hypercube. It is NP-hard to decide whether the Chvátal rank of P is 1.*

We reduce the equality knapsack problem, which is formally stated below, to the problem of deciding emptiness of the Chvátal closure of a rational polytope given by its linear description.

Equality Knapsack Problem (see [21]). Given positive integers $a_1, \ldots, a_n, b$, is there a set of nonnegative integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = b$?

Without loss of generality, we assume that $a_1, \ldots, a_n$ are relatively prime. We follow the idea behind Cornuéjols and Li's construction ([10, 11], Lemma 1), where they first construct some points using the input data for an instance of the equality knapsack problem and then take their convex hull to construct a rational polytope. Although the polytopes generated from their construction are not necessarily contained in the unit hypercube, we are able to refine their idea and choose our points in the unit hypercube as described in the next lemma. Theorem 26 immediately follows from it.

Lemma 28. *Given an equality knapsack instance of n positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time generate the linear description of a rational polytope $P \subseteq [0,1]^{n+4}$ contained in the unit hypercube satisfying the following:*

(a) P can be chosen to be the convex hull of $n + 10$ points in $[0,1]^{n+4}$.

(b) $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \in P$ but P contains no integer point.

(c) P is full-dimensional.

(d) There exists a solution to the equality knapsack instance if and only if there exists a Chvátal inequality of P that separates $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.

(e) There exists a solution to the equality knapsack instance if and only if the Chvátal closure of P is empty and the number of Chvátal inequalities to certify this is exactly 2.

Proof. Let a rational polytope $P \subseteq [0,1]^{n+4}$ be defined as the convex hull of the following $n + 10$

points $v^1, \cdots, v^{n+10} \in [0,1]^{n+4}$:

$$
\begin{array}{rclllllllll}
v^1 & := & (& \frac{1}{2b}, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^2 & := & (& 0, & \frac{1}{2b}, & \cdots, & 0, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
 & & & & & & & \vdots & & & & & \\
v^n & := & (& 0, & 0, & \cdots, & 0, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^{n+1} & := & (& 0, & 0, & \cdots, & 0, & 0, & 0, & 1/2, & 1/2, & 1/2 &) \\
v^{n+2} & := & (& 1, & 1, & \cdots, & 1, & 1, & 1, & 1/2, & 1/2, & 1/2 &) \\
v^{n+3} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1, & 1 &) \\
v^{n+4} & := & (& 1/4, & 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 &) \\
v^{n+5} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1, & 1/2 &) \\
v^{n+6} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0, & 1/2 &) \\
v^{n+7} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1/2, & 1, & 1 &) \\
v^{n+8} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1/2, & 0, & 0 &) \\
v^{n+9} & := & (& \frac{a_1}{2b}, & \frac{a_2}{2b}, & \cdots, & \frac{a_{n-1}}{2b}, & \frac{a_n}{2b}, & 0, & 0, & \frac{1}{2}-\frac{1}{4b}, & 0 &) \\
v^{n+10} & := & (& 1-\frac{a_1}{2b}, & 1-\frac{a_2}{2b}, & \cdots, & 1-\frac{a_{n-1}}{2b}, & 1-\frac{a_n}{2b}, & 1, & \frac{1}{2}+\frac{1}{4b}, & 0, & 0 &)
\end{array}
$$

Let $u := \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Notice that $u = \frac{1}{2}v^{n+1} + \frac{1}{2}v^{n+2}$, so u is contained in P. In addition, none of $v^1, \ldots, v^{n+10}$ is contained in $\{0,1\}^{n+4}$, so P contains no integer point. This shows that P satisfies (b).

Claim 1. *P is full-dimensional.*

Proof of Claim. It is easy to show that the $n+4$ vectors in $\{v^i - v^{n+1} : i = 1, \ldots, n, n+2, n+3, n+5, n+7\}$ are linearly independent. Then the $n+5$ points $v^1, \ldots, v^n, v^{n+1}, v^{n+2}, v^{n+3}, v^{n+5}, v^{n+7}$ are affinely independent, thereby proving that the dimension of P is $n+4$, as required. $\Diamond$

By Claim 1, we know that P satisfies (c). Claim 1 also implies that we can compute the linear description of P in polynomial time, as stated in the following claim.

Claim 2. *The linear description of P can be obtained in polynomial time.*

Proof of Claim. Since P is full-dimensional, the number of facets of P is at most $\binom{n+10}{n+4} \leq n^6$. Given $n+4$ affinely independent points among $v^1, \cdots, v^{n+10}$, we can compute the hyperplane containing these $n+4$ points using the Gaussian elimination method. Since the encoding size of each v^i is polynomial in $\log a_1, \cdots, \log a_n, \log b$, and n, the complexity of the hyperplane is also polynomially bounded by the input encoding size. Therefore, we can find each facet of P in polynomial time. $\Diamond$

To prove that P satisfies (d) and (e), we need the following two claims:

Claim 3. *If there exists a solution to the equality knapsack instance, then the Chvátal closure of P is empty and the number of Chvátal inequalities to certify this is exactly 2.*

Proof of Claim. Let $(w_1, \cdots, w_n)$ be a solution to the knapsack instance. Then $\sum_{i=1}^n a_i w_i = b$ and $w_i \geq 0$ for $i \in [n]$. Let $d := (w_1, \cdots, w_n, -\sum_{i=1}^n w_i, 1, -1, 1) \in \mathbb{Z}^{n+4}$. Notice that $w_k \leq a_k w_k \leq \sum_{i=1}^n a_i w_i = b$, so we get $\frac{w_k}{2b} \leq \frac{1}{2}$. Since $b > 1$, we know that $0 < \frac{1}{2b} \leq \frac{1}{4}$. Thus, $0 < dv^k = \frac{w_k}{2b} + \frac{1}{2b} < 1$ for $k \in [n]$. It is easy to show that $dv^{n+1} = dv^{n+2} = dv^{n+5} = dv^{n+6} = dv^{n+7} = dv^{n+8} = \frac{1}{2}$, $dv^{n+4} = \frac{1}{4}$, and $dv^{n+3} = 1$. In addition, $dv^{n+9} = dv^{n+10} = \frac{1}{4b}$. That means $0 < dv^i < 1$ for $i \neq n+3$ and $dv^{n+3} = 1$. Then, $dx > 0$ is valid for P, and we obtain its corresponding Chvátal inequality $dx \geq 1$. In fact, $P \cap \{x \in \mathbb{R}^{n+4} : dx \geq 1\} = \{v^{n+3}\}$, because v^{n+3} is the only vertex of P that is not cut off by $dx \geq 1$. Notice that $x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} \leq \frac{7}{2}$ is also valid for P. Then $x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} \leq 3$ is valid for P', and v^{n+3} violates this inequality. Therefore, $P \cap \{x \in \mathbb{R}^n : dx \geq 1,\ x_{n+3} + x_{n+2} + x_{n+3} + x_{n+4} \leq 3\}$ is empty. Hence, the Chvátal closure of P is empty and the number of Chvátal inequalities to certify this is 2. $\Diamond$

Claim 4. *If there exists a Chvátal inequality separating $u = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, then there exists a solution to the equality knapsack instance.*

Proof of Claim. There is a valid inequality $dx \leq d_0 + \epsilon$ for P such that $(d, d_0) \in \mathbb{Z}^{n+5}$, $0 < \epsilon < 1$, and $du > d_0$. We claim that d and d_0 satisfy the following five properties:

1) $d_{n+1} = -\sum_{i=1}^{n} d_i$.

2) $d_0 = -1$.

3) $d_{n+2} = d_{n+4} = -1$ and $d_{n+3} = 1$.

4) $\sum_{i=1}^{n} a_i d_i = -b$.

5) $d_i \leq 0$ for $i \in [n]$.

Then, $(-d_1, \cdots, -d_n)$ is a solution to the equality knapsack instance.

Since $d_0 < du \leq d_0 + \epsilon < d_0 + 1$, we get $d_0 < \frac{1}{2}\sum_{i=1}^{n+4} d_i < d_0 + 1$. In addition, we know that $dv^k \leq d_0 + \epsilon < d_0 + 1$ for $k \in [n+10]$. The integrality of $\sum_{i=1}^{n+4} d_i$ implies that $\frac{1}{2}\sum_{i=1}^{n+4} d_i$ should be equal to $d_0 + \frac{1}{2}$, and thus we get $\sum_{i=1}^{n+4} d_i = 2d_0 + 1$ and $du = d_0 + \frac{1}{2}$. Consider dv^{n+1} and dv^{n+2}:

$$d_0 + 1 > dv^{n+1} = du - \frac{1}{2}\sum_{i=1}^{n+1} d_i = d_0 + \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{n+1} d_i, \tag{1}$$

$$d_0 + 1 > dv^{n+2} = du + \frac{1}{2}\sum_{i=1}^{n+1} d_i = d_0 + \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n+1} d_i. \tag{2}$$

By (1) and (2), we get $-1 < \sum_{i=1}^{n+1} d_i < 1$. Since $\sum_{i=1}^{n+1} d_i$ is an integer, $\sum_{i=1}^{n+1} d_i = 0$ and the first property is satisfied. Then we know that $d_{n+2} + d_{n+3} + d_{n+4} = 2d_0 + 1$. Now, consider dv^{n+3} and dv^{n+4}:

$$d_0 + 1 > dv^{n+3} = du + \frac{1}{2}(d_{n+2} + d_{n+3} + d_{n+4}) = 2d_0 + 1, \tag{3}$$

$$d_0 + 1 > dv^{n+4} = \frac{1}{2}du = \frac{1}{2}d_0 + \frac{1}{4}. \tag{4}$$

By (3) and (4), we obtain $-\frac{3}{2} < d_0 < 0$ and thus $d_0 = -1$. So the second property holds and $d_{n+2} + d_{n+3} + d_{n+4} = -1$. Consider dv^{n+5} and dv^{n+6}:

$$d_0 + 1 > dv^{n+5} = du + \frac{1}{2}(d_{n+2} + d_{n+3}) = d_0 + \frac{1}{2} + \frac{1}{2}(d_{n+2} + d_{n+3}), \tag{5}$$

$$d_0 + 1 > dv^{n+6} = du - \frac{1}{2}(d_{n+2} + d_{n+3}) = d_0 + \frac{1}{2} - \frac{1}{2}(d_{n+2} + d_{n+3}). \tag{6}$$

By (5) and (6), we know that $-1 < d_{n+2} + d_{n+3} < 1$. So, $d_{n+2} + d_{n+3} = 0$. Similarly, we get $d_{n+3}+d_{n+4} = 0$ by considering dv^{n+7} and dv^{n+8}. Together with the observation $d_{n+2}+d_{n+3}+d_{n+4} = -1$, we get $d_{n+3} = 1$ and $d_{n+2} = d_{n+4} = -1$. Hence, the third property is satisfied. To prove the fourth property, we consider dv^{n+9} and dv^{n+10}:

$$dv^{n+9} = \frac{1}{2b}\sum_{i=1}^{n} a_i d_i + (\frac{1}{2} - \frac{1}{4b}) < d_0 + 1 = 0, \tag{7}$$

which implies that $\sum_{i=1}^{n} a_i d_i < -b + \frac{1}{2}$, so $\sum_{i=1}^{n} a_i d_i \leq -b$ since the sum is an integer;

$$dv^{n+10} = \sum_{i=1}^{n+1} d_i - \frac{1}{2b}\sum_{i=1}^{n} a_i d_i - (\frac{1}{2} + \frac{1}{4b}) = -\frac{1}{2b}\sum_{i=1}^{n} a_i d_i - (\frac{1}{2} + \frac{1}{4b}) < d_0 + 1 = 0, \tag{8}$$

which implies that $\sum_{i=1}^{n} a_i d_i > -b - \frac{1}{2}$, so $\sum_{i=1}^{n} a_i d_i \geq -b$ since the sum is an integer. Therefore, $\sum_{i=1}^{n} a_i d_i = -b$. Lastly, consider dv^k for $k \in [n]$:

$$dv^k = \frac{1}{2b} d_k - \frac{1}{2b} < d_0 + 1 = 0. \tag{9}$$

By (9), $d_k < 1$ and thus $d_k \leq 0$. $\Diamond$

Claim 3 proves one direction of (d) and that of (e), and Claim 4 proves the other directions of (d) and (e). Therefore, (d) and (e) are also satisfied, as required. $\square$

4.2 The case of simplices

Theorem 29. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational simplex. It is NP-complete to decide whether the Chvátal closure of P is empty, even when P contains no integer point.*

To prove Theorem 26, we constructed a polytope that is the convex hull of $n+10$ points in $[0,1]^{n+4}$, but a simplex in $\mathbb{R}^{n+4}$ has less vertices. By allowing to choose some points sitting outside the hypercube, we are able to reduce the number of points so that we can construct rational simplices as described in the following lemma. Lemma 30 is very similar to Lemma 28, but its proof is more technical and involves a longer argument. Instead of adding the whole proof in this paper, we just give our construction here and we refer the reader to Lee's dissertation [30] for more details.

Lemma 30. *Given an equality knapsack instance of n positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time generate the linear description of a rational simplex $P \subseteq \mathbb{R}^{n+1}$ and a point $u \in P$ satisfying the following:*

(a) P contains no integer point.

(b) There exists a solution to the equality knapsack instance if and only if there exists a Chvátal inequality of P that separates u.

(c) There exists a solution to the equality knapsack instance if and only if the Chvátal closure of P is empty and the number of Chvátal inequalities to certify this is exactly 2.

Proof. Let $P \in \mathbb{R}^{n+1}$ be a rational polytope defined as the convex hull of the following $n+2$ points $v^1, \ldots, v^{n+2} \in \mathbb{R}^{n+1}$:

$$\begin{array}{rlllllll}
v^1 := & (& \frac{1}{2rB}, & 0, & \cdots, & 0, & \frac{1}{2r} - \frac{b}{2rBA} &) \\
v^2 := & (& 0, & \frac{1}{2rB}, & \cdots, & 0, & \frac{1}{2r} - \frac{b}{2rBA} &) \\
 & & & & \vdots & & & \\
v^n := & (& 0, & 0, & \cdots, & \frac{1}{2rB}, & \frac{1}{2r} - \frac{b}{2rBA} &) \\
v^{n+1} := & (& ra_1, & ra_2, & \cdots, & ra_n & -rb + \frac{1}{2} &) \\
v^{n+2} := & (& -ra_1, & -ra_2, & \cdots, & -ra_n & rb + 1 &)
\end{array}$$

where A and B denote $\sum_{i=1}^{n} a_i$ and the smallest integer greater than $\frac{b}{A}$, respectively and $r := 2016b + \frac{1}{2b}$. It is easy to show that $v^1, \ldots, v^{n+2}$ are affinely independent, thereby proving that P is a rational simplex. Let $u := (\frac{a_1}{6rBA}, \cdots, \frac{a_n}{6rBA}, \frac{1}{6r} + \frac{1}{2} - \frac{b}{6rBA})$. Then, u is contained in P, and P together with u satisfies (a), (b), and (c). $\square$

Theorem 29 follows Lemma 30 and implies the following corollary.

Corollary 31. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational simplex. It is NP-hard to decide whether the Chvátal rank of P is 1.*

4.3 Optimization and separation over Chvátal closure

Eisenbrand [19] showed that the separation problem over the Chvátal closure of a rational polyhedron given by its linear description is NP-hard, answering an early question of Schrijver [36]. He derived this result as an extension of a result by Caprara and Fischetti [8].

Separation problem over the Chvátal closure. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, and let $\bar{x} \in \mathbb{Q}^n$ be a rational point. Then either show that $\bar{x} \in P'$ or find a valid Chvátal inequality $dx \leq d_0$ for P' such that $d\bar{x} > d_0$.

According to a general result given by Grötschel, Lovász and Schrijver [23], this problem is equivalent to its optimization version up to a polynomial time overhead.

Optimization problem over the Chvátal closure. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, and let $c \in \mathbb{Q}^n$ be a rational objective coefficient vector. Then find a point $x^* \in P'$ satisfying $cx^* = \max\{cx : x \in P'\}$, or show $P' = \emptyset$, or find a ray z of the recession cone of P' for which cz is positive.

As an immediate corollary of Theorem 26 and Theorem 29, we obtain the following, which answers an open question raised by Letchford, Pokutta, and Schulz [32].

Theorem 32. *The optimization and separation problems over the Chvátal closure of a rational polytope given by its linear description are NP-hard, even when the input polytope is contained in the unit hypercube or is a rational simplex.*

4.4 Deciding whether adding a certain number of Chvátal cuts can yield the integer hull

Theorem 32 indicates that the number of Chvátal cuts of a rational polytope to obtain its Chvátal closure can be, in general, super-polynomial in the encoding size of the polytope. It seems rare that the Chvátal closure of a rational polytope is obtained by adding a constant number of (rank-1) Chvátal cuts. Besides, we know that the Chvátal rank of a rational polytope can be larger than 1, so it seems rarer that we can obtain the integer hull of a rational polytope by adding a constant number of Chvátal cuts. Given a rational polytope, can we easily decide whether its integer hull 'cannot' be obtained by adding a fixed number of (rank-1) Chvátal cuts? The answer to this question is probably 'no'. We remark the following, which can be derived from Lemma 28 and a result of Mahajan and Ralphs ([33], Proposition 3.4).

Remark 33. *Let $P = \{x \in [0,1]^n : Ax \le b\}$ be a rational polytope contained in the unit hypercube, and let k be a positive integer. Deciding whether we can obtain the integer hull of P by adding at most k (rank-1) Chvátal inequalities to the linear description of P is NP-hard.*

Proof. If $k \ge 2$, we know from Lemma 28 that the decision problem is NP-hard. To prove that the problem is still NP-hard even when $k = 1$, we borrow the construction of Mahajan and Ralphs [33]. They constructed a polytope using the data for an instance of the partition problem, which is NP-hard and stated below.

Partition Problem (see [21]). Given positive integers $a_1, \cdots, a_n$, is there a subset K of the set of indices $[n]$ such that $\sum_{i\in K} a_i = \sum_{j\in[n]\setminus K} a_j$?

Let $a_1, \cdots, a_n$ be the input for an instance of the partition problem. Let $\widetilde{a_k} := \frac{1}{\sum_{j=1}^n a_j} a_k$ for $k \in [n]$. Let P be the convex hull of the following $n+4$ points in $[0,1]^{n+2}$:

$$
\begin{array}{rlllllll}
v^1 := & (& \frac{1}{2} + \frac{1}{2(n+1)}, & \frac{1}{2(n+1)}, & \cdots, & \frac{1}{2(n+1)}, & 0, & 0 \quad) \\
v^2 := & (& \frac{1}{2(n+1)}, & \frac{1}{2} + \frac{1}{2(n+1)}, & \cdots, & \frac{1}{2(n+1)}, & 0, & 0 \quad) \\
 & & & & \vdots & & & \\
v^n := & (& \frac{1}{2(n+1)}, & \frac{1}{2(n+1)}, & \cdots, & \frac{1}{2} + \frac{1}{2(n+1)}, & 0, & 0 \quad) \\
v^{n+1} := & (& \widetilde{a_1}, & \widetilde{a_2}, & \cdots, & \widetilde{a_n}, & 1, & 1 \quad) \\
v^{n+2} := & (& \widetilde{a_1}, & \widetilde{a_2}, & \cdots, & \widetilde{a_n}, & \frac{1}{2} - \frac{1}{2\sum_{j=1}^n a_j}, & 0 \quad) \\
v^{n+3} := & (& \widetilde{a_1}, & \widetilde{a_2}, & \cdots, & \widetilde{a_n}, & 0, & \frac{1}{2} - \frac{1}{2\sum_{j=1}^n a_j} \quad) \\
v^{n+4} := & (& 0, & 0, & \cdots, & 0, & \frac{1}{2}, & 0 \quad)
\end{array}
$$

We show that the Chvátal closure of P is empty, meaning that the integer hull of P is empty. Let $d := (1, \cdots, 1, 1, -1)$. Then $dv^i = 1 - \frac{1}{2(n+1)}$ for $i \in [n]$. Besides, we get $dv^{n+1} = 1$, $dv^{n+2} = \frac{3}{2} - \frac{1}{2\sum_{j=1}^n a_j}$, $dv^{n+3} = \frac{1}{2} + \frac{1}{2\sum_{j=1}^n a_j}$, and $dv^{n+4} = \frac{1}{2}$. Then $0 < dx < 2$ is valid for all $x \in P$, and thus $dx = 1$ is valid for P'. Since $0 < a_1 < \sum_{j=1}^n a_j$, $0 < \widetilde{a_1} < 1$. This implies that the first component of each v^i be less than 1, so $x_1 \le 0$ is valid for P'. Notice that $P \cap \{x \in [0,1]^{n+2} : x_1 \le 0\} = \{v^{n+4}\}$. Besides, $dv^{n+4} = \frac{1}{2} \ne 1$. Since $P' \subseteq P \cap \{x \in [0,1]^{n+2} : dx = 1,\ x_1 \le 0\} = \emptyset$, we have that $P' = \emptyset$, as required.

The integer hull of P, which is empty, is obtained by adding a Chvátal inequality $\pi x \le \pi_0$ if and only if $\pi x < \pi_0 + 1$ is valid for P and every point in P violates $\pi x \le \pi_0$ (or equivalently,

$P \subseteq \{x \in \mathbb{R}^{n+2} : \pi_0 < \pi x < \pi_0 + 1\}$). Mahajan and Ralphs ([33], Proposition 3.4) proved that there is $(\pi, \pi_0) \in \mathbb{Z}^{n+3}$ such that $P \subseteq \{x \in \mathbb{R}^{n+2} : \pi_0 < \pi x < \pi_0 + 1\}$ if and only if there exists a subset K of $[n]$ such that $\sum_{i \in K} a_i = \sum_{j \in [n] \setminus K} a_j$. Therefore, the problem of deciding if we can obtain the integer hull of a rational polytope by adding at most k Chvátal inequalities to the linear description of P is NP-hard, even when $k = 1$. $\square$

Note from the proof of Remark 33 that k is not necessarily a constant. Observe that the construction of Mahajan and Ralphs used to prove Remark 33 is in the spirit of our constructions in Lemmas 28 and 30, but one difference is that the Chvátal closure of a polytope from their construction is always empty.

The decision problem remains NP-hard, even when the input polytope is a rational simplex, as stated in the following remark. It follows from Lemma 30 and Proposition 3.2 in [33].

Remark 34. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational simplex, and let k be a positive integer. Deciding if we can obtain the integer hull of P by adding at most k Chvátal inequalities to the linear description of P is NP-hard.*

Acknowledgments: We would like to thank two anonymous referees for their excellent suggestions to improve an earlier version of this paper. In particular, we are grateful to one of the referees for pointing out a missing assumption in the statement of Theorem 18 and providing Example 19. This work was supported in part by NSF grant CMMI 1560828 and ONR grant 00014-18-12129

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