On Some Polytopes Contained in the 0,1 Hypercube that Have a Small Chvátal Rank

Gérard Cornuéjols* Dabeen Lee*

April 2016, revised July 2017

Abstract

In this paper, we consider polytopes P that are contained in the unit hypercube. We provide conditions on the set of 0,1 vectors not contained in P that guarantee that P has a small Chvátal rank. Our conditions are in terms of the subgraph induced by these infeasible 0,1 vertices in the skeleton graph of the unit hypercube. In particular, we show that when this subgraph contains no 4-cycle, the Chvátal rank is at most 3; and when it has tree width 2, the Chvátal rank is at most 4. We also give polyhedral decomposition theorems when this graph has a vertex cutset of size one or two.

1 Introduction

Let $H_n := [0,1]^n$ denote the 0,1 hypercube in $\mathbb{R}^n$. Let $P \subseteq H_n$ be a polytope. Let $S := P \cap \{0,1\}^n$ denote the set of 0,1 vectors in P. If an inequality $cx \geq d$ is valid for P for some $c \in \mathbb{Z}^n$, then $cx \geq \lceil d \rceil$ is valid for $\mathrm{conv}(S)$ since it holds for any $x \in P \cap \mathbb{Z}^n$. Chvátal [4] introduced an elegant notion of closure as follows.

$$P' = \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \geq \lceil \min\{cx : \ x \in P\} \rceil\}$$

is the *Chvátal closure* of P. Chvátal's cuts are equivalent to Gomory's fractional cuts [13] and the closure defined by Gomory's fractional cuts is the same as the Chvátal closure. Chvátal [4] proved that the closure of a rational polytope is, again, a rational polytope, and Schrijver [17] later proved the polyhedrality of the closure of general rational polyhedron. Recently, Dunkel and Schulz [10] proved that the Chvátal closure of an irrational polytope is a rational polytope, answering a long-standing open question raised by Schrijver [17]. Independently and almost simultaneously to this work, Dadush, Dey, and Vielma [9] showed the polyhedrality of the Chvátal closure of any convex compact set. Let $P^{(0)}$ denote P and $P^{(t)}$ denote $(P^{(t-1)})'$ for $t \geq 1$. Then $P^{(t)}$ is the tth Chvátal closure of P, and the smallest k such that $P^{(k)} = \mathrm{conv}(S)$ is called the *Chvátal rank* of P. Chvátal [4] proved that the Chvátal rank of every rational polytope is finite, and Schrijver [17] later proved that the Chvátal rank of every rational polyhedron is also finite.

An upper bound of $O(n^3 \log n)$ on the Chvátal rank of a polytope $P \subseteq H_n$, which only depends on the dimension n, was first given by Bockmayr, Eisenbrand, Hartmann, and Schulz [3].

*Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15206, USA, {gc0v,dabeenl}@andrew.cmu.edu

Later, Eisenbrand and Schulz [11] improved this to $O(n^2 \log n)$. Recently, Rothvoss and Sanitá [16] constructed a polytope $P \subseteq H_n$ whose Chvátal rank is $\Omega(n^2)$. However, some special polytopes arising in combinatorial optimization problems have small Chvátal rank; for example, the fractional matching polytope has Chvátal rank 1. Hartmann, Queyranne and Wang [14] gave a necessary and sufficient condition for a facet-defining inequality of $\text{conv}(S)$ to have rank 1. In this paper, we investigate 0,1 polytopes with a Chvátal rank that is a small constant or grows slowly with n.

The *skeleton* of H_n is the graph $G := (V, E)$ whose vertices correspond to the 2^n extreme points of H_n and whose edges correspond to the 1-dimensional faces of H_n, namely the $n2^{n-1}$ line segments joining 2 extreme points of H_n that differ in exactly 1 coordinate. Let $\bar{S} := \{0,1\}^n \setminus S$ denote the set of 0,1 vectors that are not in P. Consider the subgraph $G(\bar{S})$ of G induced by the vertices in $\bar{S}$. In this paper, we give conditions on $G(\bar{S})$ that guarantee a small Chvátal rank. For example, we show that when $\bar{S}$ is a stable set in G, the Chvátal rank of P is at most 1; when each connected component of $G(\bar{S})$ is a cycle of length greater than 4 or a path, the Chvátal rank is at most 2; when $G(\bar{S})$ contains no 4-cycle, the Chvátal rank is at most 3; when the tree width of $G(\bar{S})$ is 2, the Chvátal rank is at most 4. In Section 5, we give polyhedral decomposition theorems for $\text{conv}(S)$ when $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2. These decomposition theorems are used to prove the results on forests and on graphs of tree width two mentioned above. In section 7, we give an upper bound on the Chvátal rank of P that depends on the cardinality of $\bar{S}$. In particular, we show that if only a constant number of 0,1 vectors are infeasible, then the Chvátal rank of P is also a constant. We then give a superpolynomial range on the number of infeasible 0,1 vectors where the upper bound of $O(n^2 \log n)$ on the Chvátal rank obtained by Eisenbrand and Schulz can be slightly improved to $O(n^2 \log \log n)$. Finally, in Section 8, we show that optimizing a linear function over S is polynomially solvable when the Chvátal rank of a canonical polytope for S is constant.

Although our results are mostly of theoretical interest, we mention two applications. The first is to the theory of clutters with the packing property. Abdi, Cornuéjols and Pashkovich [1] constructed minimal nonpacking clutters from 0,1 polytopes with Chvátal rank at most 2. In particular, a 0,1 polytope in $[0,1]^5$ where the infeasible 0,1 vectors induce 2 cycles of length 8 and the remaining 16 points are feasible lead to the discovery of a new minimally nonpacking clutter on 10 elements. Another application occurs when S is the set of 0,1 vectors whose sum of entries is congruent to i modulo k. The cases $k = 2$ and $k = 3$ are discussed in Sections 3.1 and 4.

2 Preliminaries

In this section, we present two results, totally unimodularity, and some notation that will be used later in the paper.

2.1 Two results

Lemma 1. *Consider a half-space $D := \{x \in \mathbb{R}^n : \ dx \geq d_0\}$. Let $T := D \cap \{0,1\}^n$ and $\bar{T} := \{0,1\}^n \setminus T$. For every face F of H_n, the graph $G(F \cap \bar{T})$ is connected. In particular $G(\bar{T})$ is a connected graph.*

Proof. Suppose that $G(F \cap \bar{T})$ is disconnected. Let $\bar{x}$ and $\bar{y}$ be vertices in distinct connected components of $G(F \cap \bar{T})$ with the property that the number of distinct coordinate values in the vectors $\bar{x}$ and $\bar{y}$ is as small as possible. Let j be a coordinate in which $\bar{x}$ and $\bar{y}$ differ and assume that $\bar{x}_j = 0$ and $\bar{y}_j = 1$. If $d_j < 0$, then $\bar{x} + e_j \in \bar{T}$ and is contained in the same component as $\bar{x}$. Besides, it has one more component in common with $\bar{y}$ than $\bar{x}$. Similarly, if $d_j \geq 0$, then $\bar{y} - e_j \in \bar{T}$ and has one more component in common with $\bar{x}$ than $\bar{y}$. In either case, we get a contradiction. $\square$

Theorem 2 (AADK [2])**.** *Let P be a polytope and let $G = (V, E)$ be its skeleton. Let $S \subset V$, $\bar{S} = V \setminus S$, and $\bar{S}_1, \ldots, \bar{S}_t$ be a partition of $\bar{S}$ such that there are no edges of G connecting $\bar{S}_i, \bar{S}_j$ for all $1 \leq i < j \leq t$. Then $\mathrm{conv}(S) = \bigcap_{i=1}^{t} \mathrm{conv}(V \setminus \bar{S}_i)$.*

Theorem 2, due to Angulo, Ahmed, Dey and Kaibel [2], shows that we can consider each connected component of $G(\bar{S})$ separately when studying $\mathrm{conv}(S)$. In Sections 5.1 and 5.2, we give similar theorems in the case where $P \subset [0,1]^n$ and $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2.

2.2 Totally unimodularity

A matrix A is totally unimodular if every square submatrix has determinant -1, 0, or 1. It is known that both duplicating a row and multiplying a row by -1 preserve totally unimodularity. If A is totally unimodular, it is easy to observe that $P := \{x \in \mathbb{R}^n : \; Ax \geq b\}$ for any vector b with integer entries is always integral. In fact, replacing an inequality $a^i x \geq b_i$ of the system $Ax \geq b$ by either $a^i x \leq b_i$ or $a^i x = b_i$ preserves the integrality of P. We can easily observe the following, using a characterization of totally unimodular matrices due to Ghouila-Houri [12].

Remark 3. *Let A be a 0,1 matrix.*

- *If A has at most 2 rows, then A is totally unimodular.*
- *If* $A = \begin{pmatrix} 1 & ? & ? & 0\cdots 0 & 1\cdots 1 \\ 0 & 1 & 0 & 0\cdots 0 & 1\cdots 1 \\ 0 & 0 & 1 & 0\cdots 0 & 1\cdots 1 \end{pmatrix}$*, then A is totally unimodular.*
- *If* $A = \begin{pmatrix} 1 & 1 & 0 & 0\cdots 0 & 1\cdots 1 \\ 0 & 1 & 0 & 1\cdots 1 & 1\cdots 1 \\ 0 & 0 & 1 & 1\cdots 1 & 1\cdots 1 \end{pmatrix}$*, then A is totally unimodular.*
- *If A is totally unimodular, then so is* $\begin{pmatrix} A \\ I \\ -I \end{pmatrix}$.

In particular, if a system of linear inequalities consists of $0 \leq x \leq 1$ plus two additional constraints which have only 0,1 coefficients, then its constraint matrix is totally unimodular by Remark 3 and thus the linear system defines an integral polyhedron.

2.3 Notation

Throughout the paper, we will use the following notation. Let $N := \{1, \ldots, n\}$. For a 0,1 vector $\bar{x}$, we denote by $\bar{x}^i$ the 0,1 vector that differs from $\bar{x}$ only in coordinate $i \in N$, and more generally, for $J \subseteq N$, we denote by $\bar{x}^J$ the 0,1 vector that differs from $\bar{x}$ exactly in the coordinates J. Besides, let e^i denote the ith unit vector for $i \in N$.

3 Some polytopes with small Chvátal rank

To prove results on $P \subset [0,1]^n$, we will work with a canonical polytope Q_S that has exactly the same set S of feasible $0,1$ vectors. The description of Q_S is as follows.

$Q_S := \{x \in [0,1]^n : \sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \frac{1}{2} \text{ for } \bar{x} \in \bar{S}\}$.

The reason for working with Q_S is that the Chvátal rank of P is always less than or equal to the Chvátal rank of Q_S. Indeed, in Lemma 7, we show that their kth Chvátal closures satisfy $P^{(k)} \subseteq Q_S^{(k)}$ for $k \geq 1$. Furthermore, we have a good handle on $Q_S^{(k)}$ because of the following lemma. The middle point of a k-dimensional 0,1 hypercube $[0,1]^k$ is defined as the vector in $\mathbb{R}^k$ all of whose entries are equal to $\frac{1}{2}$.

Lemma 4 (CCH [5])**.** *The middle points of all $(k+1)$-dimensional faces of H_n belong to the kth Chvátal closure $Q_S^{(k)}$ for $0 \leq k \leq n-1$.*

Chvátal, Cook and Hartmann [5] proved this result when $S = \emptyset$. The result clearly follows for general $S \subseteq \{0,1\}^n$ since $Q_\emptyset \subseteq Q_S$ implies $Q_\emptyset^{(k)} \subseteq Q_S^{(k)}$. We also remark that Q_S when $S = \emptyset$ was studied by Cornuéjols and Li [7] and that it is in the spirit of Pokutta and Schulz' work [15]. In this section, we provide the descriptions for $Q_S^{(1)}$, $Q_S^{(2)}$, $Q_S^{(3)}$.

3.1 Chvátal rank 1

Theorem 5. *The polytope Q_S has Chvátal rank 1 if and only if $\bar{S}$ is a nonempty stable set in G.*

In particular, if S contains all the 0,1 vertices of H_n with an even (odd resp.) number of 1s, then P has Chvátal rank at most 1. Theorem 5 is proved by first characterizing $Q_S^{(1)}$. For each $\bar{x} \in \bar{S}$, we call

$$\sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{1}$$

the *vertex inequality* corresponding to $\bar{x}$. For example, when $\bar{x} = 0$, the corresponding vertex inequality is $x_1 + x_2 + \ldots + x_n \geq 1$. Note that each vertex inequality cuts off exactly the vertex $\bar{x}$ and it goes through all the neighbors of $\bar{x}$ on H_n.

Theorem 6. $Q_S^{(1)}$ *is the intersection of $[0,1]^n$ with the half-spaces defined by the vertex inequalities* (1) *for $\bar{x} \in \bar{S}$.*

Proof. Let e be an edge of H_n. Because the middle point of edge e belongs to Q_S, any valid inequality $dx \geq d_0$ for Q_S cuts off at most one of the two endpoints of e. Let $\bar{T}$ denote the set of $0,1$ vectors that satisfy $dx < d_0$. Since $G(\bar{T})$ is a connected graph by Lemma 1, it follows that every valid inequality $dx \geq d_0$ for Q_S cuts off at most one vertex $\bar{x}$ of H_n. Applying the Chvátal rounding procedure to $dx \geq d_0$, the resulting Chvátal inequality cannot cut off any vertex of H_n other than $\bar{x}$. In particular it cannot cut off the neighbors of $\bar{x}$ on H_n. The inequalities that cut off $\bar{x}$ but none of its neighbors on H_n are implied by $\sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$ and $0 \leq x \leq 1$. Furthermore, this inequality is a rank 1 Chvátal cut for Q_S since it is obtained by rounding $\sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \frac{1}{2}$. This shows that $Q_S^{(1)} = \{x \in [0,1]^n : \sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \text{ for } \bar{x} \in \bar{S}\}$. □

Proof of Theorem 5. Assume all connected components of $G(\bar{S})$ have cardinality 1. By Theorem 2, $\text{conv}(S) = \bigcap_{\bar{x} \in \bar{S}} \{x \in H_n \,:\, \sum_{j=1}^n \bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j \geq 1\}$, which is $Q_S^{(1)}$ by Theorem 6.

Assume some connected component of $G(\bar{S})$ has cardinality 2 or greater, i.e. $G(\bar{S})$ contains at least 1 edge. Without loss of generality, we may assume that $\{0, e^1\} \subseteq \bar{S}$ where e^1 denotes the first unit vector. Then the point $(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$ belongs to $Q_S^{(1)}$ by Lemma 4 but not to $\text{conv}(S)$ since $\sum_{j=2}^n x_j \geq 1$ is valid for $\text{conv}(S)$. This shows $Q_S^{(1)} \neq \text{conv}(S)$. □

Next we show that the Chvátal rank of Q_S is an upper bound on the Chvátal rank of P.

Lemma 7. *The polytopes P and Q_S have the same set S of feasible 0,1 solutions, and the Chvátal rank of P is always less than or equal to the Chvátal rank of Q_S.*

Proof. The inequalities defining Q_S cut off the 0,1 vectors in $\bar{S}$ and no other. Therefore $S = Q_S \cap \{0,1\}^n$.

To prove the second part of the lemma, note that if two polytopes P and R have the same set of integer solutions and $P \subseteq R$, then the Chvátal rank of P is always less than or equal to the Chvátal rank of R. We will construct such a polytope R from P. For each $\bar{x} \in \bar{S}$, the linear program $\min_P \sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j)$ has a positive objective value. Therefore there exists $0 < \epsilon_{\bar{x}} \leq \frac{1}{2}$ such that the inequality $\sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \epsilon_{\bar{x}}$ is valid for P. Let $R := \{x \in [0,1]^n : \sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \epsilon_{\bar{x}} \text{ for } \bar{x} \in \bar{S}\}$.

Now the lemma follows by observing that R and Q_S have the same first Chvátal closure. Indeed $Q_S \subseteq R$ implies $Q_S^{(1)} \subseteq R^{(1)}$ and, applying the Chvátal procedure to the inequalities defining R, we get that $R^{(1)} \subseteq \{x \in [0,1]^n : \sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \text{ for } \bar{x} \in \bar{S}\} = Q_S^{(1)}$. □

3.2 Chvátal rank 2

Theorem 8. *For $n \geq 3$, the Chvátal rank of Q_S is 2 if and only if $G(\bar{S})$ contains a connected component of cardinality at least 2, and each connected component of $G(\bar{S})$ is either a cycle of length greater than 4 or a path.*

We postpone the proof of this theorem until the end of Section 3.3. First we provide an explicit characterization of $Q_S^{(2)}$. Let $\bar{x}, \bar{y} \in \bar{S}$ be two vertices of $G(\bar{S})$ such that they differ in exactly one coordinate. Using the notation introduced in Section 2.3, we write $\bar{y} = \bar{x}^i$, where i indexes the coordinate where $\bar{x}$ and $\bar{y}$ differ. The inequality

$$\sum_{j \in N \setminus \{i\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{2}$$

is called the *edge inequality* corresponding to edge $\bar{x}\bar{y}$. For example, when $\bar{x} = 0$ and $\bar{y} = e^1$, the corresponding edge inequality is $x_2 + x_3 + \ldots + x_n \geq 1$. The inequality (2) is the strongest inequality that cuts off $\bar{x}$ and $\bar{y}$ but no other vertex of H_n. Indeed, its boundary contains all $2(n-1)$ neighbors of $\bar{x}$ or $\bar{y}$ on H_n (other than $\bar{x}$ and $\bar{y}$ themselves). The next theorem states that vertex and edge inequalities are sufficient to describe the second Chvátal closure of Q_S.

Theorem 9. *$Q_S^{(2)}$ is the intersection of $Q_S^{(1)}$ with the half-spaces defined by the edge inequalities (2) for $\bar{x}, \bar{y} \in \bar{S}$ such that $\bar{x}\bar{y}$ is an edge of H_n.*

Proof. The 2-dimensional faces of H_n are 4-cycles, namely, squares. Because the center of each square belongs to $Q_S^{(1)}$ by Lemma 4, any valid inequality for $Q_S^{(1)}$ cuts off at most two vertices of each 2-dimensional face of H_n, and these two vertices are adjacent. Indeed, by Lemma 1, the graph induced by the vertices that are cut off is connected and this graph cannot contain a subpath of length 2 since any such path belongs to a square of H_n. This proves the claim. The tightest such valid inequalities are the edge inequalities.

Next we show that they are valid for $Q_S^{(2)}$. The edge inequalities can be obtained by applying the Chvátal procedure to vertex inequalities valid for $Q_S^{(1)}$ as follows. Let $\bar{x}\bar{y}$ be an edge in $G(\bar{S})$. Say $\bar{x}_i = 0$ and $\bar{y}_i = 1$. Then $x_i + \sum_{j \in N\setminus\{i\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$ and $-x_i + \sum_{j \in N\setminus\{i\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 0$ are valid for $Q_S^{(1)}$. Adding them and multiplying by $\frac{1}{2}$, it follows that the inequality $\sum_{j \in N\setminus\{i\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \frac{1}{2}$ is valid for $Q_S^{(1)}$. Applying the Chvátal procedure, $\sum_{j \in N\setminus\{i\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$ is valid for $Q_S^{(2)}$. □

Note that the edge inequality (2) dominates the vertex inequalities for $\bar{x} \in \bar{S}$ and for $\bar{y} \in \bar{S}$. Thus vertex inequalities are only needed for the isolated vertices of $G(\bar{S})$.

3.3 Chvátal rank 3

Theorem 11 below is the main result of this section. It characterizes $Q_S^{(3)}$.

4-cycles of $G(\bar{S})$ correspond to 2-dimensional faces of H_n that are squares. If $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell} \in \bar{S}$, then we say that $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$ is a square. Note that

$$\sum_{j \in N\setminus\{i,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{3}$$

is the strongest inequality cutting off exactly the four points of the square $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$. Indeed, the $4(n-2)$ neighbors of $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}$ in H_n (other than $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}$ themselves) all satisfy (3) at equality. We call (3) a *square inequality.* As an example, if $(0, e^1, e^2, e^1+e^2)$ is a square contained in $G(\bar{S})$, the corresponding square inequality is $x_3 + x_4 + \ldots + x_n \geq 1$.

If $\bar{x}$ and $t \geq 3$ of its neighbors $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$ all belong to $\bar{S}$, then we say that $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ is a *star.* The following *star inequality* is valid for $\mathrm{conv}(S)$.

$$\sum_{r=1}^{t}(\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \neq i_1,\ldots,i_t} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2. \tag{4}$$

Indeed, it cuts off exactly the vertices of the star, and goes through the other $n-t$ neighbors of $\bar{x}$ on H_n and the $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$. For example, if $(0, e^1, \ldots, e^t)$ is a star, then (4) is $x_1 + \ldots + x_t + 2(x_{t+1} + \ldots + x_n) \geq 2$.

The proof of Theorem 11 uses the following lemma.

Lemma 10. *Assume $\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t} \in \bar{S}$ for $t \geq 1$. If $t \geq 3$, i.e., $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ is a star, then* $\mathrm{conv}(S)$ *is completely defined by the corresponding star inequality together with the edge inequalities and the bounds* $0 \leq x \leq 1$. *If $t = 1$ or 2, then* $\mathrm{conv}(S)$ *is defined by edge inequalities and the bounds* $0 \leq x \leq 1$.

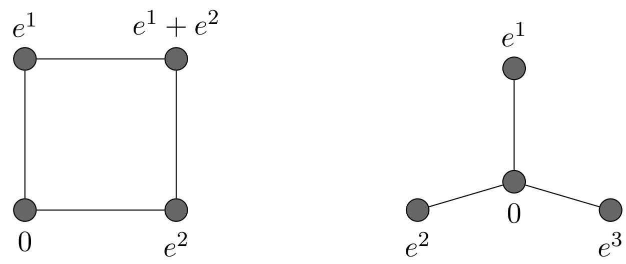

Figure 1: Square and star with $\bar{x} = 0$

Proof. We may assume that $\bar{x} = 0$, $\bar{S} = \{0, e^1, \ldots, e^t\}$ and $I := \{1, \ldots, t\}$.

If $t = n$, then S is the set of 0,1 vectors satisfying the system $\sum_{j=1}^{n} x_j \geq 2$ with $0 \leq x \leq 1$. This constraint matrix is totally unimodular by Remark 3. Therefore it defines an integral polytope, which must be $\mathrm{conv}(S)$.

If $t = 2$, we observe that $\{x \in [0,1]^n : \sum_{j \in N \setminus \{r\}} x_j \geq 1 \text{ for } r = 1, 2\}$ is an integral polytope. Indeed, the corresponding constraint matrix is also totally unimodular by Remark 3. Similarly if $t = 1$.

If $3 \leq t < n$, it is sufficient to show that $A := \{x \in [0,1]^n : \sum_{i \in I} x_i + 2\sum_{j \in N \setminus I} x_j \geq 2, \sum_{j \in N \setminus \{r\}} x_j \geq 1 \text{ for } 1 \leq r \leq t\}$ is an integral polytope. Let v be an extreme point of A. We will show that v is an integral vector. Since we assumed $n \geq 3$, A has dimension n and there exist n linearly independent inequalities active at v.

First, consider the case when the star inequality is active at v. If no edge inequality is active at v, then $n-1$ inequalities among $0 \leq x \leq 1$ are active at v. Since $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j = 2$, it follows that all coordinates of v are integral. Thus we may assume that an edge inequality $\sum_{j \in N \setminus \{1\}} x_j \geq 1$ is active at v. Consider the face F of A defined by setting this edge inequality and the star inequality as equalities. Clearly v is a vertex of F. Observe that the two equations defining F can be written equivalently as $\sum_{j \in N \setminus \{1\}} x_j = 1$ and $x_1 + \sum_{j \in N \setminus I} x_j = 1$. Furthermore, any other edge inequality $\sum_{j \in N \setminus \{r\}} x_j \geq 1$ is implied by $x \geq 0$ since it can be rewritten as $\sum_{j \in I \setminus \{1,r\}} x_j \geq 0$ using $x_1 + \sum_{j \in N \setminus I} x_j = 1$. This means that F is entirely defined by $0 \leq x \leq 1$ and the two equations $x_1 + \sum_{j \in N \setminus I} x_j = 1$ and $\sum_{j \in N \setminus \{1\}} x_j = 1$. This constraint matrix is totally unimodular by Remark 3, showing that v is an integral vertex.

Assume now that the star inequality is not active at v, namely $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j > 2$. If at most one edge inequality is tight at v, then v is obviously integral. Thus, we may assume that $k \geq 2$ edge inequalities are tight at v, say $\sum_{j \in N \setminus \{r\}} x_j \geq 1$ for $1 \leq r \leq k$. Then $v_1 = \ldots = v_k$. If v_1 is fractional, v has at least k fractional coordinates. We assumed that only k inequalities other than $0 \leq x \leq 1$ are active at v, so the other coordinates are integral. If $v_j = 1$ for some $j \geq k+1$, then $\sum_{j \in N \setminus \{r\}} v_j > 1$ for each $1 \leq r \leq k$, which contradicts the assumption that $\sum_{j \in N \setminus \{r\}} v_j = 1$. Hence, $v_j = 0$ for $j \notin \{1, \ldots, k\}$ and $v_1 = \ldots = v_k = \frac{1}{k-1}$. Then $\sum_{r=1}^{t} v_r + 2\sum_{j \in N \setminus I} v_j = \frac{k}{k-1} \leq 2$. However, this contradicts the assumption that $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j > 2$. □

Theorem 11. *$Q_S^{(3)}$ is the intersection of $Q_S^{(2)}$ with the half-spaces defined by the square inequalities* (3) *and the star inequalities* (4).

Proof. Applying the Chvátal procedure to inequalities defining $Q_S^{(2)}$, it is straightforward to show the validity of the inequalities (3) and (4) for $Q_S^{(3)}$.

To complete the proof of the theorem, we need to show that all other valid inequalities for $Q_S^{(3)}$ are implied by those defining $Q_S^{(2)}$, (3) and (4). Consider a valid inequality for $Q_S^{(3)}$ and let $\bar{T}$ denote the set of 0,1 vectors cut off by this inequality.

If $\bar{T} = \emptyset$, then the inequality is implied by $0 \leq x \leq 1$. Thus, we assume that $\bar{T} \neq \emptyset$. Let $T := \{0,1\}^n \setminus \bar{T}$. By the definition of a Chvátal inequality, there exists a valid inequality $ax \geq b$ for $Q_S^{(2)}$ that cuts off exactly the vertices in $\bar{T}$. By Lemma 4, the center points of the cubes of H_n all belong to $Q_S^{(2)}$. This means $ax \geq b$ does not cut off any of them. By Lemma 1, $G(\bar{T})$ is a connected graph. We claim that the distance between any 2 vertices in $G(\bar{T})$ is at most 2. Indeed, otherwise $G(\bar{T})$ contains two opposite vertices of a cube, and therefore its center satisfies $ax < b$, a contradiction.

We consider 3 cases: $|\bar{T}| \leq 3$, $G(\bar{T})$ contains a square, and $G(\bar{T})$ contains no square.

If $|\bar{T}| \leq 3$, then $G(\bar{T})$ is either an isolated vertex, an edge, or a path of length two. Then vertex and edge inequalities with the bounds $0 \leq x \leq 1$ are sufficient to describe $\text{conv}(T)$ by Lemma 10.

If $G(\bar{T})$ contains a square $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$, it cannot cut off any other vertex of H_n (otherwise, by Lemma 1 there would be another vertex of $\bar{T}$ adjacent to the square, and thus in a cube, a contradiction). Thus, $\bar{T} = \{\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}\}$. Since $\text{conv}(T) = \{x \in [0,1]^n : \sum_{j \in N \setminus \{i,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1\}$, a Chvátal inequality derived from $ax \geq b$ will therefore be implied by the square inequality that corresponds to $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$ and the bounds $0 \leq x \leq 1$.

Assume that $G(\bar{T})$ contains no square and $|\bar{T}| \geq 4$. Note that a cycle of H_n that is not a square has length at least six. Since the distance between any two vertices in $G(\bar{T})$ is at most two, $G(\bar{T})$ contains no cycle of H_n. Thus, $G(\bar{T})$ is a tree. In fact, $G(\bar{T})$ is a star since the distance between any two of its vertices is at most two. Thus $\bar{T} = \{\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t}\}$ for some $t \geq 3$. By Lemma 10, $\text{conv}(T)$ is described by edge and star inequalities with the bounds $0 \leq x \leq 1$. $\square$

Note that, if an edge $\bar{x}\bar{y}$ of $G(\bar{S})$ belongs to a square of $G(\bar{S})$, the corresponding inequality is not needed in the description of $Q_S^{(3)}$ since it is dominated by the square inequality. On the other hand, if an edge belongs to a star $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ of $G(\bar{S})$ with $t < n$, there is no domination relationship between the corresponding edge inequality and the star inequality.

Proof of Theorem 8. We first prove the "if" part. Since $G(\bar{S})$ contains neither a 4-cycle nor a star, Theorem 11 implies that $Q_S^{(3)} = Q_S^{(2)}$. It follows that $Q_S^{(2)} = \text{conv}(S)$. Since $G(\bar{S})$ contains a connected component of size greater than 1, $Q_S^{(1)} \neq \text{conv}(S)$ by Theorem 5. Thus Q_S has Chvátal rank exactly 2.

We now show the "only if" part. Suppose a connected component of $G(\bar{S})$ contains a cycle of length 4 or a vertex of degree greater than 2.

Consider first the 4-cycle case, say $\{0, e^1, e^2, e^1 + e^2\} \subseteq \bar{S}$. Then the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$ belongs to $Q_S^{(2)}$ by Lemma 4 but not to $\text{conv}(S)$ since $\sum_{j=3}^n x_j \geq 1$ is valid for $\text{conv}(S)$.

Now consider a vertex of degree greater than 2, say $\{0, e^1, e^2, e^3\} \subseteq \bar{S}$ where e^1, e^2, e^3 denote the first 3 unit vectors. Then the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$ belongs to $Q_S^{(2)}$ by Lemma 4 but not to $\text{conv}(S)$ since $\sum_{j=1}^3 x_j + 2\sum_{j=4}^n x_j \geq 2$ is valid for $\text{conv}(S)$. $\square$

4 Chvátal rank 4

In this section, we give the characterization of $Q_S^{(4)}$. It is somewhat more involved than the results for $Q_S^{(1)}$, $Q_S^{(2)}$ and $Q_S^{(3)}$, but it is in the same spirit.

Consider any cube with vertices in $G(\bar{S})$. Specifically, for $\bar{x} \in \{0,1\}^n$, recall that we use the notation $\bar{x}^i$ to denote the 0,1 vertex that differs from $\bar{x}$ only in coordinate i, and more generally, for $J \subseteq N$, let $\bar{x}^J$ denote the 0,1 vector that differs from $\bar{x}$ exactly in the coordinates J. If the 8 points $\bar{x}, \bar{x}^i, \bar{x}^k, \bar{x}^\ell, \bar{x}^{ik}, \bar{x}^{i\ell}, \bar{x}^{k\ell}, \bar{x}^{ik\ell}$ all belong to $\bar{S}$, then we say that these points form a *cube*. Note that

$$\sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{5}$$

is a valid inequality for $\mathrm{conv}(S)$ and that it cuts off exactly 8 vertices of H_n, namely the 8 corners of the cube. In fact, it is the strongest such inequality since it is satisfied at equality by all $8(n-3)$ of their neighbors in H_n. We call (5) a *cube inequality.*

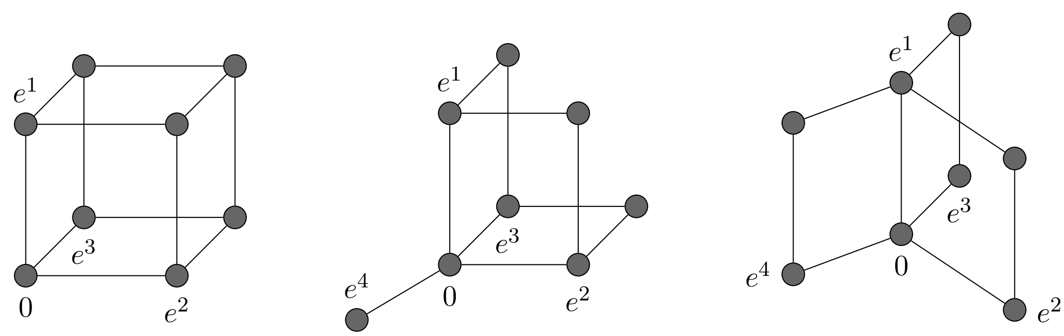

Figure 2: Cube, tulip, and propeller with $\bar{x} = 0$

If $\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \bar{x}^{i_3}, \bar{x}^{i_1 i_2}, \bar{x}^{i_2 i_3}, \bar{x}^{i_3 i_1}, \bar{x}^{i_4}, \ldots, \bar{x}^{i_t}$ all belong to $\bar{S}$ for some $t \geq 4$, then we say that these points form a *tulip*. Let $I_T := \{i_1, \ldots, i_t\}$. Note that

$$\sum_{k=1}^{3}(\bar{x}_{i_k}(1-x_{i_k}) + (1-\bar{x}_{i_k})x_{i_k}) + 2\sum_{r=4}^{t}(\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 3\sum_{j \notin I_T}(\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 3 \tag{6}$$

is a valid inequality for $\mathrm{conv}(S)$ that cuts off exactly these points. We call it a *tulip inequality.* For example, if $\bar{x} = 0$, and $\bar{x}^{i_k} = e^k$ for $k = 1, 2, 3$, (6) is $x_1 + x_2 + x_3 + 2\sum_{r=4}^{t} x_{i_r} + 3\sum_{j \notin I_T} x_j \geq 3$.

If $\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \ldots, \bar{x}^{i_t}, \bar{x}^{i_{t+1}}, \bar{x}^{i_1 i_{t+1}}, \bar{x}^{i_2 i_{t+1}}, \ldots, \bar{x}^{i_t i_{t+1}}$ all belong to $\bar{S}$ for some $t \geq 3$, then we say that these points form a *propeller*. Besides, we say the edge $\bar{x}\bar{x}^{i_{t+1}}$ is the *axis* of the propeller. Let $I_P := \{i_1, \ldots, i_{t+1}\}$. Note that

$$\sum_{r=1}^{t}(\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P}(\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2 \tag{7}$$

is a valid inequality that cuts off exactly the above points. We call it a *propeller inequality.* It goes through $2(n-t-1)$ neighbors of $\bar{x}$ and $\bar{x}^{i_{t+1}}$, $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$, and another $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1 i_{t+1}}, \ldots, \bar{x}^{i_t i_{t+1}}$. For example, if $\bar{x} = 0$, $\bar{x}^{i_{t+1}} = e^1$ and $\bar{x}^{i_k} = e^{k+1}$ for $k = 1, \ldots, t$, the propeller inequality is $x_2 + \ldots + x_{t+1} + 2(x_{t+2} + \ldots + x_n) \geq 2$.

Theorem 12. *$Q_S^{(4)}$ is the intersection of $Q_S^{(3)}$ and the half spaces defined by all cube, tulip, and propeller inequalities.*

Theorem 12 is the main result of this section. Before proving it, we present some its consequences.

Corollary 13. *The Chvátal rank of Q_S is 3 if and only if $G(\bar{S})$ contains no cube, tulip or propeller but it contains a star or a square.*

Proof. This follows from Theorems 8, 11 and 12. $\square$

Corollary 14. *Let $P \subseteq [0,1]^n$ be a polytope, $S = P \cap \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. If $G(\bar{S})$ contains no 4-cycle (i.e., the girth of $G(\bar{S})$ is at least 6), then P has Chvátal rank at most 3.*

Corollary 15. *Let $n \geq 3$ and $i = 0, 1$ or 2. For $S := \{x \in \{0,1\}^n : \sum_{j=1}^n x_j = i \text{ (mod 3) }\}$, the set $\operatorname{conv}(S)$ is entirely described by vertex, edge, star inequalities and bounds $0 \leq x \leq 1$.*

Proof. If $G(\bar{S})$ contains a square Σ, let $\bar{x}$ be the vertex with the fewest number of 1's in Σ. We can write $\Sigma = (\bar{x}, \bar{x}^i, \bar{x}^k, \bar{x}^{ik})$. If $\sum_{j \in N} \bar{x}_i = p$, then we get that $\sum_{j \in N} \bar{x}_j^i = \sum_{j \in N} \bar{x}_j^k = p + 1$ and $\sum_{j \in N} \bar{x}_j^{ik} = p + 2$. Then at least one of these four vertices is in S, but this contradicts the assumption that Σ is in $G(\bar{S})$. $\square$

Remark 16. *For $S := \{x \in \{0,1\}^n : \sum_{j=1}^n x_j = i \text{ (mod 4) }\}$, the linear description of $\operatorname{conv}(S)$ might contain inequalities with Chvátal rank 5.*

Proof. Let $n = 5$ and $i = 3$. Then $\bar{S}$ contains $\{0, e^1, \ldots, e^5, e^1 + e^2, e^1 + e^3, e^1 + e^4, e^2 + e^3, e^2 + e^4\}$. Then $x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 \geq 4$ is valid for $\operatorname{conv}(S)$. The rank of this inequality is at least 5, because it cuts off $(\frac{1}{2}, \ldots, \frac{1}{2})$ which belongs to $Q_S^{(4)}$. Note that tulip inequalities $x_1 + x_2 + x_3 + 2x_4 + 2x_5 \geq 3$ and $x_1 + x_2 + 2x_3 + x_4 + 2x_5 \geq 3$ have Chvátal rank 4, and propeller inequalities $x_2 + x_3 + x_4 + 2x_5 \geq 2$ and $x_1 + x_3 + x_4 + 2x_5 \geq 2$ also have Chvátal rank 4. Adding them all up and dividing by 3 we get that $x_1 + x_2 + \frac{5}{3}x_3 + \frac{5}{3}x_4 + \frac{8}{3}x_5 \geq \frac{10}{3}$, and we generate $x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 \geq 4$ by rounding up the resulting inequality. Hence, the inequality has Chvátal rank 5. $\square$

To prove Theorem 12, we need the following technical lemmas.

Lemma 17. *Let $\bar{S} = \{0, e^1, \ldots, e^k, e^1 + e^2\}$ for some $k \geq 3$. Then $\operatorname{conv}(S)$ is described by a square inequality for the square $(0, e^1, e^2, e^1 + e^2)$, a star inequality for the star $(0, e^1, \ldots, e^k)$, edge inequalities for the edges connecting 0 to $e^3, \ldots, e^k$ and the bounds $0 \leq x \leq 1$.*

Proof. If $k = n$, then by Lemma 10 it is sufficient to show that $A := \{x \in [0,1]^n : \sum_{i=3}^n x_i \geq 1, \sum_{i=1}^n x_i \geq 2\}$ is an integral polytope. This is the case since the constraint matrix of A is totally unimodular by Remark 3.

If $k < n$, then it is sufficient to show that $A' := \{x \in [0,1]^n : \sum_{i=3}^n x_i \geq 1, \sum_{i=1}^k x_i + 2\sum_{j=k+1}^n x_j \geq 2, \sum_{i \in N \setminus \{j\}} x_i \geq 1 \text{ for } 3 \leq j \leq k\}$ is integral. Let v be an extreme point of A'. By Lemma 10, we know that $\{x \in [0,1]^n : \sum_{i=1}^k x_i + 2\sum_{j=k+1}^n x_j \geq 2, \sum_{i \in N \setminus \{j\}} x_i \geq 1 \text{ for } 1 \leq j \leq k\}$

is integral. That means that v is integral if the square inequality $\sum_{i=3}^{n} x_i \geq 1$ is not active at v. Thus we may assume that v belongs to the face of A' defined by $\sum_{i=3}^{n} x_i = 1$. Then $\sum_{i=1}^{k} x_i + 2\sum_{j=k+1}^{n} x_j \geq 2$ is equivalent to $x_1 + x_2 \geq \sum_{i=3}^{k} x_i$. Furthermore, each edge inequality $\sum_{i \in N\setminus\{j\}} x_i \geq 1$ is equivalent to $x_1+x_2 \geq x_j$ for $j \geq 3$. Note that $x \geq 0$ and $x_1+x_2 \geq \sum_{i=3}^{k} x_i$ imply $x_1+x_2 \geq x_j$ for $j \geq 3$. Therefore the face F is completely defined by $0 \leq x \leq 1$, $x_1+x_2 \geq \sum_{i=3}^{k} x_i$, and $\sum_{i=3}^{n} x_i = 1$. Notice that $x_1 + x_2 \geq \sum_{i=3}^{k} x_i$ can be rewritten as $x_1 + x_2 + \sum_{i=k+1}^{n} x_i \geq 1$, so the constraint matrix for this system is totally unimodular by Remark 3. Therefore F is an integral polytope and its vertex v is integral. $\square$

Lemma 18. *Let $\bar{S} = \{0, e^1, \ldots, e^k, e^1+e^2, e^1+e^3\}$ for some $k \geq 4$. Then $\operatorname{conv}(S)$ is described by two square inequalities for $(0, e^1, e^2, e^1+e^2)$ and $(0, e^1, e^3, e^1+e^3)$, a star inequality for the star $(0, e^1, \ldots, e^k)$, edge inequalities for the edges connecting 0 to $e^4, \ldots, e^k$ and the bounds $0 \leq x \leq 1$.*

Proof. If $k = n$, it is sufficient to show that $A = \{x \in [0,1]^n :\ x_3 + \sum_{i=4}^{n} x_i \geq 1,\ x_2 + \sum_{i=4}^{n} x_i \geq 1,\ \sum_{i=1}^{n} x_i \geq 2\}$ is integral. The constraint matrix of A is totally unimodular by Remark 3, so A is an integral polytope.

If $k < n$, it is sufficient to show that $A' = \{x \in [0,1]^n :\ x_3 + \sum_{i=4}^{n} x_i \geq 1,\ x_2 + \sum_{i=4}^{n} x_i \geq 1,\ \sum_{i=1}^{k} x_i + 2\sum_{j=k+1}^{n} x_j \geq 2, \sum_{i \in N\setminus\{j\}} x_i \geq 1 \text{ for } 4 \leq j \leq k\}$ is integral. Let v be an extreme point of A'. If at most one of the two square inequalities is active at v, then v is integral by Lemma 17. Assume now that both square inequalities are active at v. Let F be the face of A' defined by $x_3 + \sum_{i=4}^{n} x_i = 1$ and $x_2 + \sum_{i=4}^{n} x_i = 1$.

As we proved in Lemma 17, $\sum_{i=3}^{n} x_i = 1$ and $\sum_{i=1}^{k} x_i + 2\sum_{j=k+1}^{n} x_j \geq 2$ imply all the edge inequalities. That means that F is completely defined by $0 \leq x \leq 1$, $x_3 + \sum_{i=4}^{n} x_i = 1$, $x_2 + \sum_{i=4}^{n} x_i = 1$, and $\sum_{i=1}^{k} x_i + 2\sum_{j=k+1}^{n} x_j \geq 2$. Using the first equation, this last inequality can be rewritten as $x_1 + x_2 + \sum_{i=k+1}^{n} x_i \geq 1$. Then the constraint matrix of the resulting system for F is totally unimodular by Remark 3. Therefore v is an integral vector. $\square$

Lemma 19. *Consider the tulip $\bar{S} = \{0, e^1, e^2, e^3, \ldots, e^k, e^1+e^2, e^2+e^3, e^3+e^1\}$ for some $k \geq 4$. Then $\operatorname{conv}(S)$ is described by the tulip inequality, the three square inequalities, a star inequality for the star $(0, e^1, e^2, \ldots, e^k)$, edge inequalities for the edges connecting 0 to $e^4, \ldots, e^k$, and the bounds $0 \leq x \leq 1$.*

Proof. It is sufficient to show that $A := \{x \in [0,1]^n :\ \sum_{i=1}^{3} x_i + 2\sum_{i=4}^{k} x_i + 3\sum_{i=k+1}^{n} x_i \geq 3,\ \sum_{i=1}^{k} x_i + 2\sum_{i=k+1}^{n} x_i \geq 2,\ x_j + \sum_{i=4}^{n} x_i \geq 1 \text{ for } j = 1,2,3,\ \sum_{i \in N\setminus\{j\}} x_i \geq 1 \text{ for } 4 \leq j \leq k\}$ is an integral polytope. Let v be an extreme point of A.

We consider several cases. If all three square inequalities are tight, then the tulip inequality becomes $-\sum_{i=4}^{k} x_i \geq 0$. This implies $x_4 = \ldots = x_k = 0$. But then the square equalities become $x_j + \sum_{i=k+1}^{n} x_i = 1$ for $j = 1,2,3$. Subtracting two of them from the star inequality shows that it is implied by $x \geq 0$. Furthermore the edge inequalities are implied by $x \geq 0$ and the equality $x_1 + \sum_{i=k+1}^{n} x_i = 1$. Therefore, the system reduces to the three above equations and the bounds $0 \leq x \leq 1$. The constraint matrix is totally unimodular by Remark 3. Therefore we may assume that at most two square inequalities are tight. Lemma 18 shows that the system comprising these two square inequalities, the star inequality and the edge inequalities defines an integral polytope. Therefore we may assume in the remainder that the tulip inequality is tight.

Consider the case where two square inequalities are tight, say $x_1 + \sum_{i=4}^{n} x_i = 1$ and $x_2 + \sum_{i=4}^{n} x_i = 1$ but the third is not. Since the tulip inequality is tight $\sum_{i=1}^{3} x_i + 2\sum_{i=4}^{k} x_i +$

$3\sum_{i=k+1}^n x_i = 3$, it follows that $x_3 + \sum_{i=k+1}^n x_i = 1$. Note that the star inequality becomes $x_1 + x_2 + \sum_{i=4}^n x_i \geq 1$, which is implied by either of the square equations and $x \geq 0$. Furthermore the edge inequalities are implied by $x \geq 0$ and the equality $x_3 + \sum_{i=k+1}^n x_i = 1$. Therefore the system reduces to the three equations $x_1 + \sum_{i=4}^n x_i = 1$, $x_2 + \sum_{i=4}^n x_i = 1$, $x_3 + \sum_{i=k+1}^n x_i = 1$ and $0 \leq x \leq 1$. The constraint matrix is totally unimodular by Remark 3.

Consider the case where one square inequality is tight, say $x_1 + \sum_{i=4}^n x_i = 1$, but the other two are not. Then the tulip equality $\sum_{i=1}^3 x_i + 2\sum_{i=4}^k x_i + 3\sum_{i=k+1}^n x_i = 3$ can be rewritten as $-x_1 + x_2 + x_3 + \sum_{i=k+1}^n x_i = 1$. Note that the star inequality is implied by $x_1 \geq 0$ and the equation $x_2 + x_3 + \sum_{i=4}^k x_i + 2\sum_{i=k+1}^n x_i = 2$ obtained from the above equalities. Furthermore the edge inequalities are implied by $x \geq 0$ and the equality $-x_1 + x_2 + x_3 + \sum_{i=k+1}^n x_i = 1$. Therefore the system reduces to the two equations $x_1 + \sum_{i=4}^n x_i = 1$, $-x_1 + x_2 + x_3 + \sum_{i=k+1}^n x_i = 1$, and the bound conditions $0 \leq x \leq 1$. If $x_1 = 1$, then the two equations can be rewritten as $\sum_{i=4}^n x_i = 0$ and $x_2 + x_3 + \sum_{i=k+1}^n x_i = 2$ and we know that the constraint matrix of the resulting system is totally unimodular by Remark 3. If $x_1 = 0$, we get that $\sum_{i=4}^n x_i = 1$ and $x_2 + x_3 + \sum_{i=k+1}^n x_i = 1$ and thus the constraint matrix for the resulting system is also totally unimodular in this case by Remark 3. This indicates that v_1 should be fractional if v is a fractional vertex. Notice that $\sum_{i=4}^k v_i + \sum_{i=k+1}^n v_i = 1 - v_1$ and $v_2 + v_3 + \sum_{i=k+1}^n v_i = 1 + v_1$. Since v has at most 2 fractional coordinates, so $\sum_{i=k+1}^n v_i$ is fractional whereas both $\sum_{i=4}^k v_i$ and $v_2 + v_3$ are integral. Since $0 < 1 - v_1 < 1$, we get $\sum_{i=k+1}^n v_i = 1 - v_1$ and this implies $\sum_{i=4}^k v_i = 0$ and $v_2 + v_3 = 2v_1 \in \mathbb{Z}$. Therefore, $v_1 = \frac{1}{2}$ and $\sum_{i=k+1}^n v_i = 1 - v_1 = \frac{1}{2}$. This means $v_i = \frac{1}{2}$ for exactly one $k+1 \leq i \leq n$, say $v_{k+1} = \frac{1}{2}$ and $v_i = 0$ for $k+2 \leq i \leq n$. Since the square inequalities $x_j + \sum_{i=4}^n x_i \geq 1$ for $j = 2, 3$ can be rewritten as $x_2, x_3 \geq \frac{1}{2}$, the only possibility is $v_2 = v_3 = \frac{1}{2}$. But then all three square inequalities are tight, contradicting our assumption.

Finally, consider the case where no square inequality is tight. The tulip inequality is tight. If the star inequality is also tight, then we can rewrite these two equalities as $\sum_{i=4}^k x_i + \sum_{i=k+1}^n x_i = 1$ and $\sum_{i=1}^3 x_i + \sum_{i=k+1}^n x_i = 1$. The second equality and $x \geq 0$ imply all the edge inequalities. Therefore the system reduces to the two above equations and $0 \leq x \leq 1$. The constraint matrix is totally unimodular by Remark 3 and v is an integral vertex. Thus we may assume that the star inequality is not tight. In other words, the system reduces to the tulip equation $\sum_{i=1}^3 x_i + 2\sum_{i=4}^k x_i + 3\sum_{i=k+1}^n x_i = 3$, the edge inequalities and the bounds $0 \leq x \leq 1$. If no edge inequality is tight, the vector v is clearly integral. So we may assume that at least one edge inequality is tight, say $\sum_{i \in N\setminus\{k\}} x_i = 1$. From this equation and the tulip equation, we get $x_k = x_1 + x_2 + x_3 + \frac{1}{2}\sum_{i=4}^{k-1} x_i$. Using the edge equality, the star inequality becomes $x_k \geq \sum_{i=1}^{k-1} x_i$. This implies that $x_4 = \ldots = x_{k-1} = 0$ and that the star inequality is tight. This case was already considered. This completes the proof. □

Lemma 20. *Let $\bar{S} = \{0, e^1, \ldots, e^\ell, e^1 + e^2, \ldots, e^1 + e^k\}$ for some $k \geq 4$ and $\ell \geq k+1$. Note that $\bar{S}$ is a propeller which consists of k squares and $(0, e^1, e^2, \ldots, e^\ell)$ is a star. Then* $\mathrm{conv}(S)$ *is described by the star inequality for the star $(0, e^1, e^2, \ldots, e^\ell)$, edge inequalities for the edges connecting 0 to $e^{k+1}, \ldots, e^\ell$, the square and propeller inequalities that correspond to the propeller $(0, e^1, \ldots, e^k, e^1 + e^2, \ldots, e^1 + e^k)$, and the bounds $0 \leq x \leq 1$.*

Proof. We assume that $\ell < n$. Proof for the case when $\ell = n$ is similar to this case. It is sufficient to show that the polytope $A := \{x \in [0,1]^n : \sum_{i=2}^k x_i + 2\sum_{j=k+1}^n x_j \geq 2,\ \sum_{i=1}^\ell x_i + 2\sum_{j=\ell+1}^n x_j \geq 2,\ \sum_{i \in N\setminus\{1,j\}} x_i \geq 1 \text{ for } 2 \leq j \leq k,\ \sum_{i \in N\setminus\{j\}} x_i \geq 1 \text{ for } k+1 \leq j \leq \ell\}$ is integral.

Let v be an extreme point of A. Suppose that no edge inequality is tight at v, for $k+1 \leq j \leq \ell$. Then the only constraint relating x_1 to the other variables is the star inequality, and therefore the vector $(v_2, \dots, v_n)$ is an extreme point of the system

$$0 \leq x_i \leq 1 \text{ for } 2 \leq i \leq n, \ \sum_{i=2}^{k} x_i + 2 \sum_{j=k+1}^{n} x_j \geq 2, \ \sum_{i \in N \setminus \{1,j\}} x_i \geq 1 \text{ for } 2 \leq j \leq k. \tag{8}$$

By Lemma 10, the system (8) is integral; this implies that v is integral. Therefore we can assume that at least one edge inequality is tight, say $\sum_{N \setminus \{k+1\}} x_j = 1$. This equation implies that the star inequality can be rewritten as $x_{k+1} + \sum_{i=\ell+1}^{n} x_i \geq 1$. This together with $x \geq 0$ implies the propeller and square inequalities. Therefore the face F of A defined by $\sum_{N \setminus \{k+1\}} x_j = 1$ is completely described by this equation, $0 \leq x \leq 1$, the star inequality and the remaining edge inequalities. By Lemma 10, this polytope is integral. Therefore v is integral.

□

Proof of Theorem 12. We first show that the inequalities stated in the theorem are valid for $Q_S^{(4)}$.

A cube can be decomposed into two vertex-disjoint squares, and $x_\ell + \sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$ and $-x_\ell + \sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 0$ are the corresponding square inequalities which are valid for $Q_S^{(3)}$. Adding them, dividing by 2, and applying the Chvátal procedure generates the cube inequality, so it is valid for $Q_S^{(4)}$.

A tulip contains a star with $\bar{x}$ as its root, and the corresponding star inequality is $\sum_{r=1}^{t} (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_T} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2$. In addition, it has three squares containing $\bar{x}$, and the corresponding square inequalities are $\sum_{j \in N \setminus \{i_1,i_2\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$, $\sum_{j \in N \setminus \{i_2,i_3\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$, and $\sum_{j \in N \setminus \{i_1,i_3\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$. These four inequalities are all valid for $Q_S^{(3)}$. Adding them, dividing by 2, and applying the Chvátal procedure shows the validity of the tulip inequality for $Q_S^{(4)}$.

A propeller contains two stars with $\bar{x}$, $\bar{x}^{i_{t+1}}$ as their roots, respectively, and the corresponding star inequalities are $x_{i_{t+1}} + \sum_{r=1}^{t} (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2$ and $-x_{i_{t+1}} + \sum_{r=1}^{t} (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$. These are valid for $Q_S^{(3)}$. Adding them, dividing by 2, and applying the Chvátal procedure shows the validity of the propeller inequality for $Q_S^{(4)}$.

To complete the proof of the theorem, we need to show that every valid inequality for $Q_S^{(4)}$ is a consequence of the inequalities defining $Q_S^{(3)}$ and cube, tulip and propeller inequalities.

Consider any valid inequality for $Q_S^{(4)}$ and let $\bar{T}$ denote the set of 0,1 vectors cut off by this inequality. Let $T := \{0,1\}^n \setminus \bar{T}$. We will show that vertex, edge, square, star, cube, tulip and propeller inequalities are sufficient to describe $\text{conv}(T)$.

It follows from the definition of a Chvátal inequality that there exists a valid inequality $ax \geq b$ for $Q_S^{(3)}$ that cuts off the same set $\bar{T}$. We know that $G(\bar{T})$ is a connected graph by Lemma 1. We claim that $G(\bar{T})$ satisfies the following three properties: 1) if a path of length three appears in $G(\bar{T})$, then either the square of G containing the first three vertices of the path or the square containing the last three vertices belongs to $G(\bar{T})$; 2) the maximum distance in H_n between two vertices in $G(\bar{T})$ is at most three; 3) if $G(\bar{T})$ contains two squares, then either they share a common edge or $G(\bar{T})$ is a 3-dimensional cube and the two squares are opposite 2-dimensional faces of it. The following three paragraphs prove these claims.

To show the first claim, consider a path of length three in $G(\bar{T})$. We may assume without loss of generality that the path is $(e^1, 0, e^2, e^2 + e^3)$. Suppose both $e^1 + e^2$ and e^3 satisfy $ax \geq b$. Then their middle point m also satisfies $ax \geq b$, contradicting the fact that e^1 and $e^2 + e^3$ (and therefore their middle point, which is m) satisfy $ax < b$. Therefore $e^1 + e^2$ or e^3 is in $\bar{T}$, forming a square with either $e^1, 0, e^2$ or $0, e^2, e^2 + e^3$.

Let $u, v \in \bar{T}$. Since u and v are connected in $G(\bar{T})$, there is a path between u and v in $G(\bar{T})$. If the distance between u and v on H_n is at least 4, then there exists a vertex w on the path such that the distance on H_n between u and w is 4. Their middle point is also cut off by $ax \geq b$. Since they are opposite vertices of a 4-dimensional face of H_n, the middle point of the face is cut off by the inequality. However, this contradicts Lemma 4 for $Q_S^{(3)}$. Hence, the maximum distance on H_n between two points in $\bar{T}$ is at most three.

Assume that $G(\bar{T})$ contains two squares. Without loss of generality, we may assume that one of them is $(0, e^1, e^2, e^1 + e^2)$. Suppose that the second square does not share an edge with it. If they share a vertex, we may assume that the second square is $(0, e^3, e^4, e^3 + e^4)$. Note that the distance on H_n between $e^1 + e^2$ and $e^3 + e^4$ is 4, contradicting the second claim. Thus, the two squares do not share any vertex. Because $G(\bar{T})$ is connected and no path of length greater than three exists, it easy to verify that $G(\bar{T})$ must be a 3-dimensional cube.

We now consider different cases according to the number of squares contained in $G(\bar{T})$.

First, consider the case when $G(\bar{T})$ has no square. then the distance on H_n between any two vertices in $G(\bar{T})$ is at most two by the first claim. Then $G(\bar{T})$ can be a single vertex, an edge, two consecutive edges, or a star. Hence, vertex, edge, and star inequalities with the bounds $0 \leq x \leq 1$ are sufficient to describe $\mathrm{conv}(T)$ by Lemma 10.

Second, consider the case when $G(\bar{T})$ contains exactly one square. Without loss of generality, we may assume that it is $(0, e^1, e^2, e^1 + e^2)$. If $\bar{T}$ consists of just this square, then the square inequality $\sum_{j=3}^{n} x_j \geq 1$ suffices. If not, the square is adjacent to at least one 0,1 point in $\bar{T}$ and thus we may assume that e^3 is in $\bar{T}$. Note that the other points in $\bar{T}$ (if any) are not adjacent to any of $e^1, e^2, e^1 + e^2$, by the first claim and the assumption that only one square exists in $G(\bar{T})$. Therefore, we may assume that $\bar{T}$ is $\{0, e^1, e^2, \ldots, e^k, e^1 + e^2\}$ for some $k \geq 3$. In this case, edge, star, and square inequalities are sufficient by Lemma 17.

Third, assume that $G(\bar{T})$ contains exactly two squares. By the symmetry of H_n, we may assume that $\bar{T}$ contains $0, e^1, e^2, e^3, e^1{+}e^2, e^1{+}e^3$. If no other vertex belongs to $\bar{T}$, then $x_3 + \sum_{j=4}^{n} x_j \geq 1$ and $x_2 + \sum_{j=4}^{n} x_j \geq 1$ together with $0 \leq x \leq 1$ suffice since the constraint matrix for this system is totally unimodular by Remark 3. So we may assume that there exists $v \in \bar{T} \setminus \{0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3\}$. By connectivity of $G(\bar{T})$ we may assume that v is adjacent to at least one of $0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3$. Since $G(\bar{T})$ contains only two squares, v is adjacent to exactly one of these vertices. If v is adjacent to e^2, then v can be written as $e^2 + e^k$ for some $k \geq 4$. However, this is impossible by the second claim since the distance on H_n between $e^2{+}e^k$ and $e^1{+}e^3$ is 4. Thus, v cannot be adjacent to e^2. Likewise, v cannot be adjacent to e^3, $e^1{+}e^2$, and $e^1{+}e^3$. Without loss of generality, v is adjacent to 0. If there exists $u \in \bar{T}$ adjacent to e^1, then $G(\bar{T})$ should contain an additional square containing either u or v by the first claim. Therefore, all the vertices in $\bar{T} \setminus \{0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3\}$ are adjacent to 0. Namely $\bar{T} = \{0, e^1, e^2, e^3, \ldots, e^k, e^1 + e^2, e^1 + e^3\}$ for some $k \geq 4$. In this case, edge, star, and square inequalities are sufficient by Lemma 18.

Finally, assume that $G(\bar{T})$ contains at least three squares. If $G(\bar{T})$ contains a cube, then $G(\bar{T})$

contains no other vertex by the third claim and therefore we may assume that $\bar{T} = \{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1, e^1 + e^2 + e^3\}$. In this case $\sum_{j=4}^n x_j \geq 1$ together with $0 \leq x \leq 1$ suffices. Now, assume that $G(\bar{T})$ contains no cube. Any two of the squares should share a common edge by the third claim. There are two possibilities: All squares share a common edge or three squares are the three 2-dimensional faces incident to a vertex of H_n. Thus we may assume that $\bar{T}$ contains either $\{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$ or $\{0, e^1, e^2, e^3, e^4, e^1 + e^2, e^1 + e^3, e^1 + e^4\}$.

First, assume that $\{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\} \subseteq \bar{T}$. If $\bar{T} = \{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$, then $x_1 + \sum_{j=4}^n x_j \geq 1$, $x_2 + \sum_{j=4}^n x_j \geq 1$, and $x_3 + \sum_{j=4}^n x_j \geq 1$ together with $0 \leq x \leq 1$ gives $\text{conv}(T)$. This is because the constraint matrix of the system is totally unimodular by Remark 3. Thus, we may assume that there exists $v \in \bar{T} \setminus \{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$, and by the connectivity of $G(\bar{T})$ we may assume that v is adjacent to at least one of vertices $0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3$, and $e^3 + e^1$. If v is adjacent to e^1, then v can be written as $e^1 + e^k$ for some $k \geq 4$. Then the distance on H_n between v and $e^2 + e^3$ is 4. If v is adjacent to $e^1 + e^2$, then v is $e^1 + e^2 + e^k$ for some $k \geq 4$. Then the distance on H_n between v and e^3 is 4. Therefore, v is adjacent to 0. Hence, $\bar{T}$ is a tulip $\{0, e^1, e^2, e^3, \ldots, e^k, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$ for some $k \geq 4$. In this case, edge, star, square, and tulip inequalities are sufficient by Lemma 19.

Second, assume that $\{0, e^1, e^2, e^3, e^4, e^1 + e^2, e^1 + e^3, e^1 + e^4\} \subseteq \bar{T}$. It is possible that $\bar{T}$ contains more than three squares. Then, the other squares contain the edge $0, e^1$. As shown in the case when $G(\bar{T})$ contains exactly two squares, all vertices which are not in any square but in $\bar{T}$ should be adjacent to a single common vertex which can be either 0 or e^1. Hence, we may assume that $\bar{T} = \{0, e^1, e^2, e^3, \ldots, e^k, e^{k+1}, \ldots, e^\ell, e^1 + e^2, \ldots, e^1 + e^k\}$ for some $k \geq 3$ and $\ell \geq k + 1$. In this case, edge, star, square, and propeller inequalities are sufficient by Lemma 20. □

5 Vertex cutsets

In this section, we give polyhedral decomposition theorems for $\text{conv}(S)$ when the graph $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2.

5.1 Cut vertex

Theorem 21 below shows that $\text{conv}(S)$ can be decomposed when $G(\bar{S})$ contains a vertex cut. This result is in the spirit of the theorem of Angulo, Ahmed, Dey and Kaibel (Theorem 2) but it is specific to polytopes contained in the unit hypercube. At the end of this section, we give an example showing that the result does not extend to general polytopes. Before we state Theorem 21, let us illustrate an example first.

Let $G = (V, E)$ be a graph and let $X \subseteq V$. For $v \in X$, let $N_X[v]$ denote the closed neighborhood of v in the graph $G(X)$. That is $N_X[v] := \{v\} \cup \{u \in X : uv \in E\}$.

Example 1. Let $S = \{e^2, e^1 + e^2, e^1 + e^3\} \subset \{0,1\}^3$, and we consider $\text{conv}(S) \subset [0,1]^3$. In Figure 3, $\text{conv}(S)$ is a triangle which can be viewed as the intersection of the two tetrahedrons in the figure. Notice that e^3 is a cut vertex in $G(\bar{S})$, whose deletion leaves $\bar{S}_1 := \{0, e^1\}$ and $\bar{S}_2 := \{e^2 + e^3, e^1 + e^2 + e^3\}$ as two separate components. The set of 0,1 points that do not belong to the left tetrahedron is exactly $N_{\bar{S}}[e^3] \cup \bar{S}_2$, whereas that of the right one is $N_{\bar{S}}[e^3] \cup \bar{S}_1$.

Theorem 21. *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let v be a cut vertex in $G(\bar{S})$ and let $\bar{S}_1, \ldots, \bar{S}_t$ denote the connected components of $G(\bar{S} \setminus \{v\})$. Then* $\text{conv}(S) = \bigcap_{i=1}^t \text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i))$.

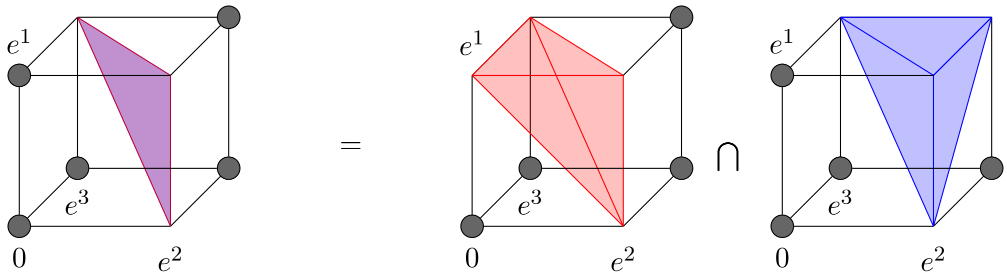

Figure 3: An example of decomposition around a cut vertex in $\mathbb{R}^3$

Furthermore, if v does not belong to any 4-cycle in $G(\bar{S})$, then $\operatorname{conv}(S) = \operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \bigcap_{i=1}^t \operatorname{conv}(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$.

Proof. To ignore trivial cases, we assume $n \geq 3$ and $t \geq 2$. Lemma 10 gives a linear description of the polytope $\operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$: Star and edge inequalities together with $0 \leq x \leq 1$ are sufficient to describe $\operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$.

First, we will show that the skeleton of $\operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ contains no edge connecting a vertex of $\bar{S}_i \setminus N_{\bar{S}}[v]$ to a vertex of $\bar{S}_j \setminus N_{\bar{S}}[v]$ if $i \neq j$. Then, by Theorem 2, we get that

$$\operatorname{conv}(S) = \bigcap_{i=1}^{t} \operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i)).$$

To prove the second statement of Theorem 21, we will show that $\operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i)) = \operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \operatorname{conv}(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$ if v does not belong to any 4-cycle of $G(\bar{S})$.

Let $u \in \bar{S}_i \setminus N_{\bar{S}}[v]$ and $w \in \bar{S}_j \setminus N_{\bar{S}}[v]$ where $i \neq j$. Suppose that u and w are adjacent in the skeleton of $\operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$. Then we can find $n-1$ linearly independent inequalities in the description of $\operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ that are active at both u and w. If t coordinates of u and w are different, then $n-t$ inequalities among $0 \leq x \leq 1$ are active at both u and w. Then there are $t-1$ linearly independent inequalities other than $0 \leq x \leq 1$ that are active at both u and w.

Consider an edge vr in the star $G(N_{\bar{S}}[v])$. Suppose that the corresponding edge inequality is active at both u and w. Then each of u and w is adjacent on H_n to an endpoint of the edge. Since u and w cannot be adjacent to v by the definition of $G(N_{\bar{S}}[v])$, both are adjacent to r. Then (u, r, w) is a path contained in $G(\bar{S} \setminus \{v\})$, contradicting the assumption that u and w are disconnected in $G(\bar{S} \setminus \{v\})$. Hence, no edge inequality is active at both u and w.

Now, the only candidates are the star inequality and the bounds $0 \leq x \leq 1$. If the star inequality is active at both u and w, then each of u and w is adjacent to two vertices of $N_{\bar{S}}[v] \setminus \{v\}$ on H_n. Since u and w cannot have a common neighbor vertex, there exist four distinct vertices in $N_{\bar{S}}[v] \setminus \{v\}$ such that u is adjacent to two of them and w is adjacent to the other two vertices. That means $n-4$ inequalities among $0 \leq x \leq 1$ are active at both u and w, so only $n-3$ linearly independent inequalities are active at both u and w. Thus, we may assume that the star inequality is not active at both u and w. Since u and w are at distance at least 2 on H_n, at most $n-2$ among $0 \leq x \leq 1$ are active at both. Thus the skeleton of $\operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ contains no edge connecting a vertex of $\bar{S}_i \setminus N_{\bar{S}}[v]$ to a vertex of $\bar{S}_j \setminus N_{\bar{S}}[v]$ if $i \neq j$.

Let $W := N_{\bar{S}}[v] \setminus (\bar{S}_i \cup \{v\})$ be the pendant vertices of the star that are not in $\bar{S}_i$. To prove the second part, it is sufficient to show that the skeleton of $\operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \setminus W))$ contains no edge connecting a vertex of $\bar{S}_i \setminus N_{\bar{S}}[v]$ to a vertex of W. Let $w \in W$ and $s \in \bar{S}_i \setminus N_{\bar{S}}[v]$. By the assumption that v does not belong to any square in $G(\bar{S})$, s is adjacent to at most one pendent vertex of $N_{\bar{S}}[v]$ on H_n. That means the star inequality is not active at s. We consider two cases. Suppose first that s is adjacent to a vertex r in $N_{\bar{S}}[v] \setminus W$. Then the edge inequality for vr is active at s, but no other edge inequality is active at s. Since w is adjacent to v, the edge inequality is also active at w. However, the distance in H_n between s and w is exactly 3 in this case. Thus at most $n-3$ bound inequalities are active at both s and w, for a total of at most $n-2$ linearly independent inequalities active at both. But we need $n-1$. So s and w are not connected by an edge of the skeleton in this case. Now consider the case where s is not adjacent to any pendant vertex of $N_{\bar{S}}[v] \setminus W$. Then no edge inequality is active at s. Since s and w are not adjacent in H_n, at most $n-2$ inequalities among $0 \le x \le 1$ are active at both u and w. Therefore, s and w are not adjacent in the skeleton of $\operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \setminus W))$ in this case, either. Thus the assertion holds by Lemma 2. □

Corollary 14 implies that if $G(\bar{S})$ induces a forest, the Chvátal rank of P is at most 3. This can also be proved directly using Theorem 21.

Corollary 22. *Let $P \subseteq [0,1]^n$, $S = P \cap \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. If $G(\bar{S})$ is a forest, then the Chvátal rank of P is at most 3.*

Proof. By Theorem 2 we may assume that $G(\bar{S})$ is connected, that is $G(\bar{S})$ induces a tree. We will prove the result by induction on the size of the tree. The result holds if $|\bar{S}| \le 3$. Let $G(\bar{S})$ induce a tree T and assume that the result holds for all trees with fewer vertices. The theorem holds if T is a star by Lemma 10, so we may assume that T is not a star. Let v be a non-pendant vertex of T and let $\bar{S}_1, \ldots, \bar{S}_t$ denote the connected components of $G(\bar{S} \setminus \{v\})$. Since v does not belong to any 4-cycle in $G(\bar{S})$, Theorem 21 implies that $\operatorname{conv}(S) = \operatorname{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \bigcap_{i=1}^t \operatorname{conv}(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$. Note that the sets $N_{\bar{S}}[v]$ and $\{v\} \cup \bar{S}_i$ for $i = 1, \ldots, t$ have smaller cardinality than $\bar{S}$. Therefore the result holds by induction. □

Unlike the result of Angulo, Ahmed, Dey and Kaibel (Theorem 2), Theorem 21 cannot be extended to general polytopes, as shown by the following example.

Example 2. Let P be the polytope in $\mathbb{R}^2$ shown in Figure 4. Let $V := \{v_1, \ldots, v_8\}$ denote its vertex set and let $G = (V, E)$ be its skeleton graph. Let $S := \{v_5, v_6, v_7\}$ and $\bar{S} := V \setminus S$. In the figure the set of white vertices is S, while the set of black vertices is $\bar{S}$. Note that v_2 is a cut vertex of $G(\bar{S})$, and $N_{\bar{S}}[v_2] = \{v_1, v_2, v_3\}$. Therefore, $\bar{S}_1 := \{v_1, v_8\}$ and $\bar{S}_2 := \{v_3, v_4\}$ induce two distinct connected components of $G(\bar{S} \setminus \{v_2\})$.

Note that $\operatorname{conv}(S) \ne \operatorname{conv}(V \setminus (N_{\bar{S}}[v_2] \cup \bar{S}_1)) \cap \operatorname{conv}(V \setminus (N_{\bar{S}}[v_2] \cup \bar{S}_2))$ since $\operatorname{conv}(S)$ is a triangle but the intersection of $\operatorname{conv}(V \setminus \{v_1, v_2, v_3, v_4\})$ and $\operatorname{conv}(V \setminus \{v_1, v_2, v_3, v_8\})$ is a parallelogram. □

5.2 2-vertex cut

The next theorem generalizes Theorem 21 to vertex cuts of cardinality 2. It will play a key role in proving the main result of Section 6.

Theorem 23. *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let $\{v_1, v_2\}$ be a vertex cut of size 2 in $G(\bar{S})$. Let $\bar{S}_1, \ldots, \bar{S}_t$ denote the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. Then $\operatorname{conv}(S) = \bigcap_{i=1}^t \operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] \cup \bar{S}_i))$.*

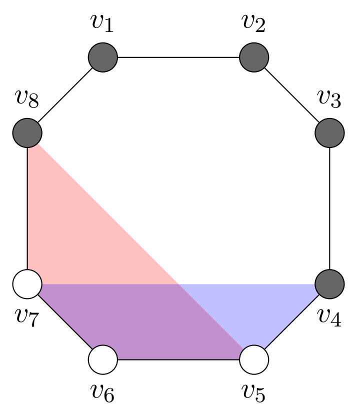

Figure 4: An example in $\mathbb{R}^2$

It is natural to ask whether this theorem can be extended to vertex cuts of larger sizes. The 3-vertex cut case is open, but it turns out that Theorem 23 cannot be generalized to 4-vertex cutsets as shown by the following example.

Example 3. Consider $\bar{S} = ((\{0,1\}^4 \times \{0\}) \setminus \{e^1 + e^2 + e^3 + e^4\}) \cup \{e^5\}$. Then $x_1 + x_2 + x_3 + x_4 + 3x_5 \geq 4$ is a facet-defining inequality for $\operatorname{conv}(S)$. Note that it cuts off all points in $\bar{S}$. In addition, $\bar{C} := \{e^1, e^2, e^3, e^4\}$ is a vertex cut of cardinality four in $\bar{S}$. Then $\bar{S}_1 := \{0, e^5\}$ and $\bar{S}_2 := \{e^1+e^2+e^3, e^1+e^2+e^4, e^1+e^3+e^4, e^2+e^3+e^4, e^1+e^2, e^1+e^3, e^1+e^4, e^2+e^3, e^2+e^4, e^3+e^4\}$ induce two connected components of $G(\bar{S} \setminus \bar{C})$. However,

$$\operatorname{conv}(S) \neq \bigcap_{i=1}^{2} \operatorname{conv}(\{0,1\}^5 \setminus (N_{\bar{S}}[e^1] \cup \ldots \cup N_{\bar{S}}[e^4] \cup \bar{S}_i))$$

since $x_1 + x_2 + x_3 + x_4 + 3x_5 \geq 4$ is not valid for $\operatorname{conv}(\{0,1\}^5 \setminus (N_{\bar{S}}[e^1] \cup \ldots \cup N_{\bar{S}}[e^4] \cup \bar{S}_i))$ for $i = 1, 2$. $\square$

To prove Theorem 23, we will use Theorem 2. This entails analyzing the adjacency on the skeleton of $\operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ between two points in different connected components of the graph $G(\bar{S} \setminus \{v_1, v_2\})$. To do this, we need a linear description of $\operatorname{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. Theorem 27 will give such a characterization. Its proof requires several lemmas.

Lemma 24. *Let $\bar{S} = \{0, e^1, \ldots, e^k, e^1 + e^2, e^2 + e^3, e^3 + e^1, e^1 + e^2 + e^3\}$ for some $k \geq 4$. Note that $(0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1, e^1 + e^2 + e^3)$ is a cube and $e^4, \ldots, e^k$ are pendant vertices adjacent to 0 on H_n. Then $\operatorname{conv}(S)$ is described by the star inequality for the star $(0, e^1, e^2, \ldots, e^k)$, the cube inequality for the cube $(0, e^1, e^2, e^3, e^1+e^2, e^2+e^3, e^3+e^1, e^1+e^2+e^3)$, the edge inequalities for the edges connecting 0 to $e^4, \ldots, e^k$, and the bounds $0 \leq x \leq 1$.*

Proof. If $k = n$, then $A := \{x \in [0,1]^n : \sum_{i=1}^n x_i \geq 2, \sum_{i=4}^n x_i \geq 1\}$ is an integral polytope since the constraint matrix of A is totally unimodular by Remark 3. Thus we may assume that $k < n$. It suffices to show that $A' = \{x \in [0,1]^n : \sum_{i=4}^n x_i \geq 1, \sum_{i=1}^k x_i + 2\sum_{j=k+1}^n x_i \geq 2, \sum_{i \in N \setminus \{j\}} x_i \geq 1 \text{ for } 4 \leq j \leq k\}$ is an integral polytope. Let v be an extreme point of A'. If the cube inequality is not active, v is integral by Lemma 10. Thus we may assume that $\sum_{i=4}^n v_i = 1$. Then the edge inequalities for $j = 1, 2, 3$ are implied by the cube equality. Subtracting the equation $\sum_{i=4}^{n} v_i = 1$ from the star inequality gives $\sum_{i=1}^{3} v_i + \sum_{j=k+1}^{n} v_j \geq 1$. Note that $\sum_{i=1}^{3} v_i + \sum_{j=k+1}^{n} v_j \geq 1$ and $0 \leq v \leq 1$ imply the edge inequalities for $4 \leq j \leq k$. Therefore v is a vertex of the polytope defined by $\sum_{i=4}^{n} v_i = 1$, $\sum_{i=1}^{3} v_i + \sum_{j=k+1}^{n} v_j \geq 1$, and $0 \leq x \leq 1$. This system is totally unimodular by Remark 3, which implies that v is integral. $\square$

Lemma 25. *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let $\bar{x}$ and $\bar{y}$ be 2 points at distance 2 in H_n, i.e., $\bar{y} = \bar{x}^{ij}$ for some i, j. Then $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of* $\mathrm{conv}(S)$ *if and only if $\bar{x}^i$ or $\bar{x}^j$ is in $\bar{S}$.*

Proof. Without loss of generality, we may assume that $\bar{x} = 0$ and $\bar{y} = e^1 + e^2$.

If $e^1 \in \bar{S}$, then the corresponding vertex inequality $-x_1 + \sum_{i=2}^{n} x_i \geq 0$ is valid for $\mathrm{conv}(S)$ and active at both $\bar{x}$ and $\bar{y}$. We also know that $x_i \geq 0$ for $i \geq 3$ are all active at both $\bar{x}$ and $\bar{y}$. Since these $n-1$ inequalities are linearly independent, $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of $\mathrm{conv}(S)$. Likewise if $e^2 \in \bar{S}$.

If all four points $0, e^1, e^2$ and e^1+e^2 belong to S, then the corresponding square is a 2-dimensional face of $\mathrm{conv}(S)$. The center of the square can be obtained as a nontrivial convex combination of 4 distinct vertices of $\mathrm{conv}(S)$, and therefore it does not lie on any 1-dimensional face of $\mathrm{conv}(S)$. Thus the diagonal connecting 0 to $e^1 + e^2$ is not a face of $\mathrm{conv}(S)$. $\square$

Lemma 26. *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let $\bar{x}, \bar{y} \in S$ be 2 points at distance 3 in H_n, i.e., $\bar{y} = \bar{x}^{ijk}$ for some i, j, k. Note that $(\bar{x}^i, \bar{x}^{ij}, \bar{x}^j, \bar{x}^{jk}, \bar{x}^k, \bar{x}^{ki})$ is a cycle of length 6 in H_n. Then $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of* $\mathrm{conv}(S)$ *if and only if there exist 3 consecutive vertices in the cycle that are contained in $\bar{S}$.*

Proof. Without loss of generality, assume that $\bar{x} = 0$ and $\bar{y} = e^1 + e^2 + e^3$.

Assume that there exist three consecutive vertices in the cycle which are in $\bar{S}$. We would like to show that $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of $\mathrm{conv}(S)$. Without loss of generality, we may assume that e^1, $e^1 + e^2$, and e^2 are in $\bar{S}$. Then two edge inequalities $-x_1 + \sum_{i=3}^{n} x_i \geq 0$ and $-x_2 + \sum_{i=3}^{n} x_i \geq 0$ are valid for $\mathrm{conv}(\{0,1\}^n \setminus \bar{S})$. Furthermore, they are active at both $\bar{x}$ and $\bar{y}$. In addition, $x_i \geq 0$ is active at both $\bar{x}$ and $\bar{y}$ for $i \geq 4$. Since these $n-1$ inequalities are linearly independent, $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of $\mathrm{conv}(S)$.

Assume that we can find a vertex contained in S among every three consecutive vertices of the 6-cycle. Then S contains either two vertices of the cycle at distance three or three vertices of the cycle such that the distance is two between any pair. In both cases, the center of the cube is a convex combination of two or three points of S in the cycle. Since the segment joining $\bar{x}$ to $\bar{y}$ also contains the center of the cube, this segment cannot be a 1-dimensional face of $\mathrm{conv}(S)$. $\square$

Theorem 27. *Let $\bar{S} \subseteq \{0,1\}^n$ and $v_1, v_2 \in \bar{S}$. Then* $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ *is described by edge, star, square, cube, propeller inequalities and the bounds* $0 \leq x \leq 1$.

Proof. The theorem holds when $n = 3$ since any polytope in $[0,1]^3$ has Chvátal rank at most 3. Thus we assume $n \geq 4$. Consider $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. The structure of the graph varies according to the distance between v_1 and v_2 on H_n. Without loss of generality, we may assume that $v_1 = 0$.

Distance 1 :

We may assume that $v_2 = e^1$. Note that $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ consists of squares containing $v_1 v_2$ as a common edge and pendant vertices that are adjacent to either v_1 or v_2. If the number of squares is less then 3, then $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ can be described by edge, star, and square inequalities by Theorems 11 and 12. If the number of squares is at least 3, then $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ consists of a propeller and pendant vertices adjacent to v_1 or v_2. Let $\bar{P}$ the set of vertices in the propeller. Let $\bar{U}$ and $\bar{W}$ denote the sets of pendant vertices adjacent to v_1 and v_2, respectively. If one of $\bar{U}$ and $\bar{W}$ is empty, then $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ satisfies the theorem by Lemma 20. So we assume there are two vertices $u \in \bar{U}$ and $w \in \bar{W}$. In fact, $u = e^i$ and $w = e^1 + e^j$ for some i and j with $i \neq j$. Since both e^i and e^1+e^j are pendant vertices of $N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$, e^1+e^i and e^j are not in $\bar{P}$. It follows that u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus \bar{P})$ by Lemma 26. Now, by Theorem 2, we get $\mathrm{conv}(\{0,1\}^n \setminus (\bar{P} \cup \bar{U} \cup \bar{W})) = \mathrm{conv}(\{0,1\}^n \setminus (\bar{P} \cup \bar{U})) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{P} \cup \bar{W}))$. Thus, by Lemma 20 and Theorem 12, the theorem holds in this case.

Distance 2 :

If the distance in H_n between v_1 and v_2 is 2, there exist j, ℓ such that $1 \leq j, \ell \leq n$ and $v_2 = e^j + e^\ell$. Note that the square $(0, e^j, e^\ell, e^j + e^\ell)$ in H_n contains v_1 and v_2 as its diagonal. If both e^j and e^ℓ are in $\bar{S}$, they are contained in $N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$. In this case, $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains the square $(0, e^j, e^\ell, e^j + e^\ell)$. Note that the other squares of H_n cannot be contained in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. Indeed, every vertex in $N_{\bar{S}}[v_1] \setminus \{v_1, e^j, e^\ell\}$ is a pendant vertex in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. Therefore, $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ consists of, possibly, a square containing v_1 and v_2 as its diagonal and some pendant vertices adjacent to either v_1 or v_2. Then $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains no cube, tulip or propeller. By Theorems 11 and 12, $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ satisfies the theorem.

Distance 3 :

Now we assume that the distance in H_n between v_1 and v_2 is 3. Then there exist j, k, ℓ such that $1 \leq j, k, \ell \leq n$ and $v_2 = e^j + e^k + e^\ell$. Note that the cube $\bar{C} := (0, e^j, e^k, e^\ell, e^j + e^k, e^k + e^\ell, e^\ell + e^j, e^j + e^k + e^\ell)$ in H_n contains v_1 and v_2 as its diagonal, and that each vertex in $N_{\bar{S}}[v_1] \setminus \bar{C}$ and $N_{\bar{S}}[v_2] \setminus \bar{C}$ is a pendant vertex. We claim that if $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains a tulip, then edge, star, and cube inequalities with $0 \leq x \leq 1$ are sufficient. If a tulip exists in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$, there exists a vertex with degree at least 4 incident to 3 squares in this graph. Only v_1 and v_2 can have degree at least 4. If the tulip contains v_1, then all vertices of $\bar{C} \setminus \{v_2\}$ are in $\bar{S}$. In fact, $\bar{C} \subseteq \bar{S}$, because $v_2 \in \bar{S}$. Then $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ may consist of a cube with v_1, v_2 as its diagonal plus some pendant vertices adjacent to v_1 or v_2. If all the pendant vertices are adjacent to a single vertex, then $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ is described by the cube inequality for $\bar{C}$, and the edge and star inequalities for $N_{\bar{S}}[v_1]$ by Lemma 24. Thus we may assume each of v_1 and v_2 is connected to a pendant vertex. Note that $\mathrm{conv}(\{0,1\}^n \setminus \bar{C}) = \{x \in [0,1]^n : \sum_{i \in N \setminus \{j,k,\ell\}} x_i \geq 1\}$. Let $\bar{U}$ and $\bar{W}$ denote the sets of pendent vertices adjacent to v_1 and v_2, respectively. If $u \in \bar{U}$ and $w \in \bar{W}$, it is easy to show that u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus \bar{C})$. Therefore, Theorem 2 implies $\mathrm{conv}(\{0,1\}^n \setminus (\bar{C} \cup \bar{U} \cup \bar{W})) = \mathrm{conv}(\{0,1\}^n \setminus (\bar{C} \cup \bar{U})) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{C} \cup \bar{W}))$. By Lemma 24, edge, star, and cube inequalities with $0 \leq x \leq 1$ is sufficient in this case.

If $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ does not contain a tulip, then a point in the cube is not in $\bar{S}$, so $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ does not contain any tulip and cube. Obviously, propellers do not exist in the graph, either. Then we get that $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ satisfies the theorem by Corollary 13.

Distance at least 4 :

Assume that the distance on H_n between v_1 and v_2 is at least 4. There is no edge between a vertex of $N_{\bar{S}}[v_1]$ and a vertex of $N_{\bar{S}}[v_2]$. By Theorem 2, $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])) = \mathrm{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v_1]) \cap \mathrm{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v_2])$ and by Lemma 10, it satisfies the theorem. □

To prove Theorem 23, we first delete two star cutsets $N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ from $\{0,1\}^n$. If we can prove that no edge connects a vertex of $\bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ to a vertex of $\bar{S}_j \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ for $i \neq j$, the theorem follows by Theorem 2. Theorem 27 provides us with the linear description of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. Therefore, we only need to consider edge, star, square, propeller, cube inequalities and the bounds $0 \leq x \leq 1$ in order to analyze the adjacency of vertices on the polytope $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Proof of Theorem 23. We assume that $n \geq 4$ to ignore trivial cases. If at most one set $\bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ is nonempty, the theorem trivially holds. So assume that at least two such sets exist. Let $u \in \bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ and $w \in \bar{S}_j \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ for some $i \neq j$. We will show that no edge connects u and w in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$.

Let pq be an edge in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. Suppose that the edge inequality for pq is active at both u and w. Then both u and w are adjacent to the edge pq. If $p \in \{v_1, v_2\}$, then u and w are not adjacent to p by the definition of $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$. Then, we get that $q \notin \{v_1, v_2\}$ and both u and w are adjacent to q. But (u, q, w) is a path contained in $G(\bar{S} \setminus \{v_1, v_2\})$, contradicting the assumption that u and w belong to distinct sets $\bar{S}_i$ and $\bar{S}_j$ in $G(\bar{S} \setminus \{v_1, v_2\})$. Hence, we may assume that $p, q \in \bar{S} \setminus \{v_1, v_2\}$. But then u and w are connected in $G(\bar{S} \setminus \{v_1, v_2\})$, which is again a contradiction. Therefore no edge inequality is active at both u and w.

Since u and w are disconnected in $G(\bar{S} \setminus \{v_1, v_2\})$, the distance in H_n between u and w is at least 2. If the distance is exactly 2, w can be written as u^{ij} for some i, j. Since u cannot be adjacent to v_1 and v_2, we get $u^i, u^j \notin \{v_1, v_2\}$. Besides, $u^i, u^j \notin \bar{S}$. Otherwise, u and w are connected in $G(\bar{S} \setminus \{v_1, v_2\})$, which contradicts the assumption. Then, u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ by Lemma 25. Therefore, we may assume that the distance in H_n between u and w is at least 3.

In Theorem 27, we showed that the structure of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ depends on the distance on H_n between v_1 and v_2. To prove Theorem 23, we consider different cases according to this distance. Without loss of generality, we may assume $v_1 = 0$.

<u>Distance 1</u> :

Without loss of generality, we may assume that $v_2 = e^1$ since $v_1 = 0$. We showed in Theorem 27 that $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ can be described by edge, star, square, and propeller (if it exists) inequalities.

Note that each square in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains v_1v_2 as an edge. Pick one square and consider the corresponding square inequality. Let p, q denote the other two vertices in the square. If the inequality is active at both u and w, then u and w are adjacent to a vertex in the square. Since u and w cannot be adjacent to any of v_1 and v_2, they are adjacent to either p or q. In this case, u and w are connected by the edge pq in $G(\bar{S} \setminus \{v_1, v_2\})$ which contradicts the assumption that u and w are disconnected. Hence no square inequality is active at both u and w.

Consider the star inequality for $N_{\bar{S}}[v_1]$. If it is active at both, then each of u and w is adjacent to two vertices in $N_{\bar{S}}[v_1] \setminus \{v_1\}$. Since u and w cannot have a common neighbor vertex in $N_{\bar{S}}[v_1] \setminus \{v_1\}$, there exist four distinct vertices $e^p, e^q, e^r, e^s \in N_{\bar{S}}[v_1] \setminus \{v_1\}$ such that $u = e^p + e^q$ and $w = e^r + e^s$. In addition, we know that $p, q, r, s > 1$, because u and w cannot be adjacent to v_2. That means the star inequality for $N_{\bar{S}}[v_2]$ cannot be active at u and w. This implies that at most one star inequality is active at both u and w.

If a star inequality is active at both u and w, we observed that $n-4$ among $0 \leq x \leq 1$ are active at both. Besides, the other star inequality is not active at both. Even if the propeller inequality is active at both u and w, we have only $n-2$ inequalities active at both u and w. In no star inequality is active at both, then we know that at most $n-3$ among $0 \le x \le 1$ are active at both. Then we have at most $n-2$ inequalities active at both u and w. Therefore, u and w are not adjacent in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

<u>Distance 2</u> :

Without loss of generality, we may assume that $v_2 = e^1 + e^2$. By Theorem 27, $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ can be described by edge, star, and square (if it exists) inequalities.

Consider the star inequality for $N_{\bar{S}}[v_1]$. If it is active at both u and w, we know that u and w can be written as $e^p + e^q$ and $e^r + e^s$, respectively, for some distinct p, q, r, s. Besides, $n-4$ inequalities among $0 \le x \le 1$ are active at both u and w. We need two more active inequalities. Then the other star inequality and the square inequality should be active at both u and w. Then we may assume that $p = 1$ and $r = 2$, so u and w can be written as $e^1 + e^q$ and $e^2 + e^s$, respectively. Without loss of generality, assume that $q = 3$ and $s = 4$. Note that $\{e^1, e^2, e^3, e^4\} \subseteq N_{\bar{S}}[v_1]$ and $\{e^1, e^2, e^1 + e^2 + e^3, e^1 + e^2 + e^4\} \subseteq N_{\bar{S}}[v_2]$. In this case, the followings are $n-1$ inequalities that are active at both u and w.

$$x_i \ge 0 \text{ for } i \ge 5,\ \sum_{i=3}^{n} x_i \ge 1,\ \sum_{i=1}^{k_1} x_i + 2\sum_{j=k_1+1}^{n} x_j \ge 2,\ -x_1 - x_2 + \sum_{i=3}^{k_2} x_i + 2\sum_{j=k_2+1}^{n} x_j \ge 0$$

for some $k_1, k_2 \ge 4$. Note that $x_i = 0$ for $i \ge 5$ and $\sum_{i=1}^{k_1} x_i + 2\sum_{j=k_1+1}^{n} x_j = 2$ imply that $x_1 + x_2 + x_3 + x_4 = 2$. Besides, $x_i = 0$ for $i \ge 5$ and $-x_1 - x_2 + \sum_{i=3}^{k_2} x_i + 2\sum_{j=k_2+1}^{n} x_j = 0$ imply that $-x_1 - x_2 + x_3 + x_4 = 0$. Then we get that $x_3 + x_4 = 1$ by adding the two equations. Since $x_3 + x_4 = 1$ and $x_i = 0$ for $i \ge 5$ imply $\sum_{i=3}^{n} x_i = 1$, it follows that at most $n-2$ linearly independent inequalities are active at both u and w in this case.

Therefore we may assume that no star inequality is active at both u and w. The only remaining candidates are at most $n-3$ inequalities among $0 \le x \le 1$ and the square inequality, so we have at most $n-2$ linearly independent inequalities active at both. Therefore, u and w are not adjacent in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

<u>Distance 3</u> :

Without loss of generality, we may assume that $v_2 = e^1 + e^2 + e^3$. Each square contains either v_1 or v_2 but not both. Suppose that a square inequality is active at both u and w. Without loss of generality, assume that the square is $(0, e^1, e^2, e^1 + e^2)$. Since u and w cannot be adjacent to $v_1 (= 0)$, they are adjacent to either e^1, e^2, or $e^1 + e^2$. However, this contradicts the assumption that u and w are disconnected. Hence, no square inequality is active at both u and w.

First, consider the case when a vertex in the cube $(0, e^1, e^2, e^3, e^1+e^2, e^2+e^3, e^3+e^1, e^1+e^2+e^3)$ is not in $\bar{S}$. In this case, $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ is described by edge, star, and square inequalities together with $0 \le x \le 1$ by Theorem 27.

Consider a star contained in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. If the star is not $N_{\bar{S}}[v_1]$ or $N_{\bar{S}}[v_2]$, then we showed in Theorem 27 that it is entirely contained in the cube. If the corresponding star inequality is active at u, then either u is in the cube or u is a vertex outside of the cube adjacent to the root r of the star. Note that a vertex in the cube is adjacent to either v_1 or v_2. This means that u cannot be in the cube, and u is adjacent to r. If the inequality is also active at w, then w is adjacent to r as well. Hence, we get that (u, r, w) is a path contained in $G(\bar{S} \setminus \{v_1, v_2\})$. Therefore, the star inequality is not active at both u and w.

Thus, we only need to consider two star inequalities for $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$. Consider the star inequality for $N_{\bar{S}}[v_1]$. If it is active at both u and w, then $n-4$ inequalities among $0 \le x \le 1$ are active at both u and w. But then at most $n-2$ inequalities are active at both u and w since no edge and square inequality is active at both u and w.

If no star inequality is active at both u and w, then no inequality other than $0 \le x \le 1$ is active at both in fact. Since at most $n-3$ inequalities among $0 \le x \le 1$ are active at both u and w, we cannot find $n-1$ linearly independent inequalities active at both in this case, either.

Now consider the case when all the vertices in the cube are in $\bar{S}$. By Theorem 27, the cube inequality and the two star inequalities that correspond to $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$ together with $0 \le x \le 1$ describe $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Suppose that the cube inequality is active at both u and w. Then u and w are adjacent to at least one vertex in the cube on H_n distinct from v_1 and v_2. That means u and w are connected by six vertices $(e^1, e^2, e^3, e^1+e^2, e^2+e^3, e^3+e^1)$ in $G(\bar{S} \setminus \{v_1, v_2\})$. However, this contradicts the assumption that u and w are disconnected. Therefore, the cube inequality is not active at both u and w.

If a star inequality is active at both u and w, then we know that the distance on H_n between them is 4. As in the previous case, at most $n-2$ inequalities are active at both u and w. This leads to a contradiction. If no star inequality is active at both u and w, then no inequality other than $0 \le x \le 1$ is active at both u and w.

Therefore, u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Distance at least 4 :

We know that $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$ are two separated stars. By Theorem 27, we know that edge and star inequalities together with $0 \le x \le 1$ describe $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Consider the star inequality corresponding to $N_{\bar{S}}[v_1]$. If it is active at both u and w, then $n-4$ inequalities among $0 \le x \le 1$ are active at both u and w. Since no edge inequality is active at both u and w, we have at most $n-2$ inequalities that are active at both u and w since the only candidates are two star inequalities and the bounds. This contradicts to observation that there exist $n-1$ linearly independent inequalities that are active both u and w.

Therefore, u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. □

5.3 Implication for the Chvátal rank

Theorems 21 and 23 imply bounds on the Chvátal rank of P when $G(\bar{S})$ has a vertex cutset of size one or two.

Corollary 28. *Let $P = \cap_{i=1}^t P_i$, where $P_i \subseteq [0,1]^n$ are polytopes. Let $V_i = P_i \cap \{0,1\}^n$, $S = P \cap \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$.*

(i) Let v be a cut vertex in $G(\bar{S})$, let $\bar{S}_1, \ldots, \bar{S}_t$ induce the connected components of $G(\bar{S} \setminus \{v\})$. Assume $V_i = \{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i)$. Then the Chvátal rank of P is no greater than the maximum Chvátal rank of P_i, $i = 1, \ldots, t$.

(ii) Let $\{v_1, v_2\}$ be a vertex cut of size two in $G(\bar{S})$. Let $\bar{S}_1, \ldots, \bar{S}_t$ induce the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. Assume $V_i = \{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] \cup \bar{S}_i)$. Then the Chvátal rank of P is no greater than the maximum Chvátal rank of P_i, $i = 1, \ldots, t$.

6 Graphs of tree width 2

Trees can be generalized using the notion of tree width. A connected graph has tree width one if and only if it is a tree. Next, we focus our attention on the case when $G(\bar{S})$ has tree width two. Instead of working directly with the definition of tree width, we will use the following characterization: A graph has tree width at most two if and only if it contains no K_4-minor; furthermore a graph with no K_4-minor and at least four vertices always has a vertex cut of size two. The main result of this section is that P has Chvátal rank at most 4 when $G(\bar{S})$ has tree width 2.

Theorem 29. *Let $P \subseteq [0,1]^n$, $S = P \cap \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. If $G(\bar{S})$ has tree width 2, the Chvátal rank of P is at most 4.*

We first prove two lemmas.

Lemma 30. *Consider a star $\bar{N} = (\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ for some $\bar{x}$ and $t \geq 3$. Take a subset $\bar{T}$ of $\{\bar{x}^{i_j i_k} : 1 \leq j < k \leq t\}$ such that $\bar{x}^{i_j i_k}, \bar{x}^{i_k i_\ell} \in \bar{T}$ implies $\bar{x}^{i_j i_\ell} \notin \bar{T}$. Let $\bar{S}$ be the union of $\bar{N}$ and $\bar{T}$. Then $\mathrm{conv}(S)$ is described by the star inequality for $\bar{N}$, edge inequalities for the edges connecting $\bar{x}$ and pendant vertices of $G(\bar{S})$, square inequalities for all squares, propeller inequalities for all propellers and the bounds $0 \leq x \leq 1$.*

Proof. Note that the distance on H_n between $\bar{x}$ and any $\bar{y} \in \bar{T}$ is 2, and adding $\bar{y}$ to the star $\bar{N}$ creates a square. Furthermore $G(\bar{S})$ does not contain cubes or tulips. We argue by induction on $|\bar{T}|$. If $|\bar{T}| = 0$, the assertion is true by Lemma 10. If $|\bar{T}| = 1$, it holds by Lemma 17. Assume that the assertion is true when $|\bar{T}| = m$ for some $m \geq 1$.

Consider the case when $|\bar{T}| = m+1$. Let $\bar{x}^{i_j i_k} \in \bar{T}$. By the induction hypothesis, $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup (\bar{T} \setminus \{\bar{x}^{i_j i_k}\}))) = \mathrm{conv}(S \cup \{\bar{x}^{i_j i_k}\})$ is described by edge inequalities, the star inequality for $\bar{N}$, the square and propeller inequalities for all squares and propellers contained in $G(\bar{S} \setminus \{\bar{x}^{i_j i_k}\})$ with $0 \leq x \leq 1$. Since pendant vertices of $G(\bar{S})$ also have degree 1 in $G(\bar{S} \setminus \{\bar{x}^{i_j i_k}\})$, the edge inequalities for the edges connecting $\bar{x}$ and pendant vertices of $G(\bar{S})$ appear in the description of $\mathrm{conv}(S \cup \{\bar{x}^{i_j i_k}\})$. Adding $\bar{x}^{i_j i_k}$ to $G(\bar{S} \setminus \{\bar{x}^{i_j i_k}\})$ will create a square $(\bar{x}, \bar{x}^{i_j}, \bar{x}^{i_k}, \bar{x}^{i_j i_k})$ in $G(\bar{S})$ which might be contained in two propellers of $G(\bar{S})$ (one has $\bar{x}\bar{x}^{i_j}$ as its axis, while the other one has $\bar{x}\bar{x}^{i_k}$). We know that the edge inequalities for $\bar{x}\bar{x}^{i_j}$ and $\bar{x}\bar{x}^{i_k}$ are dominated by the square inequality for $(\bar{x}, \bar{x}^{i_j}, \bar{x}^{i_k}, \bar{x}^{i_j i_k})$. We will prove that to get $\mathrm{conv}(S)$ from $\mathrm{conv}(S \cup \{\bar{x}^{i_j i_k}\})$, we just add the square inequality for the square $(\bar{x}, \bar{x}^{i_j}, \bar{x}^{i_k}, \bar{x}^{i_j i_k})$ and two propeller inequalities for two propellers of $G(\bar{S})$ that contain $(\bar{x}, \bar{x}^{i_j}, \bar{x}^{i_k}, \bar{x}^{i_j i_k})$ in $G(\bar{S})$ to the description of $\mathrm{conv}(S \cup \{\bar{x}^{i_j i_k}\})$.

Let $\bar{T}_j$ and $\bar{T}_k$ denote $\{\bar{x}^{i_j i_{j'}} \in \bar{T} : j' \neq k\}$ and $\{\bar{x}^{i_k i_{k'}} : k' \neq j\}$, respectively. Let $\bar{x}^{i_j i_{j'}} \in \bar{T}_j$ and $\bar{x}^{i_k i_{k'}} \in \bar{T}_k$. Then we know that $\bar{x}^{i_k i_{j'}} \notin \bar{T}$ and $\bar{x}^{i_j i_{k'}} \notin \bar{T}$, because $\bar{x}^{i_j i_k} \in \bar{T}$. Therefore $\bar{x}$ is a cut vertex of $G(\bar{N} \cup \bar{T}_j \cup \bar{T}_k)$, since $\bar{T}_j$ and $\bar{T}_k$ are separated in $G((\bar{N} \cup \bar{T}_j \cup \bar{T}_k) \setminus \{\bar{x}\})$. By Theorem 21, we get that $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k)) = \mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j)) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_k))$. Note that $\bar{N} \cup \bar{T}_j$ consists of squares that share $\bar{x}\bar{x}^{i_j}$ as a common edge and pendant vertices which are adjacent to $\bar{x}$. Then $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j))$ is described by edge, star, square, propeller inequalities and the bounds $0 \leq x \leq 1$ by Lemma 20. Similarly, the same statement is also true for $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_k))$. Now, we have the linear description for $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k))$.

We would like to show that there is no edge between $\bar{x}^{i_j i_k}$ and $\bar{T} \setminus (\bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\})$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k))$. Then theorem 2 implies

$$\mathrm{conv}(S) = \mathrm{conv}(S \cup \{\bar{x}^{i_j i_k}\}) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\})).$$

Let $\bar{x}^{i_p i_q} \in \bar{T} \setminus (\bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\})$. Then p and q are distinct from j and k, so the distance on H_n between $\bar{x}^{i_p i_q}$ and $\bar{x}^{i_j i_k}$ is 4. That means $n-4$ among $0 \le x \le 1$ are active at both. We also know that the star inequality is active at both $\bar{x}^{i_j i_k}$ and $\bar{x}^{i_p i_q}$, so $n-3$ linearly independent inequalities are active at both.

Consider two vertices $\bar{x}^{i_j i_{j_1}}$ and $\bar{x}^{i_j i_{j_2}}$ in $\bar{T}_j$. Then $(\bar{x}, \bar{x}^{i_j}, \bar{x}^{i_{j_1}}, \bar{x}^{i_j i_{j_1}})$ and $(\bar{x}, \bar{x}^{i_j}, \bar{x}^{i_{j_2}}, \bar{x}^{i_j i_{j_2}})$ are two squares contained in the propeller that has $\bar{x}\bar{x}^{i_j}$ as its axis. Note that $\bar{x}^{i_p i_q}$ is not adjacent to $\bar{x}^{i_j}$, because p and q are different from j. In addition, the distance on H_n between $\bar{x}^{i_j i_{j_1}}$ and $\bar{x}^{i_p i_q}$ is at least 2, so they are not adjacent. Hence, if $\bar{x}^{i_p i_q}$ is adjacent to those two squares, then we get that $\bar{x}^{i_p i_q}$ is adjacent to both $\bar{x}^{i_{j_1}}$ and $\bar{x}^{i_{j_2}}$ and thus $\{p, q\} = \{j_1, j_2\}$. However, this contradicts the assumption that $\bar{x}^{i_j i_{j_1}}, \bar{x}^{i_j i_{j_2}} \in \bar{T}$ implies $\bar{x}^{i_{j_1} i_{j_2}} \notin \bar{T}$. Therefore, $\bar{x}^{i_p i_q}$ is adjacent to at most one square of the propeller. This directly implies that the propeller inequality for $\bar{T}_j$ is not active at $\bar{x}^{i_p i_q}$. Likewise, we can show that $\bar{x}^{i_p i_q}$ is adjacent in H_n to at most one square of the other propeller, which has $\bar{x}\bar{x}^{i_k}$ as its axis. Besides, the corresponding propeller inequality is not active at $\bar{x}^{i_p i_q}$.

Vertex $\bar{x}^{i_j i_k}$ is adjacent to all squares since they contain either $\bar{x}\bar{x}^{i_j}$ or $\bar{x}\bar{x}^{i_k}$ as an edge. If $\bar{x}^{i_j i_k}$ and $\bar{x}^{i_p i_q}$ are adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k))$, we must find two more square inequalities active at both. That means $\bar{x}^{i_p i_q}$ is adjacent to two squares in $G(\bar{N} \cup \bar{T}_j \cup \bar{T}_k)$, one contained in the propeller with axis $\bar{x}\bar{x}^{i_j}$ and the other in the propeller with axis $\bar{x}\bar{x}^{i_k}$. We may assume without loss of generality that $\bar{x}^{i_j i_p} \in \bar{T}_j$ and $\bar{x}^{i_k i_q} \in \bar{T}_k$. Then the following two equations are satisfied by both $\bar{x}^{i_j i_k}$ and $\bar{x}^{i_p i_q}$.

$$\sum_{\ell \in N \setminus \{i_j, i_p\}} (\bar{x}_\ell(1-x_\ell) + (1-\bar{x}_\ell)x_\ell) = 1, \quad \sum_{\ell \in N \setminus \{i_k, i_q\}} (\bar{x}_\ell(1-x_\ell) + (1-\bar{x}_\ell)x_\ell) = 1.$$

Adding these two equations gives

$$\sum_{\ell \in \{i_j, i_k, i_p, i_q\}} (\bar{x}_\ell(1-x_\ell) + (1-\bar{x}_\ell)x_\ell) + 2 \sum_{\ell \notin \{i_j, i_k, i_p, i_q\}} (\bar{x}_\ell(1-x_\ell) + (1-\bar{x}_\ell)x_\ell) = 2.$$

Given that $\bar{x}^{i_j i_k}_\ell = \bar{x}^{i_p i_q}_\ell = 0$ for $\ell \notin \{i_j, i_k, i_p, i_q\}$, the above equation is equivalent to $\sum_{\ell \in \{i_j, i_k, i_p, i_q\}} (\bar{x}_\ell(1-x_\ell) + (1-\bar{x}_\ell)x_\ell) = 2$ and this implies the equation for the star inequality that is active at both $\bar{x}^{i_j i_k}$ and $\bar{x}^{i_p i_q}$. Therefore, there are at most $n-2$ linearly independent inequalities active at both $\bar{x}^{i_j i_k}$ and $\bar{x}^{i_p i_q}$, and we can conclude that $\bar{x}^{i_j i_k}$ and $\bar{x}^{i_p i_q}$ are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k))$.

To complete the proof, we need to show that edge, star, square, and propeller inequalities with $0 \le x \le 1$ are sufficient to describe $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\}))$. Notice that $\{\bar{x}, \bar{x}^{i_j i_k}\}$ is a vertex cut in $G(\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\})$ of cardinality 2, because $\bar{T}_j$ and $\bar{T}_k$ are separated in $G(\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\} \setminus \{\bar{x}, \bar{x}^{i_j i_k}\})$. By Theorem 23, we get that

$$\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\})) = \mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \{\bar{x}^{i_j i_k}\}) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\}),$$

since $\bar{x}^{i_j}$ and $\bar{x}^{i_k}$ are the only neighbor vertices of $\bar{x}^{i_j i_k}$ in $G(\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\})$. Note that $G(\bar{N} \cup \bar{T}_j \cup \{\bar{x}^{i_j i_k}\})$ consists of squares that share $\bar{x}\bar{x}^{i_j}$ as a common edge and pendant vertices adjacent to $\bar{x}$. By the definition of $\bar{T}_j$, $G(\bar{N} \cup \bar{T}_j \cup \{\bar{x}^{i_j i_k}\})$ contains all the squares of $G(\bar{S})$ that contain $\bar{x}\bar{x}^{i_j}$. By Lemma 20, $\mathrm{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \{\bar{x}^{i_j i_k}\})$ is described by the star inequality for $\bar{N}$, the propeller inequality for the propeller of $G(\bar{S})$ that has $\bar{x}\bar{x}^{i_j}$ as its axis, and square inequalities for all the squares of $G(\bar{S})$ containing $\bar{x}\bar{x}^{i_j}$ together with $0 \le x \le 1$. The same statement is true for $\text{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \{\bar{x}^{i_j i_k}\})$, and we now get the description for $\text{conv}(\{0,1\}^n \setminus (\bar{N} \cup \bar{T}_j \cup \bar{T}_k \cup \{\bar{x}^{i_j i_k}\}))$. $\square$

Let $v \in \bar{S}$. Let $M_{\bar{S}}[v]$ denote the set $N_{\bar{S}}[v] \cup \{v^{ij} \in \bar{S} : \ v^i, v^j \in N_{\bar{S}}[v]\}$. Then $M_{\bar{S}}[v]$ contains the closed neighborhood $N_{\bar{S}}[v]$ and the vertices in $\bar{S}$ at distance 2 from v that create a square when added to $N_{\bar{S}}[v]$. If $G(M_{\bar{S}}[v])$ is K_4-minor-free, $M_{\bar{S}}[v]$ is of the form $\bar{N} \cup \bar{T}$ in Lemma 30. Therefore Lemma 30 gives a description of $\text{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v])$.

Let v_1, v_2 be two vertices in $\bar{S}$ that are adjacent on H_n. The next lemma shows that $\text{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2]))$ is described by the inequalities defining $\text{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1])$ and $\text{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2])$.

Lemma 31. *Let $v_1, v_2 \in \bar{S}$ be adjacent vertices in H_n. If $G(\bar{S})$ has tree width 2, then* $\text{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])) = \text{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1]) \cap \text{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2])$.

Proof. In Theorem 27, we showed that $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ is described by square and propeller inequalities for the propeller that contains v_1v_2 as its axis, edge and star inequalities for the two stars $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$, and $0 \le x \le 1$. Let $u \in M_{\bar{S}}[v_1] \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ and $w \in M_{\bar{S}}[v_2] \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. We would like to show that no edge between u and w exists in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Without loss of generality, we may assume that $v_1 = 0$ and $v_2 = e^1$. Since $u \notin N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$, there exist $i, j > 1$ such that $u = e^i + e^j$ and thus $u \notin M_{\bar{S}}[v_2]$. By the definition of $M_{\bar{S}}[v_1]$, we know that $e^i, e^j \in \bar{S}$. In addition, one of $e^1 + e^i$ and $e^1 + e^j$ is not in $\bar{S}$. Otherwise, three squares $(0, e^1, e^i, e^1 + e^i)$, $(0, e^1, e^j, e^1 + e^j)$, and $(0, e^i, e^j, e^i + e^j)$ create a K_4-minor. Likewise, there exist k, ℓ such that $w = e^1 + e^k + e^\ell$ and one of e^k and e^ℓ is not in $\bar{S}$. By the definition of $M_{\bar{S}}[v_2]$, we know that both $e^1 + e^k$ and $e^1 + e^\ell$ are in $\bar{S}$. That means $\{i, j\} \neq \{k, \ell\}$. Thus the distance between u and w in H_n is either 3 or 5, so at most $n - 3$ inequalities among $0 \le x \le 1$ are active both u and w.

Consider an edge inequality that appears in the description of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. Then the corresponding edge has either v_1 or v_2 as its end. If the inequality is active at both u and w, then both of them are adjacent to the other end of the edge. However, this contradicts the observation that the distance on H_n between u and w is at least 3. Since $u \notin M_{\bar{S}}[v_2]$, the star inequality for $N_{\bar{S}}[v_2]$ is not active at u. Hence, no star inequality is active at both u and w. Besides, u and w cannot be adjacent to two squares of the propeller. Otherwise, $G(\bar{S})$ contains a K_4-minor. Thus the propeller inequality cannot be active at both u and w. In addition, at most one square inequality is active at both u and w.

Then we get at most $n - 2$ linearly independent inequalities active at both u and w, so u and w are not adjacent in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. By Theorem 2, we get that

$$\text{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])) = \text{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])) \cap \text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])).$$

To compute $\text{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$, we will again use Theorem 2. We will show that no vertex of $N_{\bar{S}}[v_2] \setminus M_{\bar{S}}[v_1]$ is adjacent to a vertex of $M_{\bar{S}}[v_1] \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup (N_{\bar{S}}[v_2] \cap M_{\bar{S}}[v_1])))$. Note that the vertices in $N_{\bar{S}}[v_2] \setminus M_{\bar{S}}[v_1]$ are pendant vertices in $G(M_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. Let $u \in M_{\bar{S}}[v_1] \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ and $w \in N_{\bar{S}}[v_2] \setminus M_{\bar{S}}[v_1]$. We showed in Theorem 27 that $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup (N_{\bar{S}}[v_2] \cap M_{\bar{S}}[v_1])))$ is described by edge, star, square, propeller inequalities and $0 \le x \le 1$. We proved in the above paragraphs that the propeller inequality cannot be active at u and that at most one square inequality is active at u. We also know that u and w can be written as $e^i + e^j$ for some $i, j > 1$ and $e^1 + e^k$, respectively. Since w is a pendant vertex in $G(M_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$, e^k is not in $\bar{S}$. Otherwise, $w(= e^1 + e^k)$ is adjacent to two vertices $v_2(= e^1)$ and e^k. That means k is distinct from i and j, because both e^i and e^j are in $\bar{S}$ by the definition of $M_{\bar{S}}[v_1]$. Thus, the distance on H_n between u and w is 4, so $n - 4$ among $0 \le x \le 1$ are active at both u and w. Then, it is easy to show that no edge and star inequalities are active at both u and w. Therefore, $N_{\bar{S}}[v_2] \setminus M_{\bar{S}}[v_1]$ is separated from $M_{\bar{S}}[v_1] \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup (N_{\bar{S}}[v_2] \cap M_{\bar{S}}[v_1])))$. Then by Lemma 2, we get that

$$\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])) = \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1]) \cap \mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])).$$

Now, it is sufficient to show that

$$\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1]) \cap \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2]) \subseteq \mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])).$$

The edge and star inequalities for $N_{\bar{S}}[v_1]$ in the description of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ are valid for $\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1])$, because $N_{\bar{S}}[v_1] \subseteq M_{\bar{S}}[v_1]$. Likewise, the edge and star inequalities for $N_{\bar{S}}[v_2]$ are valid for $\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2])$. Since the propeller contained in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ has v_1v_2 as its axis, it is contained in both $M_{\bar{S}}[v_1]$ and $M_{\bar{S}}[v_2]$. Thus, the square and propeller inequalities are valid for both $\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1])$ and $\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2])$ by Lemma 30. Therefore, $\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1]) \cap \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2])$ is a subset of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. □

Proof of Theorem 29. By Lemma 7, it suffices to prove the theorem for Q_S. We argue by induction on $|\bar{S}|$. If $|\bar{S}| = 1$, then the Chvátal rank of Q_S is 1. Assume that the Chvátal rank of Q_S is at most 4 if $|\bar{S}| = t$ for some $t \ge 1$. Consider the case when $|\bar{S}| = t + 1$. We may assume that $G(\bar{S})$ is a connected graph.

Note that a tulip has three squares $(\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \bar{x}^{i_1 i_2})$, $(\bar{x}, \bar{x}^{i_2}, \bar{x}^{i_3}, \bar{x}^{i_2 i_3})$, and $(\bar{x}, \bar{x}^{i_3}, \bar{x}^{i_1}, \bar{x}^{i_3 i_1})$ which are incident to a vertex $\bar{x}$. Hence, a tulip contains a K_4-minor. Likewise, a cube also contains a K_4-minor. Thus, $G(\bar{S})$ contains no tulip and cube. If there is no propeller in $G(\bar{S})$, then the Chvátal rank of Q_S is at most 3 by Theorems 11 and 12. Thus we may assume that $G(\bar{S})$ contains a propeller.

Let v_1 and v_2 denote the two vertices in the axis of the propeller. Note that the propeller contains at least three squares. Let (p, q, v_1, v_2) and (r, s, v_1, v_2) be two distinct squares contained in the propeller. If there is a path between $\{p, q\}$ and $\{r, s\}$ in $G(\bar{S} \setminus \{v_1, v_2\})$, then those two squares and the path create a K_4-minor contained in $G(\bar{S})$, a contradiction. Hence, p and q are disconnected from r and s in $G(\bar{S} \setminus \{v_1, v_2\})$. Then $\{v_1, v_2\}$ is a vertex cut of $G(\bar{S})$. Let $\bar{S}_1, \dots, \bar{S}_k$ be the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. By Theorem 23, we get that

$$\mathrm{conv}(S) = \bigcap_{i=1}^{k} \mathrm{conv}(\{0,1\}^n \setminus (\bar{S}_i \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])).$$

If $|\bar{S}_i \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]| < |\bar{S}|$ for all $i = 1, \dots, k$, then the assertion holds by the induction hypothesis. Thus, we may assume that there exists j such that $\bar{S}_j \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] = \bar{S}$. Then we can easily check that $\bar{S}_i \subset N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ for all $i \neq j$. In this case, we cannot apply the induction hypothesis.

Let $\bar{S}_i$ and $\bar{S}_j$ denote two connected components that contain $\{p,q\}$ and $\{r,s\}$, respectively. Suppose that there exist u,w such that $u \in \bar{S}_i \setminus \{p,q\}$ and $w \in \bar{S}_j \setminus \{r,s\}$. Then we can find $u_0 \in \bar{S}_i \setminus \{p,q\}$ and $w_0 \in \bar{S}_j \setminus \{r,s\}$ such that u_0 is adjacent to one of p and q and w_0 is adjacent to one of r and s. That is because, we assumed that $\bar{S}_i$ and $\bar{S}_j$ induce connected components of $G(\bar{S}\setminus\{v_1,v_2\})$. It is obvious that $u_0, w_0 \notin N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$. Then we get that $u_0 \notin \bar{S}_j \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ and $w_0 \notin \bar{S}_i \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ since $u_0 \notin \bar{S}_j$ and $w_0 \notin \bar{S}_i$. Then $|\bar{S}_i \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]| < |\bar{S}|$ and $|\bar{S}_j \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]| < |\bar{S}|$. However, this contradicts the assumption. Therefore, we may assume that $\bar{S}_i \setminus \{p,q\}$ is empty. In other words, $\bar{S}_i = \{p,q\}$, so the other vertices of $G(\bar{S} \setminus \{v_1,v_2\})$ are disconnected from p and q. Besides, p is adjacent to only v_1 and q, and q is adjacent to only v_2 and p in $G(\bar{S})$.

We would like to show that there is no edge connecting a vertex of $\{p,q\}$ and a vertex of $\bar{S} \setminus M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2]$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$. We first show that this polytope is completely described by edge, star, square, propeller inequalities and the bounds $0 \le x \le 1$.

Note that $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$ can be written as $(M_{\bar{S}}[v_1] \setminus \{p,q\}) \cup (M_{\bar{S}}[v_2] \setminus \{p,q\})$. We know that $\{p,q\}$ is contained both $M_{\bar{S}}[v_1]$ and $M_{\bar{S}}[v_2]$. Since p and q are not adjacent to any other vertices of $\bar{S} \setminus \{v_1,v_2\}$, we have

$$M_{\bar{S}}[v_\ell] \setminus \{p,q\} = M_{\bar{S}\setminus\{p,q\}}[v_\ell] \text{ for } \ell = 1,2.$$

By Lemma 31, we get that

$$\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}\setminus\{p,q\}}[v_1] \cup M_{\bar{S}\setminus\{p,q\}}[v_2])) = \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}\setminus\{p,q\}}[v_1]) \cap \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}\setminus\{p,q\}}[v_2]).$$

Therefore,

$$\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\})) = \mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \setminus \{p,q\})) \cap \mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_2] \setminus \{p,q\})).$$

By Lemma 30, this implies that the polytope $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$ is completely described by edge, star, square, propeller inequalities and the bounds $0 \le x \le 1$.

Let $w \in \bar{S} \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])$. Since p is not adjacent to vertices in $\bar{S}$ except v_1 and q, w is not adjacent to p on H_n. Thus, the distance on H_n between w and p is at least 2. If w is not adjacent to any vertex in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$, then inequalities other than $0 \le x \le 1$ cannot be active at w. That means there exist at most $n-2$ linearly independent inequalities active at both w and p, so w and p are disconnected in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$. Likewise, w is also separated from q in the skeleton. Thus, we may assume that w is adjacent to a vertex of $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$. Without loss of generality, assume that w is adjacent to a vertex in $M_{\bar{S}}[v_1] \setminus \{p,q\}$.

Without loss of generality, we may assume that $v_1 = 0$, $v_2 = e^1$, $p = e^2$, and $q = e^1 + e^2$. By the above assumption, w is adjacent to a vertex of either $N_{\bar{S}}[v_1] \setminus \{v_1\}$ or $M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$.

First, assume that w is adjacent to a vertex of $N_{\bar{S}}[v_1] \setminus \{v_1\}$. In this case, w can be written as $e^i + e^j$ for some i,j. If both e^i and e^j are in $\bar{S}$, then $e^i + e^j$ is contained in $M_{\bar{S}}[v_1]$. Thus, we may assume that $e^i \in \bar{S}$ and $e^j \notin \bar{S}$. Since w is not adjacent to v_2 and p, we get that $i,j > 2$. Consider the cube $(p, p^2, p^i, p^j, p^{2i}, p^{ij}, p^{j2}, p^{2ij})$. We know that $p^i = e^2 + e^i$, $p^j = e^2 + e^j$, $p^{2j} = e^j$ are not in $\bar{S}$. That is because p^i and p^j are both adjacent to p and $p^{2j} = e^j \notin \bar{S}$ by the assumption. Then those are not in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$, because $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\} \subseteq \bar{S}$. By Lemma 26, p and w are not adjacent in the skeleton.

It remains to show that q and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$. Note that any vertex of $N_{\bar{S}}[v_2] \setminus \{v_2, q\}$ is either 0 or $e^1 + e^\ell$ for some $\ell > 2$. Then it is obvious that $w = e^i + e^j$ is not adjacent in H_n to any vertex in $N_{\bar{S}}[v_2]$. If $e^i + e^j$ is adjacent to a vertex in $M_{\bar{S}}[v_2] \setminus N_{\bar{S}}[v_2]$, then the only possible candidates are e^i and $e^1 + e^i + e^j$ since $e^j \notin \bar{S}$. We know that $e^i + e^j$ is adjacent to e^i. If $e^1 + e^i + e^j \in M_{\bar{S}}[v_2]$, then both $e^1 + e^i$ and $e^1 + e^j$ are in $N_{\bar{S}}[v_2]$. Then three squares $(0, e^1, e^i, e^1 + e^i)$, $(e^1, e^1 + e^i, e^1 + e^j, e^1 + e^i + e^j)$, and $(e^i, e^1 + e^i, e^i + e^j, e^1 + e^i + e^j)$ are contained in $G(\bar{S})$ in this case, but they create a K_4-minor. Thus, $e^1 + e^i + e^j \notin M_{\bar{S}}[v_2]$. Therefore, e^i is the only vertex of $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$ adjacent to w in H_n. Recall that $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$ is completely described by edge, star, square, propeller inequalities and the bounds $0 \le x \le 1$. The square inequalities for squares that have $0e^i$ as an edge are active at w, and the propeller inequality for the propeller that has $0e^i$ as its axis is active at w. We know that $p^i = e^2 + e^i$ is not in $\bar{S}$, so the square $(0, e^2, e^i, e^2 + e^i)$ of H_n is not contained in the propeller. Then $q(= e^1 + e^2)$ is adjacent to at most one square of the propeller, which is possibly $(0, e^1, e^i, e^1 + e^i)$. This means that at most one square inequality is active at both q and w, and the propeller inequality is not active at both. Since the distance in H_n between q and w is 4, at most $n-3$ linearly independent inequalities are active at both q and w. Therefore q and w are not adjacent in the skeleton.

Second, assume that w is adjacent to a vertex of $M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$. In this case, w can be written as $e^i + e^j + e^k$ for some i, j, k where $e^i + e^j \in M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$. Then we know that both e^i and e^j are in $\bar{S}$. If i or j is 1, then w is adjacent to a vertex in $N_{\bar{S}}[v_2] \setminus \{v_2\}$. This reduces to the previous case. Thus, we may assume that $i, j > 1$. If i or j is 2, then p is adjacent to $e^i + e^j \in \bar{S}$. This is impossible. Therefore, i and j are greater than 2. If $k = 1$, then w is $e^1 + e^i + e^j$. Since $0, e^1, e^i, e^j, e^i + e^j, e^1 + e^i + e^j$ are all in $\bar{S}$, both $e^1 + e^i$ and $e^1 + e^j$ are not in $\bar{S}$. Otherwise, $G(\bar{S})$ contains a K_4-minor. Therefore $w(= e^1 + e^i + e^j)$ is adjacent to nothing but $e^i + e^j$ among the vertices of $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$. Then only the square inequality for the square $(0, e^i, e^j, e^i + e^j)$ is active at w. Note that the distance on H_n between p and w is 4 and the distance on H_n between q and w is 3. Then there exist at most $n-2$ linearly independent inequalities active at both w and each of p and q. Hence, neither p nor q is adjacent to w on the skeleton if $k = 1$. If $k = 2$, then $w = e^2 + e^i + e^j$. Since $p(= e^2)$ is not adjacent to any vertex other than 0 and $e^1 + e^2$, both $p^i(= e^2 + e^i)$ and $p^j(= e^2 + e^j)$ are not in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$. As the case when $k = 1$, q and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$. Besides, p and w are not adjacent in the skeleton by Lemma 25. Thus, we may assume that $k > 2$. If $e^i + e^k \in M_{\bar{S}}[v_1] \setminus \{p,q\}$, then we know that e^k also belongs to $\bar{S}$ by the definition of $M_{\bar{S}}[v_1]$. In this case, $(0, e^i, e^j, e^i + e^j)$, $(0, e^i, e^k, e^i + e^k)$, and $(e^i, e^i + e^j, e^i + e^k, e^i + e^j + e^k)$ create a K_4-minor in $G(\bar{S})$. Hence, we get that both $e^i + e^k$ and $e^j + e^k$ do not belong to $\bar{S}$. In fact, $e^i + e^j$ is the only vertex in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}$ which is adjacent to w in this case. Then only the square inequality for the square $(0, e^i, e^j, e^i + e^j)$ is active at w. Similarly, w is adjacent to neither p nor q in the skeleton in this case.

To summarize, we just proved that there is no edge connecting a vertex of $\{p,q\}$ and a vertex of $\bar{S} \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p,q\}))$. Then by Theoem 2, we get that

$$\mathrm{conv}(S) = \mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{S} \setminus \{p,q\})).$$

Since $G(\bar{S} \setminus \{p,q\})$ is a subgraph of $G(\bar{S})$, it also has tree width 2. Besides, $|\bar{S} \setminus \{p,q\}| < |\bar{S}|$. Hence, the Chvátal rank of $Q_{\{0,1\}^n \setminus (\bar{S} \setminus \{p,q\})}$ is at most 4 by induction. By Lemma 31, we also know that the Chvátal rank of $Q_{\{0,1\}^n\setminus(M_{\bar{S}}[v_1]\cup M_{\bar{S}}[v_2])}$ is at most 4. Therefore, we conclude that the Chvátal rank of Q_S is also at most 4. $\square$

7 Dependency on the cardinality of the infeasible set

One can derive an upper bound on the Chvátal rank as a function of $|\bar{S}|$ using the result of Eisenbrand and Schulz [11] showing that the Chvátal rank of a 0,1 polytope is at most $n^2(1+\log_2 n)$.

Lemma 32. *Let $\bar{S}$ be a subset of $\{0,1\}^n$ such that $|\bar{S}| \le n$ and $G(\bar{S})$ is connected. Then at least $n-|\bar{S}|+1$ coordinates of the 0,1 vectors in $\bar{S}$ are fixed at either 0 or 1.*

Proof. Without loss of generality, assume that $0 \in \bar{S}$. Let $I := \{i : x_i = 0, \forall x \in \bar{S}\}$. Suppose $|I| \le n-|\bar{S}|$, then we may assume that $1, 2, \ldots, |\bar{S}| \notin I$. For each $j \notin I$, there exists $y \in \bar{S}$ such that $y_j = 1$. Since $G(\bar{S})$ is connected, there is a path from 0 to y in $G(\bar{S})$. Then there is an edge which is parallel to e^j in the path from 0 to y, because $y_j = 1$. By the supposition, there exists at least one edge parallel to e^j for $j = 1, \ldots, |\bar{S}|$ in $G(\bar{S})$. Pick exactly one edge parallel to e^j for every j, and let $G(E)$ denote the subgraph induced by these $|\bar{S}|$ edges. Note that there is no cycle in $G(E)$. That is because $e^1, \ldots, e^{|\bar{S}|}$ are linearly independent. If $G(E)$ has ℓ connected components, then the number of vertices in $G(E)$ is at least $|\bar{S}|+\ell$. Since $\ell \ge 1$, there are at least $|\bar{S}|+1$ vertices in $G(E)$. However, this contradicts to fact that $G(E)$ is a subgraph of $G(\bar{S})$ and $G(\bar{S})$ has exactly $|\bar{S}|$ vertices. Therefore, $|I| \ge n-|\bar{S}|+1$. $\square$

Lemma 33. *Let $I := \{i : x_i = 0, \forall x \in \bar{S}\}$, and let $ax \ge b$ be a facet-defining inequality for* $\operatorname{conv}(S)$ *other than $0 \le x \le 1$. Then a_i for all $i \in I$ have the same value.*

Proof. Without loss of generality, assume that $I = \{k, \ldots, n\}$. If $ax \ge b$ does not cut off any point in $\bar{S}$, then it is implied by $0 \le x \le 1$. Thus, we may assume that there exists $v \in \bar{S}$ violating $ax \ge b$. Since $v_k = \cdots = v_n = 0$, we get that $\sum_{j=1}^{k-1} a_j v_j < b$. We know that $v + e_i \in S$ for $i \ge k$, so $\sum_{j=1}^{k-1} a_j v_j + a_i \ge b$ and thus $a_i > 0$ for $i \ge k$. Without loss of generality, assume that a_k has the minimum value among $a_k, \ldots, a_n$.

We would like to show that $\sum_{j=1}^{k-1} a_j x_j + a_k \sum_{i=k}^{n} x_i \ge b$ is valid for $\operatorname{conv}(S)$. Suppose not, then there exist $u \in S$ cut off by the above inequality. If $u_i = 0$ for all $i \ge k$, we get that $\sum_{j=1}^{k-1} a_j u_j < b$. Then u also violates $ax \ge b$, but this contradicts to the assumption that $ax \ge b$ is valid for $\operatorname{conv}(S)$. Hence, there exists $i \ge k$ such that $u_i = 1$. Then $b > \sum_{j=1}^{k-1} a_j u_j + \sum_{i=k}^{n} a_k u_i \ge \sum_{j=1}^{k-1} a_j u_j + a_k$. Then $(u_1, \ldots, u_{k-1}, 1, 0, \ldots, 0)$ violates $ax \ge b$ which is impossible by the assumption. Therefore, $\sum_{j=1}^{k-1} a_j x_j + a_k \sum_{i=k}^{n} x_i \ge b$ is valid for $\operatorname{conv}(S)$.

In addition, we know that $ax \ge \sum_{j=1}^{k-1} a_j x_j + a_k \sum_{i=k}^{n} x_i$ for all $x \ge 0$. If there are at least two distinct values among $a_k, \ldots, a_n$, then $ax \ge b$ is equivalent to a linear combination of $\sum_{j=1}^{k-1} a_j x_j + a_k \sum_{i=k}^{n} x_i \ge b$ and $x \ge 0$ and thus $ax \ge b$ is not facet-defining. Therefore a_i for $i \ge k$ have the same value. $\square$

Theorem 34. *If $|\bar{S}| = k$ for some $k \le n$, then the Chvátal rank of P is at most $k^2(1+\log_2 k)$.*

Proof. By Lemma 32, we may assume that $x_i = 0$ for all $i \ge k$ and $x \in \bar{S}$. Let $\bar{S}(k)$ denote the projection of $\bar{S}$ into the space defined by the first k coordinates. Let $Q_S(k)$ denote $\{x \in [0,1]^k : \sum_{j=1}^{k}(\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \ge \frac{1}{2} \text{ for } \bar{x} \in \bar{S}(k)\}$. Note that Q_S is equivalent to $\{x \in [0,1]^n : \sum_{j=1}^{k-1}(\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) + \sum_{i=k}^{n} x_i \ge \frac{1}{2} \text{ for } \bar{x} \in \bar{S}(k)\}$ since $\bar{x}_k = 0$ for $\bar{x} \in \bar{S}(k)$.

Let $ax \geq b$ be a facet-defining inequality for $\operatorname{conv}(S)$. Then, it is easy to show that $\sum_{j=1}^{k} a_j x_j \geq b$ is valid for $\operatorname{conv}(\{0,1\}^k \setminus \bar{S}(k))$ since $a_k = \cdots = a_n$ by Lemma 33. By the theorem of Eisenbrand and Schulz [11], the Chvátal rank of $\sum_{j=1}^{k} a_j x_j \geq b$ is at most $k^2(1 + \log_2 k)$. In other words, $Q_S(k)^{(k^2(1+\log_2 k))} = \operatorname{conv}(S(k))$ where $S(k) := \{0,1\}^k \setminus \bar{S}(k)$. Then we can generate $\sum_{j=1}^{k} a_j x_j \geq b$ from $0 \leq x_j \leq 1$ for $j \leq k$ and $x_k + \sum_{j=1}^{k-1}(\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq \frac{1}{2}$ for $\bar{x} \in \bar{S}(k)$ using at most $k^2(1 + \log_2 k)$ recursive applications of the Chvátal rounding procedure. Using the same multiplier used at each step of the Chvátal procedure applied to $Q_S(k)$, we can generate $\sum_{j=1}^{n} a_j x_j \geq b$ from $0 \leq x \leq 1$ and $\sum_{j=1}^{k-1}(\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) + \sum_{i=k}^{n} x_i \geq \frac{1}{2}$ for $\bar{x} \in \bar{S}(k)$ since we have $a_k = \cdots = a_n$. Therefore, the Chvátal rank of Q_S is at most $k^2(1 + \log_2 k)$, which is also an upper bound on the rank of P by Lemma 7. $\square$

This theorem implies that if the number of infeasible 0,1 vectors is a constant, then P is of constant Chvátal rank.

The next theorem shows that the Chvátal rank of P can be guaranteed to be smaller than the upper bound of $O(n^2 \log n)$ when the cardinality of $\bar{S}$ is bounded above by a quasi-polynomial function $n^{(\log_2 n)^k}$ where k is a constant. The proof uses a result of Eisenbrand and Schulz [11] stating that, if $cx \geq c_0$ is a valid inequality for $\operatorname{conv}(S)$, where the c_js are relatively prime integers, then the Chvátal rank of P is at most $n^2 + 2n \log_2 \|c\|_\infty$.

Theorem 35. *If $|\bar{S}| = O(f(n))$ where f is a quasi-polynomial function of n, then the Chvátal rank of P is $O(n^2 \log \log n)$.*

Proof. The hypothesis implies that there exists a constant k such that $|\bar{S}| < 2^{(\log_2 n)^k} = n^{(\log_2 n)^{k-1}}$. Let $cx \geq c_0$ be a valid inequality for $\operatorname{conv}(S)$, where the c_js are relatively prime integers. We may assume that $c_j \geq 0$ by changing variable x_j into $1 - x_j$ if necessary. We may also assume $c_0 > 0$, otherwise $cx \geq c_0$ has Chvátal rank 0. Let $\bar{R} := \{x \in \{0,1\}^n : \ cx < c_0\}$. Note that $\bar{R} \subseteq \bar{S}$. This shows that $|\bar{R}| < 2^{(\log_2 n)^k}$. Note also that $0 \in \bar{R}$.

Each point in $\bar{R}$ has at most $(\log_2 n)^k$ nonzero coordinates. Indeed, if $\bar{R}$ contains a point $\bar{x}$ with more than $(\log_2 n)^k$ 1s, the face of $[0,1]^n$ with all the remaining coordinates set at 0 has at least $2^{1+(\log_2 n)^k}$ 0,1 vectors. Since 0 and $\bar{x}$ belong to this face and are both in $\bar{R}$, the center of the face also belongs to $\bar{R}$. But then, at least half the points of the face are in $\bar{R}$, contradicting $|\bar{R}| < 2^{(\log_2 n)^k}$.

Note that every 0,1 point in the hyperplane $cx = c_0$ is adjacent to at least one point in $\bar{R}$. Therefore these points have at most $1 + (\log_2 n)^k$ coordinates equal to 1. Consider n linearly independent points $x^1, \ldots, x^n$ that define the hyperplane $cx = c_0$. The coefficients of $cx = c_0$ can be obtained as follows. Normalize this equation to $\bar{c}x = 1$. By Cramer's rule, $\bar{c}_j$ can be obtained as the ratio of two determinants, the one in the numerator comprised of a vector of all 1s and $n - 1$ vectors among $x^1, \ldots, x^n$, and the one in the denominator comprised of all n vectors $x^1, \ldots, x^n$. We can set c_0 to be the determinant in the denominator, and c_j the one in the numerator. By Hadamard's inequality, $\det(v^1 \ldots v^n) \leq \|v^1\|_2 \cdots \|v^n\|_2$. Since here $n - 1$ of the vectors have at most $1 + (\log_2 n)^k$ 1s and the remaining one is a vector of all 1s, we get that

$$|c_j| \leq (1 + (\log_2 n)^k)^{\frac{n-1}{2}} n^{\frac{1}{2}}.$$

Therefore $\log_2 \|c\|_\infty = O(n \log \log n)$. The theorem now follows from the result of Eisenbrand and Schulz [11] stated above. $\square$

8 Optimization problem under small Chvátal rank

Let $P \subseteq [0,1]^n$ and $S = P \cap \{0,1\}^n$. Even when the Chvátal rank of P is just 1, it is still an open question whether optimizing a linear function over S is polynomially solvable or not [6, 8]. In this section, we prove a weaker result.

Theorem 36. *Let $P \subseteq [0,1]^n$ and $S = P \cap \{0,1\}^n$. If the Chvátal rank of Q_S is constant, then there is a polynomial algorithm to optimize a linear function over S.*

Proof. The optimization problem is of the form $\min\{cx : x \in S\}$ where $c \in \mathbb{R}^n$. By complementing variables, we may assume $c \geq 0$. By hypothesis, $\text{conv}(S) = Q_S^{(k)}$ for some constant k. We claim that an optimal solution can be found among the 0,1 vectors with at most $k+1$ nonzero components. This will prove the theorem since there are only polynomially many such vectors. Indeed, if an optimal solution $\bar{x}$ has more than $k+1$ nonzero components, any 0,1 vector $\bar{z}$ with $\text{supp}(\bar{z}) \subset \text{supp}(\bar{x})$ and $|\text{supp}(\bar{z})| = k+1$ satisfies $c\bar{z} \leq c\bar{x}$. Because $\text{conv}(S) = Q_S^{(k)}$, Lemma 4 implies that the face of H_n of dimension $k+1$ that contains 0 and $\bar{z}$ contains a feasible point $\bar{y} \in S$. Since $c\bar{y} \leq c\bar{z} \leq c\bar{x}$, the solution $\bar{y}$ is an optimal solution. □

For example, if $G(\bar{S})$ contains no 4-cycle, then the Chvátal rank of Q_S is at most 3 by Corollary 14 and therefore Theorem 36 implies that optimizing a linear function over S can be solved in polynomial time in this case.

Acknowledgments

This work was supported in part by NSF grant CMMI1560828 and ONR grant N00014-15-12082. We would like to thank the referees for carefully reading an earlier draft of the paper and providing valuable comments.

References

[1] A. Abdi, G. Cornuéjols, and K. Pashkovich, Ideal clutters that do not pack, to appear in *Mathematics of Operations Research*.

[2] G. Angulo, S. Ahmed, S.S. Dey, and V. Kaibel, Forbidden vertices, *Mathematics of Operations Research 40* (2015) 350-360.

[3] A. Bockmayr, F. Eisenbrand, M. Hartmann, and A.S. Schulz, On the Chvátal rank of polytopes in the 0/1 cube, *Discrete Applied Mathematics 98* (1999) 21-27.

[4] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, *Discrete Mathematics 4* (1973) 305-337.

[5] V. Chvátal, W. Cook, and M. Hartmann, On cutting-plane proofs in combinatorial optimization, *Linear Algebra and its Applications 114/115* (1989) 455-499.

[6] G. Cornuéjols, D. Lee, and Y. Li, On the rational polytopes with Chvátal rank 1, Submitted.

[7] G. Cornuéjols and Y. Li, On the rank of mixed 0,1 polyhedra, *Mathematical Programming 91* (2002) 391-397.

[8] G. Cornuéjols and Y. Li, Deciding emptiness of the Gomory-Chvátal closure is NP-complete, even for a rational polyhedron containing no integer point, IPCO, Q. Louveaux and M. Skutella (Eds.), *LNCS 9682* (2016) 387-397.

[9] D. Dadush, S.S. Dey, and J.P. Vielma, On the Chvátal-Gomory closure of a compact convex set, *Mathematical Programming Ser. A 145* (2014) 327-348.

[10] J. Dunkel and A.S. Schulz, The Gomory-Chvátal closure of a nonrational polytope is a rational polytope, *Mathematics of Operations Research 38* (2013) 63-91.

[11] F. Eisenbrand and A.S. Schulz, Bounds on the Chvátal rank of polytopes in the 0/1 cube, *Combinatorica 23* (2003) 245-261.

[12] A. Ghouila-Houri, Caractérisation des matrices totalement unimodulaires, *Comptes Rendus Hebdomadaires des Scéances de l'Académie des Sciences (Paris) 254* (1962) 1192-1194.

[13] R.E. Gomory, Outline of an algorithm for integer solutions to linear programs, *Bulletin of the American Mathematical Society 64* (1958) 275-278.

[14] M.E. Hartmann, M. Queyranne, and Y. Wang, On the Chvátal rank of certain inequalities, IPCO, G. Cornuéjols, R.E. Burkard, and G.J. Woeginger (Eds.), *LNCS 1610* (1999) 218-233.

[15] S. Pokutta and A.S. Schulz, Integer-empty polytopes in the 0/1-cube with maximal Gomory-Chvátal rank, *Operations Research Letters 39* (2011) 457-460.

[16] T. Rothvoss and L. Sanitá, 0/1 polytopes with quadratic Chvátal rank, IPCO, M. Goemans and J. Correa (Eds.), *LNCS 7801* (2013) 349-361.

[17] A. Schrijver, On cutting planes, *Annals of Discrete Mathematics 9* (1980) 291-296.