Projective geometries, simplices and clutters

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Abstract

A clutter is *clean* if it has no delta or the blocker of an extended odd hole minor. There are combinatorial and geometric classes of clean clutters, namely ideal clutters, clutters without an intersecting minor, and binary clutters. Tight connections between clean clutters and the first two classes have recently been established. In this paper, we establish a deep connection between clean clutters and binary clutters, the third class. Let us elaborate.

Let $\mathcal{C}$ be a clean clutter with covering number two where every element is in a minimum cover. It was recently proved that $\mathcal{C}$ has a fractional packing of value two. Collecting the supports of all such fractional packings, we obtain what is called the *core* of $\mathcal{C}$. We see that the core is a duplication of the cuboid of a set of $0-1$ points, called the *setcore* of $\mathcal{C}$. We see that the convex hull of the setcore is a full-dimensional polytope containing the center point of the hypercube in its interior.

We define two parameters on $\mathcal{C}$, one is the optimum of a combinatorial minimization problem called the *girth*, while the other is the optimum of a geometric maximization problem called the *depth*. We prove a duality theorem between the two parameters, exposing a fascinating interplay between the combinatorics and the geometry of such clutters. Further stressing the interplay, we see that the convex hull of the setcore is a simplex if and only if the setcore is the cocycle space of a projective geometry over the two-element field.

We see that if the convex hull of the setcore is a simplex of dimension more than 3, then $\mathcal{C}$ must have the Fano plane as a minor. We also see an infinite class of ideal minimally non-packing clutters with covering number two whose setcore corresponds to the vertices of a 3-dimensional simplex.

Our results lend weight to two unpublished conjectures of Seymour from 1975 about dyadic fractional packings in ideal clutters.

Keywords. Clutters, projective geometries over the two-element field, simplices, girth, ideal minimally non-packing clutters, the Q_6 property.

Contents

1 Introduction

Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V called *members*. $\mathcal{C}$ is a *clutter* over *ground set* V if no member contains another [11]. Two distinct elements u, v are *duplicates in* $\mathcal{C}$ if for each $C \in \mathcal{C}$, $u \in C$ if and only if $v \in C$. To *duplicate an element* u *of* $\mathcal{C}$ is to introduce a new element v and replace $\mathcal{C}$ by the clutter over ground set $V \cup \{v\}$ whose members are $\{C : u \notin C \in \mathcal{C}\} \cup \{C \cup \{v\} : u \in C \in \mathcal{C}\}$. A *duplication of* $\mathcal{C}$ is any clutter obtained from $\mathcal{C}$ after duplicating some elements.

A *transversal* is a subset $B \subseteq V$ such that $|B \cap C| = 1$ for all $C \in \mathcal{C}$. A *cover* is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. A cover is *minimal* if it does not contain another cover. The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set V whose members are the minimal covers of $\mathcal{C}$ [11]. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [16, 11]. Take disjoint $I, J \subseteq V$. The *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J, denoted $\mathcal{C} \setminus I/J$, is the clutter over ground set $V - (I \cup J)$ whose members consist of the inclusion-wise minimal sets of $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. The minor is *proper* if $I \cup J \neq \emptyset$. It is well-known that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [25].

Two clutters $\mathcal{C}_1, \mathcal{C}_2$ are *isomorphic*, written as $\mathcal{C}_1 \cong \mathcal{C}_2$, if one is obtained from the other by relabeling the ground set. Take an integer $n \geq 3$. Denote by Δ_n the clutter over ground set $[n] := \{1, \ldots, n\}$ whose members are $\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}$. Observe that $b(\Delta_n) = \Delta_n$. Any clutter isomorphic to Δ_n is called a *delta of dimension* n. Given that $n \geq 5$ and is odd, an *extended odd hole of dimension* n is any clutter whose ground set can be relabeled as $[n]$ so that its minimum cardinality members are precisely $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}$. Observe that for an extended odd hole of dimension n, every cover has cardinality at least $\frac{n+1}{2}$.

We say that a clutter is *clean* if it has no delta or the blocker of an extended odd hole minor. Notice that if a clutter is clean, then so is every minor of it. Being clean is a tractable property:

Theorem 1.1 ([4], Theorem 1.11)**.** *There is an algorithm that given a clutter over* n *elements and* m *members runs in time polynomial in* n, m *and finds a delta or the blocker of an extended odd hole minor, or certifies that none exists.*

It is worth pointing out that, in stark contrast with Theorem 1.1, testing whether a clutter has a delta or an extended odd hole (rather than the blocker of an extended odd hole) as a minor is NP-complete [10, 4]. There are three important classes of clean clutters, namely *ideal clutters*, *clutters without an intersecting minor*, and *binary clutters*, as defined below.

$\mathcal{C}$ is an *ideal* clutter if *the associated set covering polyhedron*

$$\left\{x \in \mathbb{R}^V_+ : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C}\right\}$$

is integral [9] (see also [1]). Observe that $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$ and $\frac{1}{2} \cdot \mathbf{1}$ are fractional vertices of the set covering polyhedron associated with Δ_n and an extended odd hole, respectively. (Throughout the paper, $\mathbf{0}, \mathbf{1}$ refer to the all-zeros and all-ones column vector of appropriate dimension, respectively.) Thus the deltas and extended odd holes are non-ideal clutters. It is well-known that a clutter is ideal if and only if its blocker is ideal [14, 18]. In particular, the blocker of an extended odd hole is also non-ideal. Moreover, if a clutter is ideal, so is every minor of it [26]. Thus ideal clutters are clean. A natural connection between clean clutters and ideal clutters was established in [5].

A clutter is *intersecting* if every two members intersect yet no element belongs to all members [4]. Notice that every delta and the blocker of every extended odd hole is an intersecting clutter. Thus clutters without an intersecting minor are also clean. A natural connection between clean clutters and this class was recently established in [4]. This class is important because it has been conjectured that idealness of such clutters is equivalent to the total dual integrality of the associated set covering linear system [4]. This is in fact equivalent to the $\tau = 2$ *Conjecture* by Cornuéjols, Guenin and Margot [8] – we talk about this conjecture in §1.6.

$\mathcal{C}$ is a *binary* clutter if the symmetric difference of any odd number of members contains a member; equivalently, $\mathcal{C}$ is binary if $|C \cap B| \equiv 1 \pmod 2$ for all $C \in \mathcal{C}, B \in b(\mathcal{C})$ [17]. In particular, a clutter is binary if and only if its blocker is binary. Observe that the deltas, extended odd holes and their blockers are not binary. If a clutter is binary, so is every minor of it [25]. Thus binary clutters are clean. In this paper, we see a deep connection between binary clutters and clean clutters.

The *covering number* of $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. Consider the following hypothesis on a clutter:

(2CovH) the covering number is two, and every element is in a minimum cover.

Observe that every clutter with covering number at least two has a deletion minor satisfying (2CovH). An important class of clutters satisfying (2CovH) consists of cuboids. Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. We refer to the points in S as *feasible* and the points in $\{0,1\}^r - S$ as *infeasible*. The *cuboid of* S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2r]$ whose members have incidence vectors $\{(p_1, 1-p_1, \ldots, p_r, 1-p_r) : p \in S\}$ [2, 3]. Observe that for each $i \in [r]$, $|C \cap \{2i-1, 2i\}| = 1$ for all $C \in \mathrm{cuboid}(S)$, so $\{2i-1, 2i\}$ is a cover of the cuboid. Thus if the points in S do not agree on a coordinate, $\mathrm{cuboid}(S)$ satisfies (2CovH). That is, a cuboid without a cover of size one satisfies (2CovH).

Take a point $q \in \{0,1\}^r$. To *twist S by q* is to replace S by $S \triangle q := \{p \triangle q : p \in S\}$, where the second $\triangle$ denotes coordinate-wise addition modulo 2. Take a coordinate $i \in [r]$. Denote by e_i the i^{th} unit vector of appropriate dimension. To *twist coordinate i of S* is to replace S by $S \triangle e_i$. Two sets S_1, S_2 are *isomorphic*, written as $S_1 \cong S_2$, if one is obtained from the other after relabeling and twisting some coordinates. Two distinct coordinates $i, j \in [r]$ are *duplicates in S* if $S \subseteq \{x : x_i = x_j\}$ or $S \subseteq \{x : x_i + x_j = 1\}$. Observe that if two coordinates are duplicates in a set, then they are duplicates in any isomorphic set. Observe further that S has duplicated coordinates if, and only if, $\mathrm{cuboid}(S)$ has duplicated elements.

Back to general clutters. Denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimal covers of $\mathcal{C}$ of cardinality two. (This graph may have no edges.) In this paper we initiate the study of clean clutters satisfying (2CovH). The following result, proved in §2, is our starting point:

Theorem 1.2. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then $G(\mathcal{C})$ is a bipartite graph where every vertex is incident with an edge. Moreover, if $G(\mathcal{C})$ is not connected and $\{U, U'\}$ is the bipartition of a connected component, then $\mathcal{C} \setminus U/U'$ is a clean clutter satisfying (2CovH).*

If $\mathcal{C}$ is a clean clutter satisfying (2CovH), then the *rank of $\mathcal{C}$*, denoted $\mathrm{rank}(\mathcal{C})$, is the number of connected components of the bipartite graph $G(\mathcal{C})$. Let us justify our use of the term "rank".

1.1 The rank, the core and the setcore

Let $\mathcal{C}$ be a clutter over ground set V. Consider the primal-dual pair of linear programs

$$(P)\quad \begin{array}{ll} \min & \mathbf{1}^\top x \\ \text{s.t.} & \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (y_C : v \in C \in \mathcal{C}) \leq 1 \quad v \in V \\ & y \geq \mathbf{0}. \end{array}$$

The incidence vector of any cover of $\mathcal{C}$ gives a feasible solution for (P). Thus $\tau(\mathcal{C})$ is an upper bound on the optimal value of (P). A *fractional packing of $\mathcal{C}$* is any feasible solution y for (D), and its *value* is $\mathbf{1}^\top y$. Its *support*, denoted $\mathrm{support}(y)$, is the clutter over ground set V whose members are $\{C \in \mathcal{C} : y_C > 0\}$. It follows from Weak LP Duality that every fractional packing has value at most $\tau(\mathcal{C})$. The following remark is an immediate consequence of the Complementary Slackness conditions:

Remark 1.3. *Let $\mathcal{C}$ be a clutter, B a minimum cover, and y a fractional packing of value $\tau(\mathcal{C})$, if there is any. Then $|C \cap B| = 1$ for every $C \in \mathcal{C}$ such that $y_C > 0$, and $\sum (y_C : v \in C \in \mathcal{C}) = 1$ for every element $v \in B$.*

The following result was proved recently by two of us:

Theorem 1.4 ([5], Theorem 3 and [4], Lemma 1.6)**.** *Every clean clutter satisfying (2CovH) has a fractional packing of value two.*

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). The *core of $\mathcal{C}$* is defined as

$$\mathrm{core}(\mathcal{C}) := \{C \in \mathcal{C} : y_C > 0 \text{ for some fractional packing } y \text{ of value two}\};$$

the core is nonempty by Theorem 1.4. The following is an immediate consequence of Remark 1.3:

Remark 1.5. *Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH). Then every member of $\mathrm{core}(\mathcal{C})$ is a transversal of the minimum covers of $\mathcal{C}$. Moreover, for every fractional packing y of value two and for every element $v \in V$, $\sum (y_C : v \in C \in \mathcal{C}) = 1$.*

Let $G := G(\mathcal{C})$ and $r := \mathrm{rank}(\mathcal{C})$. By Theorem 1.2, G is a bipartite graph where every vertex is incident with an edge, and it has r connected components by definition. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G. As an immediate consequence of Remark 1.5,

Remark 1.6. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), let $r := \mathrm{rank}(\mathcal{C})$, and let $G := G(\mathcal{C})$. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G. Let C be a member of $\mathcal{C}$. If $C \in \mathrm{core}(\mathcal{C})$, then $C \cap (U_i \cup V_i) \in \{U_i, V_i\}$ for each $i \in [r]$.* [1]

In particular, $\mathrm{core}(\mathcal{C})$ is a duplication of a cuboid – let us elaborate. The *setcore of $\mathcal{C}$ with respect to* $(U_1, V_1; U_2, V_2; \ldots ; U_r, V_r)$ is the set $S \subseteq \{0,1\}^r$ defined as follows: start with $S = \emptyset$, and for each $C \in \mathrm{core}(\mathcal{C})$, add a point p to S such that

$$p_i = 0 \quad \Leftrightarrow \quad C \cap (U_i \cup V_i) = U_i \quad \forall i \in [r].$$

By Remark 1.6, the set S is well-defined and $\mathrm{core}(\mathcal{C})$ is a duplication of $\mathrm{cuboid}(S)$. We denote S by the notation $\mathrm{setcore}(\mathcal{C} : U_1, V_1; U_2, V_2; \ldots ; U_r, V_r)$. As the reader can imagine, we will not use this notation often, and use $\mathrm{setcore}(\mathcal{C})$ as short-hand notation. Note however that $\mathrm{setcore}(\mathcal{C})$ is defined only up to isomorphism. The following theorem is proved in §2 and hopefully justifies our choice of words for the term rank:

Theorem 1.7. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), and let $r := \mathrm{rank}(\mathcal{C})$. Then the convex hull of $\mathrm{setcore}(\mathcal{C})$ is a full-dimensional polytope contained in $[0,1]^r$ and containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior. In particular, $\mathrm{setcore}(\mathcal{C})$ does not have duplicated coordinates.*

The following remark, which is an immediate consequence of Remark 1.3, sheds light on how the hypercube center point $\frac{1}{2} \cdot \mathbf{1}$ comes into play in Theorem 1.7:

Remark 1.8. *Take an integer $r \geq 1$, a set $S \subseteq \{0,1\}^r$ and let $\mathcal{C} := \mathrm{cuboid}(S)$. Let $y \in \mathbb{R}^{\mathcal{C}}_+$ and define $\alpha \in \mathbb{R}^S_+$ as follows: for every point $p \in S$ and corresponding member $C \in \mathcal{C}$, let $\alpha_p := \frac{1}{2} \cdot y_C$. Then y is a fractional packing of $\mathcal{C}$ of value two if, and only if, $\mathbf{1}^\top \alpha = 1$ and $\sum_{p \in S} \alpha_p \cdot p = \frac{1}{2} \cdot \mathbf{1}$. In particular, $\mathcal{C}$ has a fractional packing of value two if, and only if, $\frac{1}{2} \cdot \mathbf{1} \in \mathrm{conv}(S)$.*

As an immediate but important consequence,

Remark 1.9. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). If $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ is a simplex, then $\mathcal{C}$ has a unique fractional packing of value two.*

Moving forward we need the following proposition:

[1] In §2.4 we see that the converse of this remark holds also.

Proposition 1.10. *Take an integer $r \geq 1$, a set $S \subseteq \{0,1\}^r$ without an infeasible hypercube of dimension at least $r-2$, and let $\mathcal{C} := \mathrm{cuboid}(S)$. If $\mathcal{C}$ is clean and satisfies (2CovH), then $\mathrm{rank}(\mathcal{C}) = r$ and $\mathrm{setcore}(\mathcal{C}) \cong S$.*

Proof. As S does not have an infeasible hypercube of dimension at least $r-2$, it follows that $\mathcal{C}$ has exactly r minimum covers, namely $\{2i-1, 2i\}, i \in [r]$. In particular, $G(\mathcal{C})$ is a perfect matching of cardinality r. This implies that $\mathrm{rank}(\mathcal{C}) = r$. We may therefore assume after a possible twisting that $\mathrm{setcore}(\mathcal{C}) \subseteq S \subseteq \{0,1\}^r$. By Theorem 1.7, $\frac{1}{2} \cdot \mathbf{1}$ is in the interior of $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$, so $\frac{1}{2} \cdot \mathbf{1}$ is in the interior of $\mathrm{conv}(S)$. This implies that for every point $p \in S$, $\frac{1}{2} \cdot \mathbf{1}$ can be written as a convex combination of the points in S where the coefficient of p is nonzero. It then follows from Remark 1.8 that for every $C \in \mathrm{cuboid}(S)$, there is a fractional packing of value two whose support has C as a member. That is, $\mathrm{core}(\mathcal{C}) = \mathcal{C}$ and $\mathrm{setcore}(\mathcal{C}) = S$, as required. □

1.2 The girth, the depth and duality

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). A cover of $\mathcal{C}$ is *monochromatic* if it is monochromatic in some proper 2-vertex-coloring of the bipartite graph $G(\mathcal{C})$. The *girth of* $\mathcal{C}$ is defined as

$$\mathrm{girth}(\mathcal{C}) := \min\{|B| : B \text{ is a monochromatic cover}\};$$

if there is no monochromatic cover, then $\mathrm{girth}(\mathcal{C}) := \infty$. Observe that $\mathrm{girth}(\mathcal{C}) \geq 3$. The *depth of* $\mathcal{C}$ is defined as

$$\mathrm{depth}(\mathcal{C}) := \max\{d : \mathrm{setcore}(\mathcal{C}) \text{ has a } d\text{-dimensional infeasible hypercube}\};$$

if $\mathrm{setcore}(\mathcal{C})$ has no infeasible hypercube, then $\mathrm{depth}(\mathcal{C}) := -\infty$. Observe that $\mathrm{girth}(\mathcal{C})$ is a parameter defined on all of $\mathcal{C}$ while $\mathrm{depth}(\mathcal{C})$ is a parameter defined only on the core and the setcore of $\mathcal{C}$. In §3 we prove the following two theorems:

Theorem 1.11 (Weak Duality)**.** *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then*

$$\mathrm{rank}(\mathcal{C}) - \mathrm{girth}(\mathcal{C}) \leq \mathrm{depth}(\mathcal{C}).$$

Theorem 1.12 (Strong Duality)**.** *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then*

$$\mathrm{rank}(\mathcal{C}) - \mathrm{girth}(\mathcal{C}) = \mathrm{depth}(\mathcal{C}).$$

Observe that the girth is the optimum of a *combinatorial* minimization problem, while the depth is the optimum of a *geometric* maximization problem. Theorems 1.11 and 1.12 manifest a striking interplay between the combinatorics and the geometry of clean clutters satisfying (2CovH). On the combinatorial side, we see in §4 that our notion of girth extends the conventional notion of girth for simple graphs and more generally simple binary matroids, as well as the notion of covering number for clean clutters with covering number at least three. On the geometric side, our notion of depth is intimately linked to the notion of *notch* defined recently by Benchetrit et al. [6]. More precisely, when the depth of $\mathcal{C}$ is finite, then it is equal to the notch of $\mathrm{setcore}(\mathcal{C})$ minus 1.

1.3 Dyadic fractional packings and projective geometries

For some integer $k \geq 1$, a vector y is $\frac{1}{k}$-*integral* if the vector $k \cdot y$ is integral. The vector y is *dyadic* if it is $\frac{1}{2^{k-1}}$-integral for some integer $k \geq 1$. The following result, proved in §6, strengthens Theorem 1.4 by showing that every clean clutter satisfying (2CovH) has a dyadic fractional packing of value two:

Theorem 1.13. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then for some integer k such that $1 \leq k \leq \mathrm{rank}(\mathcal{C})$, there exists a $\frac{1}{2^{k-1}}$-integral packing of value two. Moreover, if the clutter is inputted via a filter oracle and has ground set V, then the fractional packing can be found in time at most $(2^k - 1) \cdot |V|^2$.*

We conjecture that the upper bound on k can be improved by a logarithmic factor:

Conjecture 1.14. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then for some integer $k \geq 1$ such that $2^k - 1 \leq \mathrm{rank}(\mathcal{C})$, there exists a $\frac{1}{2^{k-1}}$-integral packing of value two.*

Let us illustrate where the upper bound on k comes from. Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. We say that S is a *vector space over $GF(2)$*, or simply a *binary space*, if $a \triangle b \in S$ for all points $a, b \in S$. In particular, a nonempty binary space always contains the point $\mathbf{0}$.

Remark 1.15 ([3], Remark 3.11)**.** *Let S be a binary space. Then $\mathrm{cuboid}(S)$ is a binary clutter. In particular, if the points in S do not agree on a coordinate, then $\mathrm{cuboid}(S)$ is a clean clutter satisfying (2CovH).*

Take an integer $k \geq 1$ and let A be the $k \times (2^k - 1)$ matrix whose columns are all the $0-1$ vectors of dimension k that are nonzero. Let $r := 2^k - 1$. Let $\mathrm{cocycle}(PG(k-1,2)) \subseteq \{0,1\}^r$ be the row space of A over $GF(2)$. As this set is a binary space whose points do not agree on a coordinate, it follows from Remark 1.15 that $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$ is a clean clutter satisfying (2CovH). It can be readily checked that the k rows of A are linearly independent over $GF(2)$. As a consequence, $|\mathrm{cocycle}(PG(k-1,2))| = 2^k = r+1$. In fact, as we will see in §7, $\mathrm{cocycle}(PG(k-1,2))$ forms the vertices of an r-dimensional simplex:

Theorem 1.16. *Take an integer $k \geq 1$, let $r := 2^k - 1$ and let $S := \mathrm{cocycle}(PG(k-1,2)) \subseteq \{0,1\}^r$. Then $\mathrm{conv}(S)$ is a full-dimensional simplex containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior. In particular, $\frac{1}{2^{k-1}} \cdot \mathbf{1}$ is the unique fractional packing of $\mathrm{cuboid}(S)$ of value two.*

It can be readily checked that $\mathrm{cocycle}(PG(k-1,2))$ has no infeasible hypercube of dimension at least $r-2$, so the cuboid of $\mathrm{cocycle}(PG(k-1,2))$ has rank r and setcore $\mathrm{cocycle}(PG(k-1,2))$ by Proposition 1.10. This fact, combined with Theorem 1.16, tells us that the upper bound on k in Conjecture 1.14 is tight for the cuboid of $\mathrm{cocycle}(PG(k-1,2))$. In fact, this example is the basis for our conjectured upper bound. We believe Conjecture 1.14 is a consequence of a more general phenomenon – let us elaborate.

As suggested by our notation, $\mathrm{cocycle}(PG(k-1,2))$ is the cocycle space of some binary matroid. (A primer on binary matroids is given in §4.2.) This binary matroid is called the *rank-k projective geometry over $GF(2)$* and denoted $PG(k-1,2)$.[2] Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). We say that $\mathcal{C}$ *embeds the projective geometry* $PG(k-1,2)$ if a subset of $\mathcal{C}$ is a duplication of the cuboid of $\mathrm{cocycle}(PG(k-1,2))$.

[2] In the context of binary matroids, "rank" should be read as "$GF(2)$-rank", while in the context of clutters, "rank" should be read as "$\mathbb{R}$-rank".

Conjecture 1.17. *Every clean clutter satisfying (2CovH) embeds a projective geometry.*

We see in §7 that every binary clutter satisfying (2CovH) embeds a projective geometry. Cornuéjols et al. [8] proved that ideal *minimally non-packing* clutters satisfying (2CovH) embed the projective geometry $PG(1,2)$ – we elaborate in §1.6.

Take an integer $r \geq 1$, a set $S \subseteq \{0,1\}^r$, and a coordinate $i \in [r]$. To *duplicate coordinate i of S* is to replace S by $S' := \{(x,0) : x \in S, x_i = 0\} \cup \{(x,1) : x \in S, x_i = 1\} \subseteq \{0,1\}^{r+1}$. Notice that coordinates $i, r+1$ are duplicates in S'. A *duplication of S* is any set isomorphic to a set obtained from S after duplicating some coordinates.

Remark 1.18. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) that embeds $PG(k-1,2)$ for some integer $k \geq 1$. Then $\mathcal{C}$ has a $\frac{1}{2^{k-1}}$-integral packing of value two, and $2^k - 1 \leq \mathrm{rank}(\mathcal{C})$. In particular, if Conjecture 1.17 is true, then so is Conjecture 1.14.*

Proof. Let $\mathcal{C}'$ be a subset of $\mathcal{C}$ that is equal to some duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$. It then follows from Theorem 1.16 that $\mathcal{C}'$ has a $\frac{1}{2^{k-1}}$-integral packing y of value two, where $\mathrm{support}(y) = \mathcal{C}'$. In particular, $\mathrm{core}(\mathcal{C}) \supseteq \mathcal{C}'$, implying in turn that $\mathrm{setcore}(\mathcal{C})$ contains a duplication of $\mathrm{cocycle}(PG(k-1,2))$, so $\mathrm{rank}(\mathcal{C}) \geq 2^k - 1$. □

1.4 Unique dyadic fractional packings

Consider the following hypothesis on a clutter:

(UniqH) there is a unique dyadic fractional packing of value two.

We see in §7 that every clean clutter satisfying (2CovH) and (UniqH) embeds a projective geometry:

Theorem 1.19. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) and (UniqH), and let y be the dyadic fractional packing of $\mathcal{C}$ of value two. Then there is an integer $k \geq 1$ such that y is $\frac{1}{2^{k-1}}$-integral, $\mathcal{C}$ has rank $2^k - 1$, and $\mathrm{support}(y)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$. In particular, $\mathcal{C}$ embeds a projective geometry.*

Consider the clutter over ground set $\{1,\ldots,7\}$ whose members are

$$\mathbb{L}_7 := \big\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,5,6\},\{2,4,7\},\{3,4,6\},\{3,5,7\}\big\}.$$

Notice that the elements and the members of $\mathbb{L}_7$ are the points and the lines of the Fano plane, respectively. Observe that $\mathbb{L}_7$ is equal to its blocker, is a binary clutter, and is non-ideal as $\left(\frac{1}{3},\ldots,\frac{1}{3}\right)$ is a fractional vertex of the corresponding set covering polyhedron. We prove the following in §8:

Theorem 1.20. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) and (UniqH). If $\mathcal{C}$ has rank more than 3, then it has an $\mathbb{L}_7$ minor.*

1.5 Simplices

Let us prove three applications of Theorems 1.19 and 1.20.

Theorem 1.21. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then the following statements are equivalent:*

- $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ *is a simplex,*
- $\mathrm{setcore}(\mathcal{C})$ *is isomorphic to the cocycle space of a projective geometry.*

Proof. ($\Leftarrow$) follows from Theorem 1.16. ($\Rightarrow$) By Remark 1.9, $\mathcal{C}$ has a unique fractional packing y of value two and therefore satisfies (UniqH). Thus, by Theorem 1.19, there exists an integer $k \geq 1$ such that $\mathrm{support}(y)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$. Notice however that $\mathrm{support}(y) = \mathrm{core}(\mathcal{C})$. In particular, $\mathrm{setcore}(\mathcal{C})$ is a duplication of $\mathrm{cocycle}(PG(k-1,2))$. However, as the setcore does not have duplicated coordinates by Theorem 1.7, $\mathrm{setcore}(\mathcal{C})$ must be isomorphic to $\mathrm{cocycle}(PG(k-1,2))$, as required. □

For the following theorem, notice that duplicating a coordinate of a binary space keeps the set a binary space.

Theorem 1.22. *Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$ whose convex hull is a simplex containing $\frac{1}{2} \cdot \mathbf{1}$ in its relative interior. Then exactly one of the following statements holds:*

- $\mathrm{cuboid}(S)$ *has a delta or the blocker of an extended odd hole minor, or*
- *S is a duplication of the cocycle space of a projective geometry.*

Proof. Let $\mathcal{C} := \mathrm{cuboid}(S)$. If S is a duplication of the cocycle space of a projective geometry, then up to twisting, S is a binary space, so $\mathcal{C}$ is clean by Remark 1.15. Conversely, assume that $\mathcal{C}$ is clean. As $\mathrm{conv}(S)$ is a simplex containing $\frac{1}{2} \cdot \mathbf{1}$ in its relative interior,

- the points in S do not all agree on a coordinate, so $\mathcal{C}$ satisfies (2CovH), and
- by Remark 1.8 on the connection between $\mathrm{conv}(S)$ and fractional packings of $\mathcal{C}$, $\mathcal{C}$ must have a unique fractional packing of value two, one whose support is $\mathcal{C}$.

In particular, $\mathcal{C} = \mathrm{core}(\mathcal{C})$, so S is a duplication of $\mathrm{setcore}(\mathcal{C})$. As $\mathrm{conv}(S)$ is a simplex, so is $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$, so by Theorem 1.21, $\mathrm{setcore}(\mathcal{C})$ is isomorphic to the cocycle space of a projective geometry, implying in turn that S is a duplication of the cocycle space of a projective geometry, as required. □

Consider the clutter over ground set $\{1,\ldots,6\}$ whose members are

$$Q_6 := \{\{2,4,6\},\{1,3,6\},\{1,4,5\},\{2,3,5\}\}\,.$$

Observe that the elements and the members of Q_6 correspond to the edges and the triangles of the complete graph on four vertices. Q_6 is an ideal clutter [26] and is the cuboid of $\{000,110,101,011\} = \mathrm{cocycle}(PG(1,2))$.

Theorem 1.23. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). If* $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ *is a simplex, then one of the following statements holds:*

(i) $\text{setcore}(\mathcal{C}) = \{0, 1\}$*, i.e.* $\text{core}(\mathcal{C})$ *consists of two members that partition the ground set,*

(ii) $\text{setcore}(\mathcal{C}) \cong \{000, 110, 101, 011\}$*, i.e.* $\text{core}(\mathcal{C})$ *is a duplication of* Q_6*, or*

(iii) $\mathcal{C}$ *is non-ideal.*

Proof. By Remark 1.9, $\mathcal{C}$ has a unique fractional packing of value two and therefore satisfies (UniqH). Let $r := \text{rank}(\mathcal{C})$. If $r > 3$, then $\mathcal{C}$ has an $\mathbb{L}_7$ minor by Theorem 1.20, so (iii) holds in particular. Otherwise, $1 \leq r \leq 3$. It follows from Theorem 1.21 that $\text{setcore}(\mathcal{C})$ is isomorphic to either $\text{cocycle}(PG(0, 2)) = \{0, 1\}$ or $\text{cocycle}(PG(1, 2)) = \{000, 110, 101, 011\}$, so either (i) or (ii) holds, as required. □

Though the statement of Theorem 1.23 is geometric, our current proof is purely combinatorial. We remedy this by providing a geometric proof in §5.

1.6 Ideal minimally non-packing clutters with covering number two

Let $\mathcal{C}$ be a clutter. The *packing number of* $\mathcal{C}$, denoted $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. Observe that $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$. If equality holds here, then $\mathcal{C}$ *packs*. A clutter has the *packing property* if every minor of it, including the clutter itself, packs [8]. A clutter is *minimally non-packing* if it does not pack but every proper minor does. Observe that a clutter has the packing property if and only if it has no minimally non-packing minor. A clutter is *minimally non-ideal* if it is non-ideal but every proper minor is ideal. Similarly, a clutter is ideal if and only if it has no minimally non-ideal minor. The following two results are consequences of Lehman's powerful characterization of minimally non-ideal clutters [19]:

Theorem 1.24 ([8])**.** *Every minimally non-packing clutter is either ideal or minimally non-ideal.*

Theorem 1.25 ([19], see also [18, 23])**.** *Every minimally non-ideal clutter with covering number two is either a delta or the blocker of an extended odd hole.*

As a consequence,

Corollary 1.26. *Every minimally non-packing clutter with covering number two is either ideal, a delta, or the blocker of an extended odd hole.*

Observe that,

Remark 1.27. *Every ideal minimally non-packing clutter with covering number two is clean and satisfies (2CovH).*

The so-called $\tau = 2$ *Conjecture* predicts that every ideal minimally non-packing clutter necessarily has covering number two. This conjecture was made by Cornuéjols, Guenin and Margot [8] wherein the following result was proved:

Theorem 1.28 ([8])**.** *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two. Then a subset of $\mathcal{C}$ is a duplication of Q_6, that is, $\mathcal{C}$ embeds $PG(1,2)$. In particular, Conjecture 1.17 holds for ideal minimally non-packing clutters.*

In [8] the authors provided many examples of ideal minimally non-packing clutters with covering number two. In §9 we study these examples in detail, which in turn leads us to a new infinite class of ideal minimally non-packing clutters with covering number two:

Theorem 1.29. *There is an infinite class of ideal minimally non-packing clutters with covering number two, rank three, and setcore $\{000, 110, 101, 011\}$. Moreover, every example in this class is not a cuboid.*

Observe that this infinite class necessarily belongs to category (ii) of Theorem 1.23. It is worth pointing out that prior to this work, only finitely many instances of non-cuboid ideal minimally non-packing clutters were known.

1.7 Two conjectures of Seymour

Our work is deeply related to a work by Seymour [24], which in turn was motivated by conjectures of Fulkerson [15] and Tutte [27]. Let $\mathcal{C}$ be an ideal clutter. Consider the primal-dual pair of linear programs

$$(P)\quad \begin{array}{lll} \min & \mathbf{1}^\top x & \\ \text{s.t.} & \sum_{v\in C} x_v \geq 1 & C\in\mathcal{C} \\ & x\geq \mathbf{0} & \end{array} \qquad\qquad (D)\quad \begin{array}{lll} \max & \mathbf{1}^\top y & \\ \text{s.t.} & \sum\left(y_C : v\in C\in\mathcal{C}\right)\leq 1 & v\in V \\ & y\geq \mathbf{0}. & \end{array}$$

As $\mathcal{C}$ is ideal, (P) has an integral optimal solution ([7], Theorem 4.1). As a result, by Strong LP Duality, $\mathcal{C}$ has a fractional packing of value $\tau(\mathcal{C})$. Let $\mathscr{P}(\mathcal{C})$ be the set of all integers $k\geq 1$ for which there is a $\frac{1}{k}$-integral packing of value $\tau(\mathcal{C})$ [24]. Notice that $\mathscr{P}(\mathcal{C})\neq\emptyset$, and that if an integer belongs to $\mathscr{P}(\mathcal{C})$, then so do all positive integer multiples of that integer.

Conjecture 1.30 (Seymour 1975, see [22], §79.3e)**.** *Let $\mathcal{C}$ be an ideal clutter. Then $\mathscr{P}(\mathcal{C})$ contains a power of 2, that is, $\mathcal{C}$ has a dyadic fractional packing of value $\tau(\mathcal{C})$.*

The following corollary of Theorem 1.13 lends weight to this conjecture:

Corollary 1.31. *Every ideal clutter with covering number at least two has a dyadic fractional packing of value two.*

Proof. Let $\mathcal{C}$ be an ideal clutter such that $\tau(\mathcal{C})\geq 2$. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ minimal subject to $\tau(\mathcal{C}')\geq 2$. Then $\mathcal{C}'$ is clean and necessarily satisfies (2CovH), so by Theorem 1.13, $\mathcal{C}'$ and therefore $\mathcal{C}$ has a dyadic fractional packing of value two, as required. □

Seymour in fact makes the following stronger conjecture:

Conjecture 1.32 (Seymour 1975, see [22], §79.3e)**.** *Let $\mathcal{C}$ be an ideal clutter. Then $4\in\mathscr{P}(\mathcal{C})$, that is, $\mathcal{C}$ has a $\frac{1}{4}$-integral packing of value $\tau(\mathcal{C})$.*

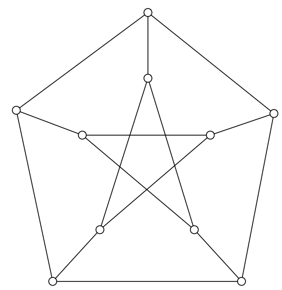

Figure 1: The Petersen graph.

The following corollary of Theorem 1.20 gives some evidence for this conjecture:

Corollary 1.33. *If an ideal clutter has a unique dyadic fractional packing of value two, then the fractional packing of value two is* $\frac{1}{2}$*-integral.*

Proof. Let $\mathcal{C}$ be an ideal clutter with a unique dyadic fractional packing of value two. In particular, $\tau(\mathcal{C}) \geq 2$. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ minimal subject to $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and satisfies (2CovH), so $\mathcal{C}'$ has a dyadic fractional packing y of value two by Theorem 1.13. Our hypothesis implies that y is the unique dyadic fractional packing of $\mathcal{C}'$ of value two, so $\mathcal{C}'$ satisfies (UniqH). As $\mathcal{C}'$ is ideal, it has no $\mathbb{L}_7$ minor, so $1 \leq \mathrm{rank}(\mathcal{C}') \leq 3$ by Theorem 1.20. It therefore follows from Theorem 1.19 that y is $\frac{1}{2}$-integral. As a result, the dyadic fractional packing of $\mathcal{C}$ of value two is $\frac{1}{2}$-integral, as required. $\square$

What happens when the uniqueness condition (UniqH) of Corollary 1.33 is dropped? It turns out that we can no longer guarantee the existence of a $\frac{1}{2}$-integral packing of value two – let us elaborate. Let $G = (V, E)$ be a graph. A *cycle* is a subset $C \subseteq E$ such that every vertex of G is incident with an even number of edges in C. The *cycle space of G* is

$$\mathrm{cycle}(G) := \{\chi_C : C \subseteq E \text{ is a cycle of } G\} \subseteq \{0,1\}^E.$$

Notice that $\mathrm{cycle}(G)$ is a binary space, so $\mathrm{cuboid}(\mathrm{cycle}(G))$ is a binary clutter by Remark 1.15. Moreover,

Theorem 1.34 ([3], Theorem 1.6, Corollary 2.6 and Theorem 2.8)**.** *The cuboid of the cycle space of a graph is an ideal clutter.*

Consider the cuboid of the cycle space of the Petersen graph Pete, as displayed in Figure 1. This clutter is denoted T_{30} by Schrijver [22], §79.3e (and is the clutter of T-joins of a graft defined by Seymour in [24] on page 440). Our discussion above implies that T_{30} is an ideal binary clutter satisfying (2CovH). This clutter has no $\frac{1}{2}$-integral packing of value two but has one that is $\frac{1}{4}$-integral [24] – let us elaborate. As Pete has no 3-edge-coloring, there does not exist 3 cycles covering each edge exactly twice, that is, T_{30} does not embed $PG(1,2)$. Thus the point $\frac{1}{2} \cdot \mathbf{1} \subseteq \{0,1\}^{E(\mathrm{Pete})}$ cannot be written as a $\frac{1}{4}$-integral convex combination of the points in $\mathrm{cycle}(\mathrm{Pete})$, in turn implying that T_{30} has no $\frac{1}{2}$-integral packing of value two by Remark 1.8. However, Pete has 7 cycles covering each edge exactly 4 times, that is, T_{30} embeds $PG(2,2)$. Thus T_{30} has a $\frac{1}{4}$-integral packing of value two.

2 The rank, the core and the setcore

In this section, we prove Theorem 1.2 in §2.1, then prove some lemmas in §2.2 about the core that is of use throughout the paper, and then prove Theorem 1.7 in §2.3. We need the following ingredient:

Lemma 2.1. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), and let $G := G(\mathcal{C})$. Then*

(i) G is bipartite. ([5], Remark 6)

Let $\{U, U'\}$ be the bipartition of a connected component of G. Then

(ii) every member of $\mathcal{C}$ disjoint from U contains U'.

Moreover, if G is connected, then the following statements hold:

(iii) Neither U nor U' is a cover. ([5], Theorem 7)

(iv) $U, U' \in \mathcal{C}$.

Proof. **(ii)** is immediate. **(iv)** follows immediately from (ii) and (iii). □

2.1 The proof of Theorem 1.2

Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH), and let $G := G(\mathcal{C})$. Then by Lemma 2.1 (i) and (2CovH), G is a bipartite graph where every vertex is incident with an edge. Let $\{U, U'\}$ be the bipartition of a connected component of G. Clearly $\mathcal{C} \setminus U/U'$ is clean and every element of it appears in a cardinality two cover. It therefore remains to prove that

$$\tau(\mathcal{C} \setminus U/U') \geq 2.$$

Suppose otherwise. Then there exists a $B \in b(\mathcal{C})$ such that $B \cap U' = \emptyset$ and $|B - U| \leq 1$. Let $\mathcal{C}'$ be the minor of $\mathcal{C}$ obtained after deleting $B - U$ and contracting $V - (U \cup U' \cup B)$. Notice that $\mathcal{C}'$ has ground set $U \cup U'$. As $G[U \cup U']$ is a connected component of G, G has no edge with an end in $U \cup U'$ and another in $B - U$.[3] Thus, as $|B - U| \leq 1$, we get that $\tau(\mathcal{C}') \geq 2$, implying in turn that $\mathcal{C}'$ is clean and satisfies (2CovH). Let $G' := G(\mathcal{C}')$. Then G' is a bipartite graph by Lemma 2.1 (i). As $G[U \cup U'] \subseteq G'$, G' is connected and its bipartition is inevitably $\{U, U'\}$. It therefore follows from Lemma 2.1 (iv) that U, U' are both members of $\mathcal{C}'$. However, $B \cap U = B \cap (U \cup U') \in b(\mathcal{C}')$ is disjoint from $U' \in \mathcal{C}'$, a contradiction. □

2.2 Three lemmas

Here we provide three lemmas about the core of a clean clutter satisfying (2CovH).

Lemma 2.2. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), and let $G := G(\mathcal{C})$. Then the following statements hold:*

[3] Given a graph $G = (V, E)$ and a subset $X \subseteq V$, $G[X]$ denotes the subgraph induced on vertices X.

(i) If G is connected and has bipartition $\{U, U'\}$, then $U, U' \in \mathrm{core}(\mathcal{C})$.

(ii) If G has exactly two connected components with bipartitions $\{U_1, V_1\}, \{U_2, V_2\}$, then $U_1 \cup U_2, U_1 \cup V_2, V_1 \cup U_2, V_1 \cup V_2 \in \mathrm{core}(\mathcal{C})$.

Proof. Denote by V the ground set of $\mathcal{C}$. **(i)** We have that $U, U' \in \mathcal{C}$ by Lemma 2.1 (iv). As $\{U, U'\}$ gives an integral packing of value two, it follows that $U, U' \in \mathrm{core}(\mathcal{C})$. **(ii)** Consider the minor $\mathcal{C}' := \mathcal{C} \setminus U_1/V_1$. Then $\mathcal{C}'$ is clean and satisfies (2CovH) by Theorem 1.2.[4] Clearly $G[U_2 \cup V_2] \subseteq G(\mathcal{C}')$, so $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{U_2, V_2\}$. It therefore follows from Lemma 2.1 (iv) that $U_2, V_2 \in \mathcal{C}'$, implying in turn that $V_1 \cup U_2, V_1 \cup V_2$ each contains a member of $\mathcal{C}$, so by Lemma 2.1 (ii), $V_1 \cup U_2, V_1 \cup V_2$ *are* members of $\mathcal{C}$. Repeating the argument on $\mathcal{C}/U_1 \setminus V_1$ instead of $\mathcal{C}'$ tells us that $U_1 \cup U_2, U_1 \cup V_2$ are also members of $\mathcal{C}$. Since $\{V_1 \cup V_2, U_1 \cup U_2\}$ and $\{V_1 \cup U_2, U_1 \cup V_2\}$ each gives an integral packing of value two, it follows that $U_1 \cup U_2, U_1 \cup V_2, V_1 \cup U_2, V_1 \cup V_2 \in \mathrm{core}(\mathcal{C})$, as required. □

Lemma 2.3. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), where $G(\mathcal{C})$ is not connected. Let $\{U, U'\}$ be the bipartition of a connected component of $G(\mathcal{C})$, and let z, z' be fractional packings of $\mathcal{C} \setminus U/U', \mathcal{C}/U \setminus U'$ of value two, respectively. Let $y, y' \in \mathbb{R}_+^{\mathcal{C}}$ be defined as follows:* [5]

$$y_C := \begin{cases} z_{C-U'} & \text{if } C \cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y'_C := \begin{cases} z'_{C-U} & \text{if } C \cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{1}{2}y + \frac{1}{2}y'$ is a fractional packing of $\mathcal{C}$ of value two. In particular, $\mathrm{core}(\mathcal{C} \setminus U/U') \subseteq \mathrm{core}(\mathcal{C}) \setminus U/U'$.

Proof. We leave this as an exercise for the reader. □

We need the following immediate remark moving forward:

Remark 2.4. *Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH). Then $\mathrm{core}(\mathcal{C})$ is a clutter over ground set V satisfying (2CovH).*

Lemma 2.5. *Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH), where for some $u, v \in V$, every member of $\mathrm{core}(\mathcal{C})$ containing u also contains v. Then u, v belong to the same part of the bipartition of a connected component of $G(\mathcal{C})$.*

Proof. By Remark 1.6, it suffices to show that u, v belong to the same connected component of G. Suppose otherwise. In particular, G is not connected. Let $\{U, U'\}$ be the bipartition of the connected component containing u where $u \in U'$. Then $\mathcal{C} \setminus U/U'$ is clean and satisfies (2CovH). Let w be a neighbor of u in G; so $w \in U$. Then $\{w, u\}$ is a cover of $\mathcal{C}$. As every member of $\mathrm{core}(\mathcal{C})$ containing u also contains v, it follows that $\{w, v\}$ is a cover of $\mathrm{core}(\mathcal{C})$, implying in turn that $\mathrm{core}(\mathcal{C}) \setminus U/U'$ has $\{v\}$ as a cover. However, $\mathrm{core}(\mathcal{C} \setminus U/U') \subseteq \mathrm{core}(\mathcal{C}) \setminus U/U'$ by Lemma 2.3, so $\mathrm{core}(\mathcal{C} \setminus U/U')$ has a cover of cardinality 1, a contradiction to Remark 2.4. □

[4] Hereinafter, Theorem 1.2 will only be used implicitly and not referenced explicitly, as it is used heavily throughout the paper.

[5] Notice that by Lemma 2.1 (ii), if $C \cap U = \emptyset$ then $U' \subseteq C$ so $C - U' \in \mathcal{C} \setminus U/U'$, and if $C \cap U' = \emptyset$ then $U \subseteq C$ so $C - U \in \mathcal{C}/U \setminus U'$.

2.3 The proof of Theorem 1.7

Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH), let $G := G(\mathcal{C})$, and let $r := \text{rank}(\mathcal{C})$. Recall that r is the number of connected components of G. For each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. Let $S \subseteq \{0,1\}^r$ be the setcore of $\mathcal{C}$ with respect to $(U_1, V_1; \ldots; U_r, V_r)$.

Claim 1. $\frac{1}{2} \cdot \mathbf{1} \in \text{conv}(S)$.

Proof of Claim. As $\text{core}(\mathcal{C})$ has a fractional packing of value two, so does $\text{cuboid}(S)$, so the claim follows from Remark 1.8. $\diamondsuit$

Claim 2. $\frac{1}{2} \cdot \mathbf{1} \pm \frac{1}{2} \cdot e_i \in \text{conv}(S)$ *for each* $i \in [r]$.

Proof of Claim. Let $\mathcal{C}' := \mathcal{C} \setminus U_i/V_i$. We know that $\mathcal{C}'$ is clean and satisfies (2CovH), and that $\text{core}(\mathcal{C}') \subseteq \text{core}(\mathcal{C}) \setminus U_i/V_i$ by Lemma 2.3. Let z be a fractional packing of $\mathcal{C}'$ of value two. We know from Remark 1.5 that

$$\sum (z_{C'} : v \in C' \in \mathcal{C}') = 1 \quad \forall\, v \in V - (U_i \cup V_i).$$

Define $y \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

$$y_C := \begin{cases} z_{C - V_i} & \text{if } C \cap U_i = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} \mathbf{1}^\top y &= 2 \\ \sum (y_C : v \in C \in \mathcal{C}) &= 1 \quad \forall\, v \in V - (U_i \cup V_i) \\ \sum (y_C : v \in C \in \mathcal{C}) &= 2 \quad \forall\, v \in V_i \\ \sum (y_C : v \in C \in \mathcal{C}) &= 0 \quad \forall\, v \in U_i. \end{aligned}$$

As $\text{support}(z) \subseteq \text{core}(\mathcal{C}') \subseteq \text{core}(\mathcal{C}) \setminus U_i/V_i$, it follows that $\text{support}(y) \subseteq \text{core}(\mathcal{C})$. Define $\alpha \in \mathbb{R}_+^S$ as follows: for every point $p \in S$ and corresponding member $C \in \text{core}(\mathcal{C})$, let $\alpha_p := \frac{1}{2} \cdot y_C$. Then the equalities above show that $\mathbf{1}^\top \alpha = 1$ and $\sum_{p \in S} \alpha_p \cdot p = \frac{1}{2} \cdot \mathbf{1} + \frac{1}{2} \cdot e_i$. In particular, $\frac{1}{2} \cdot \mathbf{1} + \frac{1}{2} \cdot e_i \in \text{conv}(S)$. Repeating the argument on $\mathcal{C}/U_i \setminus V_i$ yields $\frac{1}{2} \cdot \mathbf{1} - \frac{1}{2} \cdot e_i \in \text{conv}(S)$, thereby proving the claim. $\diamondsuit$

Claims 1 and 2 together imply that $\text{conv}(S)$ is a full-dimensional polytope containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior, thereby finishing the proof of Theorem 1.7. $\square$

2.4 A characterization of the core

Let us provide a characterization for the core of a clean clutter satisfying (2CovH). This result is a consequence of Theorem 1.7 and is the converse of Remark 1.6.

Corollary 2.6. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), let $r := \text{rank}(\mathcal{C})$, and let $G := G(\mathcal{C})$. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G. Then*

$$\text{core}(\mathcal{C}) = \{C \in \mathcal{C} : C \cap (U_i \cup V_i) = U_i \text{ or } V_i \quad \forall\, i \in [r]\}.$$

Proof. Denote by $\mathcal{C}'$ the clutter on the right-hand side. Let $S := \text{setcore}(\mathcal{C} : U_1, V_1; \ldots; U_r, V_r)$. Let S' be the subset of $\{0,1\}^r$ defined as follows: start with $S' = \emptyset$, and for each $C \in \mathcal{C}'$, add a point p to S' such that

$$p_i = 0 \quad \Leftrightarrow \quad C \cap (U_i \cup V_i) = U_i \quad \forall\, i \in [r].$$

By Remark 1.6,

$$\text{core}(\mathcal{C}) \subseteq \mathcal{C}'$$

so $S \subseteq S'$. We know from Theorem 1.7 that $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of $\text{conv}(S)$, so $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of $\text{conv}(S')$. As a result, for every $p \in S'$, $\frac{1}{2} \cdot \mathbf{1}$ can be written as a convex combination of the points in S' such that coefficient of p is nonzero. That is, by Remark 1.8, for each $C \in \mathcal{C}'$, there is a fractional packing of $\mathcal{C}'$ whose support has C as a member. As every fractional packing of $\mathcal{C}'$ is also a fractional packing of $\mathcal{C}$, it follows that

$$\mathcal{C}' \subseteq \text{core}(\mathcal{C})$$

thereby finishing the proof of Corollary 2.6. □

3 The girth, the depth and duality

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), let $G := G(\mathcal{C})$ and let $r := \text{rank}(\mathcal{C})$. For each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. Let B be a cover of $\mathcal{C}$. Recall that B is a monochromatic cover if it is monochromatic in some proper 2-vertex-coloring of G. Equivalently, B is a monochromatic cover if $B \cap U_i = \emptyset$ or $B \cap V_i = \emptyset$ for each $i \in [r]$.

Lemma 3.1. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), let $G := G(\mathcal{C})$, let $r := \text{rank}(\mathcal{C})$, and let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G for $i \in [r]$. Then the following statements are equivalent:*

- $\text{girth}(\mathcal{C}) = \infty$*, i.e. $\mathcal{C}$ does not have a monochromatic cover,*
- $\text{depth}(\mathcal{C}) = -\infty$*, i.e.* $\text{setcore}(\mathcal{C}) = \{0,1\}^r$*, i.e.* $\bigcup_{i=1}^r W_i \in \text{core}(\mathcal{C})$ *whenever* $W_i \in \{U_i, V_i\}, i \in [r]$.

Proof. $(\Rightarrow)$ Let us first prove that $\bigcup_{i=1}^r U_i \in \mathcal{C}$. For if not, then $\bigcup_{i=1}^r V_i$ is a monochromatic cover, which is not the case. Thus $\bigcup_{i=1}^r U_i \in \mathcal{C}$, and similarly, $\bigcup_{i=1}^r V_i \in \mathcal{C}$. As $\{\bigcup_{i=1}^r U_i, \bigcup_{i=1}^r V_i\}$ gives an integral packing of $\mathcal{C}$ of value two, it follows that $\bigcup_{i=1}^r U_i \in \text{core}(\mathcal{C})$. Similarly, $\bigcup_{i=1}^r W_i \in \text{core}(\mathcal{C})$ whenever $W_i \in \{U_i, V_i\}, i \in [r]$. $(\Leftarrow)$ is immediate. □

In §3.1 we prove Weak Duality, Theorem 1.11. In §3.2, we introduce irreducible monochromatic covers, and also prove that every monochromatic cover intersects at least $\text{girth}(\mathcal{C})$ many connected components of G. In §3.3, we prove that in fact every monochromatic cover of $\text{core}(\mathcal{C})$ intersects $\text{girth}(\mathcal{C})$ many connected components of G. Finally, in §3.4, we prove Strong Duality, Theorem 1.12.

3.1 Weak Duality: The proof of Theorem 1.11

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and let $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. We need to show that $\mathrm{depth}(\mathcal{C}) \geq r - g$. If $g = \infty$, then we are done. Otherwise, let B be a monochromatic minimal cover of cardinality g. After relabeling and swapping $U_i, V_i, i \in [r]$, if necessary, we may assume that $B \subseteq U_1 \cup U_2 \cup \cdots \cup U_g$. In particular, $U_1 \cup \cdots \cup U_g$ is a cover of $\mathcal{C}$. Thus for each $C \in \mathrm{core}(\mathcal{C})$,

$$C \cap (U_i \cup V_i) = U_i \quad \text{for some } i \in [g].$$

That is, if $S \subseteq \{0,1\}^r$ is the setcore of $\mathcal{C}$ with respect to $(U_1, V_1; \ldots; U_r, V_r)$, then

$$S \cap \{x : x_1 = \cdots = x_g = 1\} = \emptyset.$$

Thus $\{x : x_1 = \cdots = x_g = 1\}$ is an infeasible hypercube of S of dimension $r - g$, implying in turn that $\mathrm{depth}(\mathcal{C}) \geq r - g$. This finishes the proof of Theorem 1.11. □

3.2 Irreducible and monochromatic covers

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), and let $r := \mathrm{rank}(\mathcal{C})$. Take a subset $I \subseteq [r]$, and for each $i \in I$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of $G(\mathcal{C})$. We say that $\bigcup_{i \in I} V_i$ is an *irreducible monochromatic cover of* $\mathcal{C}$ if $\bigcup_{i \in I} V_i$ is a cover of $\mathcal{C}$, and for each $j \in I$, $\left(\bigcup_{i \in I, i \neq j} V_i\right) \cup U_j$ is not a cover. In particular, if $\bigcup_{i \in I} V_i$ an irreducible monochromatic cover, then $\bigcup_{i \in I, i \neq j} V_i$ is not a cover for any $j \in I$.

Lemma 3.2. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Let $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G for $i \in [r]$. If $V_1 \cup \cdots \cup V_k$ is a monochromatic cover for some integer $k \in [r]$, then there exists a monochromatic minimal cover B such that*

(i) $B \subseteq \bigcup_{i=1}^k (U_i \cup V_i)$ and $|B \cap (U_i \cup V_i)| \leq 1$ for each $i \in [k]$, and

(ii) if $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover, then $B \subseteq \bigcup_{i=1}^k V_i$ and $|B \cap V_i| = 1$ for each $i \in [k]$.

Proof. Denote by V the ground set of $\mathcal{C}$.

(i) Out of all the monochromatic minimal covers of $\mathcal{C}$ contained in $\bigcup_{i=1}^k (U_i \cup V_i)$, pick one B of minimum cardinality. Pick U, U' such that $\{U, U'\} = \{U_i, V_i\}$ for some $i \in [k]$ and $\emptyset \neq B \cap (U \cup U') \subseteq U'$. We claim that $|B \cap (U \cup U')| = 1$, thereby proving (i). Suppose for a contradiction that $|B \cap (U \cup U')| \geq 2$. Let $I := B - (U \cup U')$, $J := V - (U \cup U' \cup I)$ and let $\mathcal{C}' := \mathcal{C} \setminus I/J$, a minor over ground set $U \cup U'$. Assume in the first case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and satisfies (2CovH), and $G[U \cup U'] \subseteq G(\mathcal{C}')$. Thus $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{U, U'\}$. It therefore follows from Lemma 2.1 (iv) that $U, U' \in \mathcal{C}'$. However, $B \cap U' = B \cap (U \cup U') = B - I$ is a minimal cover of $\mathcal{C}'$ disjoint from U, a contradiction.

Assume in the remaining case that $\tau(\mathcal{C}') \leq 1$. That is, there is a $D \in b(\mathcal{C})$ such that $D \subseteq U \cup U' \cup I$ and $|D - I| \leq 1$. But then D is a monochromatic minimal cover of $\mathcal{C}$ contained in $\bigcup_{i=1}^{k}(U_i \cup V_i)$ and

$$|D| = |D - I| + |D \cap I| \leq 1 + |B - (U \cup U')| < |B \cap (U \cup U')| + |B - (U \cup U')| = |B|,$$

a contradiction to our minimal choice of B. As a result, $|B \cap (U \cup U')| = 1$, as desired.

(ii) The proof of this part is very similar to (i), except for our minimal choice. Out of all the monochromatic minimal covers of $\mathcal{C}$ contained in $\bigcup_{i=1}^{k} V_i$, pick one B of minimum cardinality. As $\bigcup_{i=1}^{k} V_i$ is an irreducible monochromatic cover, it follows that $B \cap V_i \neq \emptyset, i \in [k]$. To finish the proof of (ii), it suffices to show that $|B \cap V_1| = 1$. Suppose for a contradiction that $|B \cap V_1| \geq 2$. Let $I := B - V_1$, $J := V - (U_1 \cup V_1 \cup I)$ and let $\mathcal{C}' := \mathcal{C} \setminus I/J$, a minor over ground set $U_1 \cup V_1$. Assume in the first case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and satisfies (2CovH), and $G[U_1 \cup V_1] \subseteq G(\mathcal{C}')$. Thus $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{U_1, V_1\}$. It therefore follows from Lemma 2.1 (iv) that $U_1, V_1 \in \mathcal{C}'$. However, $B \cap V_1 = B - I$ is a minimal cover of $\mathcal{C}'$ disjoint from U_1, a contradiction. Assume in the remaining case that $\tau(\mathcal{C}') \leq 1$. That is, there is a $D \in b(\mathcal{C})$ such that $D \subseteq U_1 \cup V_1 \cup I$ and $|D - I| \leq 1$. As $D \subseteq (V_1 \cup \cdots \cup V_k) \cup U_1$ and $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover, it follows that $D \subseteq \bigcup_{i=1}^{k} V_i$. But then D is a monochromatic minimal cover of $\mathcal{C}$ contained in $\bigcup_{i=1}^{k} V_i$ and

$$|D| = |D - I| + |D \cap I| \leq 1 + |B - (U_1 \cup V_1)| < |B \cap (U_1 \cup V_1)| + |B - (U_1 \cup V_1)| = |B|,$$

a contradiction to our minimal choice of B. As a result, $|B \cap V_1| = 1$, as desired. □

Part (ii) will be useful later. For now, as a consequence of part (i), we get the following:

Theorem 3.3. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Then every monochromatic cover intersects at least $\mathrm{girth}(\mathcal{C})$ many connected components of $G(\mathcal{C})$. In particular,*

- *every monochromatic cover intersects at least 3 connected components of $G(\mathcal{C})$, and*
- $\mathrm{girth}(\mathcal{C}) \leq \mathrm{rank}(\mathcal{C})$.

Proof. Let B be a monochromatic cover, and let $\mathcal{K}$ be the set of the connected components intersected by B. Then by Lemma 3.2 (i) there is a monochromatic minimal cover B' that intersects only a subset of $\mathcal{K}$ and intersects every connected component at most once. As a result, $|\mathcal{K}| \geq |B'| \geq \mathrm{girth}(\mathcal{C})$, thereby finishing the proof. □

3.3 Rank drop and girth drop

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). We saw in Theorem 3.3 that every monochromatic cover of $\mathcal{C}$ intersects at least $\mathrm{girth}(\mathcal{C})$ many connected components of $G(\mathcal{C})$. Here we strengthen this result by proving that every monochromatic cover of $\mathrm{core}(\mathcal{C})$ intersects at least $\mathrm{girth}(\mathcal{C})$ many connected components of $G(\mathcal{C})$. We need two lemmas about how the rank and the girth change when the two parts of a connected component of $G(\mathcal{C})$ are deleted and contracted.

Lemma 3.4. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), where $G := G(\mathcal{C})$ is not connected, and let $\{U, U'\}$ be the bipartition of a connected component of G. Then the following statements hold:*

(i) $\mathrm{rank}(\mathcal{C} \setminus U/U') \leq \mathrm{rank}(\mathcal{C}) - 1$*, and if equality holds, then the vertex sets of the connected components of* $G(\mathcal{C} \setminus U/U')$ *are precisely the vertex sets of the connected components of G different from $G[U \cup U']$,*

(ii) equality does not *hold in (i) if and only if there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G, and*

(iii) if $\mathrm{girth}(\mathcal{C}) > 3$*, then* $\mathrm{rank}(\mathcal{C} \setminus U/U') = \mathrm{rank}(\mathcal{C}) - 1$.

Proof. Let $r := \mathrm{rank}(\mathcal{C})$, and let $\{U_i, V_i\}, i \in [r-1]$ be the bipartitions of the connected components of G different from $G[U \cup U']$. Let $\mathcal{C}' := \mathcal{C} \setminus U/U'$, which is clean and satisfies (2CovH), and let $G' := G(\mathcal{C}')$. As $G[U_i \cup V_i] \subseteq G'$ for all $i \in [r-1]$, G' has at most $r-1$ connected components, so $\mathrm{rank}(\mathcal{C}') \leq r-1$, and if equality holds, then the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r-1]$. Thus **(i)** holds.

Claim 1. *Assume that B is a monochromatic cover of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G. Then* $\mathrm{rank}(\mathcal{C}') < r - 1$.

Proof of Claim. We may assume that $B \subseteq U_{r-2} \cup U_{r-1} \cup U$. As $B - U$ is a cover of $\mathcal{C}'$, it follows that $U_{r-2} \cup U_{r-1}$ is a cover of $\mathcal{C}'$. Let us now prove that $\mathrm{rank}(\mathcal{C}') < r-1$. Suppose otherwise. Then the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r-1]$ by (i). But then $U_{r-2} \cup U_{r-1}$ is a monochromatic cover of $\mathcal{C}'$, one that intersects only 2 connected components of G', a contradiction to Theorem 3.3 applied to $\mathcal{C}'$. ♢

Claim 2. *Assume that* $\mathrm{rank}(\mathcal{C}') < r - 1$*. Then there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G.*

Proof of Claim. Observe that $r \geq 3$. As G' has fewer than $r-1$ connected components, and as $G[U_i \cup V_i] \subseteq G'$ for $i \in [r-1]$, we may assume that G' has an edge between U_{r-2} and U_{r-1}. In particular, $U_{r-2} \cup U_{r-1}$ is a cover of $\mathcal{C}'$, implying in turn that $B := U_{r-2} \cup U_{r-1} \cup U$ is the desired cover of $\mathcal{C}$. ♢

Claims 1 and 2 prove **(ii)**.

Claim 3. **(iii)** *holds.*

Proof of Claim. We prove the contrapositive. Assume that $\mathrm{rank}(\mathcal{C}') < r - 1$. Then by (ii) there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G. Theorem 3.3 now applies to $\mathcal{C}$ and tells us that $\mathrm{girth}(\mathcal{C}) = 3$, as required. ♢

This finishes the proof of Lemma 3.4. □

Lemma 3.5. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), where $G := G(\mathcal{C})$ is not connected, and let $\{U, U'\}$ be the bipartition of a connected component of G. Then*

$$\mathrm{girth}(\mathcal{C}) - 1 \leq \mathrm{girth}(\mathcal{C} \setminus U/U')\,.$$

Moreover, assuming $\mathrm{girth}(\mathcal{C})$ *is finite and greater than* 3*, equality holds above if and only if there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly* $\mathrm{girth}(\mathcal{C})$ *connected components of G.*

Proof. Let $\mathcal{C}' := \mathcal{C} \setminus U/U'$, a clean clutter satisfying (2CovH), and let $G' := G(\mathcal{C}')$. Let $g := \mathrm{girth}(\mathcal{C})$ and $g' := \mathrm{girth}(\mathcal{C}')$. If $g = 3$, then $g - 1 = 2 < g'$, so we are done. We may therefore assume that $g > 3$. Let $r := \mathrm{rank}(\mathcal{C})$ and let $\{U_i, V_i\}, i \in [r-1]$ be the bipartitions of the connected components of G different from $G[U \cup U']$. It follows from Lemma 3.4 (iii) that $\mathrm{rank}(\mathcal{C}') = r - 1$, so by Lemma 3.4 (i), the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r-1]$. This immediately implies Claim 1 below:

Claim 1. *The bipartitions of the connected components of G' are $\{U_i, V_i\}, i \in [r-1]$.*

Claim 2. *$g - 1 \leq g'$, and if g is finite and equality holds, then there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly g connected components of G.*

Proof of Claim. If $g' = \infty$, we are done. Otherwise, let B' be a monochromatic cover of $\mathcal{C}'$ of cardinality g'. Let $B := B' \cup U$, a monochromatic cover of $\mathcal{C}$. In particular, g is finite. On the one hand, B intersects at most $g' + 1$ connected components of G, as $|B'| = g'$. On the other hand, B intersects at least g connected components of G by Theorem 3.3 applied to $\mathcal{C}$. Thus $g \leq g' + 1$, and if equality holds, then B is the desired set. ♢

Claim 3. *Suppose B is a monochromatic cover of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly g connected components of G. Then $g' = g - 1$.*

Proof of Claim. As B is monochromatic, $B \cap U' = \emptyset$, so $B - U$ is a cover of $\mathcal{C}'$. In fact, $B - U$ is a monochromatic cover of $\mathcal{C}'$ by Claim 1. As $B - U$ intersects only $g - 1$ connected components of G', $g' \leq g - 1$ by Theorem 3.3 applied to $\mathcal{C}'$, so $g' = g - 1$ by Claim 2, as claimed. ♢

Claims 2 and 3 finish the proof of Lemma 3.5. □

Theorem 3.6. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), where $G := G(\mathcal{C})$, $r :=$ $\mathrm{rank}(\mathcal{C})$, and denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. Assume that for some integer $k \in \{1, \ldots, r\}$, $U_1 \cup \cdots \cup U_k$ is a cover of* $\mathrm{core}(\mathcal{C})$*. Then $k \geq 3$. In fact, $k \geq$* $\mathrm{girth}(\mathcal{C})$.

Proof. It follows from Lemma 2.2 (i)-(ii) that $r \geq 3$, and from Lemma 3.1 that $\mathrm{girth}(\mathcal{C})$ is finite.

Let us first prove that $k \geq 3$. Suppose otherwise. Then $U_1 \cup U_2$ is a cover of $\mathrm{core}(\mathcal{C})$. Pick $u_1 \in U_1, u_2 \in U_2$ and $v_2 \in V_2$. It follows from Remark 1.6 that $\{u_1, u_2\}$ is a cover of $\mathrm{core}(\mathcal{C})$, and that every member of $\mathrm{core}(\mathcal{C})$ containing v_2 also contains u_1. But then Lemma 2.5 implies that u_1, v_2 belong to the same connected component of G, a contradiction.

Let us next prove that $k \geq \mathrm{girth}(\mathcal{C})$. We proceed by induction on $r \geq 3$. For the base case $r = 3$, as the girth is finite, we may apply Theorem 3.3 and conclude that $k \geq 3 = r \geq \mathrm{girth}(\mathcal{C})$. For the induction step, assume that $r \geq 4$. If $\mathrm{girth}(\mathcal{C}) = 3$, then we are done. Otherwise, $\mathrm{girth}(\mathcal{C}) > 3$. Let $\mathcal{C}' := \mathcal{C} \setminus U_k/V_k$, which is a clean clutter satisfying (2CovH), and let $G' := G(\mathcal{C}')$. Then $\mathrm{core}(\mathcal{C}') \subseteq \mathrm{core}(\mathcal{C}) \setminus U_k/V_k$ by Lemma 2.3, so $U_1 \cup \cdots \cup U_{k-1}$ is a cover of $\mathrm{core}(\mathcal{C}')$. As $\mathrm{girth}(\mathcal{C}) > 3$, it follows from Lemma 3.4 (iii) that $\mathrm{rank}(\mathcal{C}') = r - 1$, so by Lemma 3.4 (i), the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r] - \{k\}$. We may therefore apply the induction hypothesis to conclude that $k - 1 \geq \mathrm{girth}(\mathcal{C}')$. By Lemma 3.5, $\mathrm{girth}(\mathcal{C}') \geq \mathrm{girth}(\mathcal{C}) - 1$. Putting the last two inequalities together gives us that $k \geq \mathrm{girth}(\mathcal{C})$, thereby completing the induction step. □

Notice that Theorem 3.6 non-trivially relates a core dependent parameter to a parameter defined on the whole clutter.

3.4 Strong Duality: The proof of Theorem 1.12

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH). Let $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. Let $d := \mathrm{depth}(\mathcal{C})$. We would like to show that $r - d = \mathrm{girth}(\mathcal{C})$. If $d = -\infty$, then $\mathrm{girth}(\mathcal{C}) = \infty$ by Lemma 3.1, so we are done. Otherwise, pick a subset $I \subseteq [r]$ of cardinality $r - d$ and decisions $a_1, \ldots, a_{r-d} \in \{0, 1\}$ such that $\mathrm{setcore}(\mathcal{C}) \cap \{x : x_i = a_i, i \in I\} = \emptyset$. That is, for some $W_i \in \{U_i, V_i\}, i \in I$, the union $\bigcup_{i \in I} W_i$ is a cover of $\mathrm{core}(\mathcal{C})$. It therefore follows from Theorem 3.6 that $r - d \geq \mathrm{girth}(\mathcal{C})$. By Theorem 1.11, $r - d \leq \mathrm{girth}(\mathcal{C})$, so $r - d = \mathrm{girth}(\mathcal{C})$, thereby proving Theorem 1.12. □

4 Applications of Strong Duality

In this section we discuss three applications of Strong Duality, Theorem 1.12. In §4.1, we discuss an application to irreducible monochromatic covers. In §4.2, we show how clutter girth extends the notion of girth for simple binary matroids and simple graphs. In §4.3, we show how clutter girth extends the notion of covering number for clean clutters with covering number at least three.

4.1 Irreducible monochromatic covers

Theorem 4.1. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), where $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. Assume that $g := \mathrm{girth}(\mathcal{C})$ is finite and $U_1 \cup \cdots \cup U_g$ is a cover of $\mathcal{C}$. Then $U_1 \cup \cdots \cup U_g$ is an irreducible monochromatic cover of $\mathcal{C}$.*

Proof. By symmetry, it suffices to show that $U_1 \cup \cdots \cup U_{g-1} \cup V_g$ is not a cover. Suppose otherwise. Let $S \subseteq \{0, 1\}^r$ be the setcore of $\mathcal{C}$ with respect to $(U_1, V_1; \ldots; U_r, V_r)$. Then the two covers $U_1 \cup \cdots \cup U_{g-1} \cup U_g, U_1 \cup \cdots \cup U_{g-1} \cup V_g$ yield the following infeasible hypercubes in S:

$$\{x : x_1 = \cdots = x_{g-1} = x_g = 1\} \quad \text{and} \quad \{x : x_1 = \cdots = x_{g-1} = 1, x_g = 0\},$$

implying in turn that $\{x : x_1 = \cdots = x_{g-1} = 1\}$ is an infeasible hypercube of S. This implies that $\mathrm{depth}(\mathcal{C}) \geq r - (g-1) = r - g + 1$. However, $\mathrm{depth}(\mathcal{C}) = r - g$ by Theorem 1.12, a contradiction. □

4.2 The girth of a binary matroid

Take an integer $r \geq 1$ and let $S \subseteq \{0,1\}^r$ be a binary space. Basic Linear Algebra tells us that there is a $0-1$ matrix A with r columns such that

$$S = \{x : Ax \equiv \mathbf{0} \pmod 2\}.$$

Let M be the binary matroid over ground set $EM := [r]$ that is represented by A. The *cycle space of* M is the set $\mathrm{cycle}(M) := S$ and the *cocycle space of* M, denoted $\mathrm{cocycle}(M) \subseteq \{0,1\}^r$, is the row space of A over $GF(2)$. Notice that $\mathrm{cycle}(M), \mathrm{cocycle}(M)$ are binary spaces that are orthogonal complements over $GF(2)$. Observe that the binary matroid M can be fully determined by either A, its cycle space or its cocycle space.

A *cycle of* M is a subset $C \subseteq EM$ such that $\chi_C \in \mathrm{cycle}(M)$, and a *cocycle of* M is a subset $D \subseteq EM$ such that $\chi_D \in \mathrm{cocycle}(M)$. In particular, $\emptyset$ is both a cycle and a cocycle. Notice that every cycle and every cocycle have an even number of elements in common. A *circuit of* M is a nonempty cycle that does not contain another nonempty cycle, and a *cocircuit of* M is a nonempty cocycle that does not contain another nonempty cocycle. It is well-known that every cycle is either empty or the disjoint union of some circuits, and that every cocycle is either empty or the disjoint union of some cocircuits [20].

An element $e \in EM$ is a *loop of* M if $\{e\}$ is a circuit, and two distinct elements $e, f \in EM$ are *parallel in* M if $\{e, f\}$ is a circuit. M is a *simple* binary matroid if it has no loops and no parallel elements, i.e. if every circuit has cardinality at least three. If M is simple, then its *girth*, denoted $\mathrm{girth}(M)$, is the minimum cardinality of a circuit.

Theorem 4.2. *Let M be a simple binary matroid, and let $\mathcal{C} := \mathrm{cuboid}(\mathrm{cocycle}(M))$. Then* $\mathrm{girth}(\mathcal{C}) = \mathrm{girth}(M)$.

Proof. Write $EM = [r]$, and let $S^\perp := \mathrm{cocycle}(M)$, viewed as a subset of $\{0,1\}^r$. As $S^\perp$ is a binary space, there is a $0-1$ matrix B with r columns such that $S^\perp = \left\{x \in \{0,1\}^r : Bx \equiv \mathbf{0} \pmod 2\right\}$. Observe that the row space of B over $GF(2)$, which is the orthogonal complement of $S^\perp$ over $GF(2)$, is the cycle space of M.

Claim. $\mathrm{girth}(M) = r - \max\{d : S^\perp$ *has an infeasible hypercube of dimension* $d\}$.

Proof of Claim. Let $g := \mathrm{girth}(M)$ and $d^\star := \max\{d : S^\perp$ has an infeasible hypercube of dimension $d\}$. $(\geq)$ Let $C \subseteq [r]$ be a circuit of M of length g. We may assume that $C = \{1, \ldots, g\}$. As C intersects every cocycle an even number of times, it follows that $S^\perp \cap \{x : x_1 = \cdots = x_{g-1} = 0, x_g = 1\} = \emptyset$, so there is an infeasible hypercube of dimension $r - g$, implying in turn that $g \geq r - d^\star$. $(\leq)$ Pick $I \subseteq [r]$ of cardinality $r - d^\star$ such that $S^\perp \cap \{x : x_i = a_i \ \forall i \in I\} = \emptyset$ for some decisions $a_i \in \{0,1\}, i \in I$. As a result, the following linear system of equations has no $0-1$ solution:

$$Bx \equiv \mathbf{0} \pmod 2 \quad \text{and} \quad x_i \equiv a_i \pmod 2 \quad \forall i \in I.$$

There must be an infeasibility certificate. That is, assuming B has m rows, there exist $c \in \{0,1\}^m$ and $d_i \in \{0,1\}, i \in I$ such that

$$B^\top c + \sum_{i\in I} d_i e_i \equiv \mathbf{0} \pmod 2 \quad \text{and} \quad \sum_{i\in I} d_i a_i \equiv 1 \pmod 2.$$

Let C be the cycle of M such that $\chi_C = B^\top c$. Then the first equation above tells us that $C \subseteq I$ while the second equation tells us that $C \neq \emptyset$. As a result, M has a nonempty cycle of length at most $|I| = r - d^\star$, implying in turn that there is a circuit of length at most $r - d^\star$, so $g \leq r - d^\star$, as required. ◊

As M has no loop, the points in $S^\perp$ do not agree on a coordinate, so $\mathcal{C} = \mathrm{cuboid}(S^\perp)$ is clean and satisfies (2CovH) by Remark 1.15. Moreover, as M has no parallel elements, $\mathrm{girth}(M) \geq 3$, so $S^\perp$ has no infeasible hypercube of dimension at least $r-2$, by the claim above. We may therefore apply Proposition 1.10 to conclude that $\mathrm{rank}(\mathcal{C}) = r$ and $\mathrm{setcore}(\mathcal{C}) \cong S^\perp$. It therefore follows from the claim above that $\mathrm{girth}(M) = \mathrm{rank}(\mathcal{C}) - \mathrm{depth}(\mathcal{C})$, so $\mathrm{girth}(M) = \mathrm{girth}(\mathcal{C})$ by Theorem 1.12. □

Take disjoint subsets $I, J \subseteq EM$. The *minor of* M obtained after *deleting* I and *contracting* J is the binary matroid $M \setminus I/J$ over ground set $EM - (I \cup J)$ whose cycle space is obtained from $\mathrm{cycle}(M) \cap \{x : x_i = 0 \ \forall\, i \in I\}$ by dropping from every point the coordinates in $I \cup J$. The cocycle space of $M \setminus I/J$ is obtained from $\mathrm{cocycle}(M) \cap \{x : x_j = 0 \ \forall\, j \in J\}$ by dropping from every point the coordinates in $I \cup J$ [20].

Given distinct elements $e, f, g \in EM$, if e, f are parallel and f, g are parallel, then so are e, g. A *parallel class of* M is a maximal subset of EM of parallel elements. The *simplification* of M, denoted $\mathrm{si}(M)$, is the binary matroid obtained from M after deleting all loops, and for every parallel class, keeping one representative and deleting all the other elements. Observe that $\mathrm{si}(M)$ is a simple binary matroid.

Theorem 4.3. *Let M be a binary matroid without a loop, and let $\mathcal{C} := \mathrm{cuboid}(\mathrm{cocycle}(M))$. Then $\mathrm{rank}(\mathcal{C})$ is equal to the number of parallel classes of M, and $\mathrm{girth}(\mathcal{C}) = \mathrm{girth}(\mathrm{si}(M))$.*

Proof. This is a rather immediate consequence of Theorem 4.2. We leave this as an exercise for the reader. □

An important class of binary matroids comes from graphs. Let $G = (V, E)$ be a graph. For $X \subseteq E$, denote by $G[X]$ the subgraph of G whose vertices are the ends of the edges in X, and whose edges are X. A *circuit of* G is a nonempty subset $C \subseteq E$ where $G[C]$ is connected and every vertex has degree two. Recall that a cycle of G is a subset $C \subseteq E$ where every vertex of $G[C]$ has even degree. A *cocycle*, or *cut of* G, is an edge subset of the form $\delta(U), U \subseteq V$. A *cocircuit*, or *bond of* G, is a nonempty cocycle that does not contain another nonempty cocycle. Notice that the symmetric difference of any two cycles is another cycle, implying in turn that

$$\mathrm{cycle}(G) = \{\chi_C : C \subseteq E \text{ is a cycle of } G\} \subseteq \{0,1\}^E$$

is a binary space. Notice further that the symmetric difference of any two cocycles is another cocycle, so

$$\mathrm{cocycle}(G) := \{\chi_D : D \subseteq E \text{ is a cocycle of } G\} \subseteq \{0,1\}^E$$

is another binary space. Let A be the vertex-edge incidence matrix of G. Then $\text{cycle}(G) = \{x : Ax \equiv \mathbf{0} \pmod 2\}$ while $\text{cocycle}(G)$ is the row space of A over $GF(2)$. In particular, $\text{cycle}(G), \text{cocycle}(G)$ are respectively the cycle space and the cocycle space of the binary matroid represented by A. As a result, the two theorems provided here may be interpreted graph theoretically. We leave this task to the reader.

4.3 The covering number of a clutter

Let us start with the following easy remark:

Remark 4.4. *Neither a delta nor the blocker of an extended odd hole has a transversal of cardinality two.*

Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. For a point $x \in \{0,1\}^r$, the *induced clutter of S with respect to x* is the clutter over ground set $[r]$ whose members are

$$\text{ind}(S \triangle x) := \text{ the inclusion-wise minimal sets of } \{C \subseteq [r] : \chi_C \in S \triangle x\}.$$

Observe that S has 2^r induced clutters, and that these clutters are in a one-to-one correspondence with the 2^r minors of $\text{cuboid}(S)$ obtained after contracting, for each $i \in [r]$, exactly one of $2i-1, 2i$ (see [3]). It follows from Remark 4.4 that,

Remark 4.5. *Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$ whose induced clutters are clean. Then $\text{cuboid}(S)$ is clean.*

S is *up-monotone* if for all $x, y \in \{0,1\}^r$ such that $x \geq y$, if $y \in S$ then $x \in S$. An element of a clutter is *free* if it is not contained in any member.

Remark 4.6 ([3], Remark 4.6)**.** *Take an integer $r \geq 1$, an up-monotone set $S \subseteq \{0,1\}^r$, and a point $x \in \{0,1\}^r$. Then $\text{ind}(S \triangle x)$ is, after deleting free elements, equal to $\text{ind}(S \triangle \mathbf{0}) / \{i \in [r] : x_i = 1\}$.*

Let $\mathcal{A}$ be a clutter over ground set $[r]$. The *up-monotone set associated with $\mathcal{A}$* is the up-monotone set

$$\{\chi_C : C \subseteq [r] \text{ contains a member of } \mathcal{A}\} \subseteq \{0,1\}^r.$$

Notice that the induced clutter of this set with respect to $\mathbf{0}$ is $\mathcal{A}$.

Theorem 4.7. *Let $\mathcal{A}$ be a clean clutter such that $\tau(\mathcal{A}) \geq 3$. Let S be the associated up-monotone set, and let $\mathcal{C} := \text{cuboid}(S)$. Then $\mathcal{C}$ is clean and satisfies (2CovH), and $\text{girth}(\mathcal{C}) = \tau(\mathcal{A})$.*

Proof. As $\tau(\mathcal{A}) > 1$, the points in S do not agree on a coordinate, so $\mathcal{C}$ satisfies (2CovH). As $\text{ind}(S \triangle \mathbf{0}) = \mathcal{A}$ is clean, it follows from Remark 4.6 that every induced clutter of S is clean, so $\mathcal{C}$ is clean by Remark 4.5. It remains to prove that $\text{girth}(\mathcal{C}) = \tau(\mathcal{A})$. To this end, label the ground set of $\mathcal{A}$ as $[r]$ for some integer $r \geq 1$, let $\tau := \tau(\mathcal{A})$, and let

$$d^\star := \max\{d : S \text{ has an infeasible hypercube of dimension } d\}.$$

Claim. $\tau = r - d^\star$.

Proof of Claim. ($\geq$) Let $B \subseteq [r]$ be a cover of $\mathcal{A}$ of cardinality τ. Then $S \cap \{x : x_i = 0\ \forall i \in B\} = \emptyset$, implying in turn that $d^\star \geq r - \tau$. ($\leq$) Pick $I \subseteq [r]$ of cardinality $r - d^\star$ such that $S \cap \{x : x_i = a_i\ \forall i \in I\} = \emptyset$ for some decisions $a_i \in \{0,1\}, i \in I$. As S is up-monotone, and as the infeasible hypercube above is maximum, it follows that $a_i = 0$ for all $i \in I$. As a result, I must be a cover of $\mathrm{ind}(S \triangle \mathbf{0}) = \mathcal{A}$, implying in turn that $r - d^\star \geq \tau$, as required. $\Diamond$

In particular, as $\tau \geq 3$, S has no infeasible hypercube of dimension at least $r-2$. We may therefore apply Proposition 1.10 and conclude that $\mathrm{rank}(\mathcal{C}) = r$ and $\mathrm{setcore}(\mathcal{C}) \cong S$. As a result, $\tau = \mathrm{rank}(\mathcal{C}) - \mathrm{depth}(\mathcal{C})$ by the claim above, implying that $\tau = \mathrm{girth}(\mathcal{C})$ by Theorem 1.12, thereby finishing the proof. $\square$

5 Simplices

Take an integer $r \geq 1$. A *hypercube inequality* is an inequality of the form $x_i \geq 0$ or $x_i \leq 1$ for some $i \in [r]$. A *generalized set covering inequality* is an inequality of the form

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad I, J \subseteq [r], I \cap J = \emptyset.$$

A set $S \subseteq \{0,1\}^r$ is *cube-ideal* if its convex hull can be described by hypercube and generalized set covering inequalities [3]. Since our inequalities are closed under the transformation $x_i \mapsto 1 - x_i, i \in [r]$, it follows that being cube-ideal is invariant under twisting, so if a set is cube-ideal, then so is any set isomorphic to it.

Theorem 5.1 ([3], Theorem 1.6)**.** *Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. Then* $\mathrm{cuboid}(S)$ *is ideal if, and only if, S is cube-ideal.*

In §5.1, we characterize cube-ideal simplices. In §5.2 we prove that an ideal clutter satisfying (2CovH) has an ideal core and a cube-ideal setcore. Finally, in §5.3, we provide a geometric proof of Theorem 1.23.

5.1 Cube-ideal simplices

Here we prove the following:

Theorem 5.2. *Take an integer $r \geq 1$ and a cube-ideal set $S \subseteq \{0,1\}^r$ whose convex hull is a simplex containing $\frac{1}{2} \cdot \mathbf{1}$ in its relative interior. Then after a possible twisting, either $S = \{\mathbf{0}, \mathbf{1}\}$, or $S = \{\mathbf{0}, a+b, b+c, c+a\}$ for some nonzero points $a, b, c \in \{0,1\}^r$ whose supports partition $[r]$.*

This result is a weaker version and consequence of Theorem 1.22. However, in contrast with the combinatorial proof of that theorem, we provide a purely geometric proof for Theorem 5.2:

Proof. Assume that $S = \{s_0, s_1, \ldots, s_k\}$ for some integer $k \leq r$, and $\mathrm{conv}(S) = \{x \in \mathbb{R}^r : a_i^\top x \geq b_i,\ i = 0, 1, \ldots, k\}$ where $a_i^\top x \geq b_i$ determines the facet of $\mathrm{conv}(S)$ not containing s_i. As $\frac{1}{2} \cdot \mathbf{1}$ is in the relative interior of $\operatorname{conv}(S)$, there exist $\lambda_0, \lambda_1, \ldots, \lambda_k > 0$ such that $\sum_{j=0}^k \lambda_j = 1$ and $\sum_{j=0}^k \lambda_j s_j = \frac{1}{2} \cdot \mathbf{1}$. For each $i \in \{0, 1, \ldots, k\}$,

$$\frac{1}{2} \cdot \mathbf{1}^\top a_i = \sum_{j=0}^{k} \lambda_j s_j^\top a_i = \lambda_i s_i^\top a_i + (1 - \lambda_i) b_i,$$

implying in turn that

$$\lambda_i = \frac{\mathbf{1}^\top a_i - 2b_i}{2(s_i^\top a_i - b_i)}.$$

Since $\operatorname{conv}(S)$ is cube-ideal, for each $i \in \{0, 1, \ldots, k\}$,

- $a_i^\top x \geq b_i$ is equivalent to $x_j \geq 0$ or $x_j \leq 1$ for some $j \in [r]$, or
- $a_i^\top x \geq b_i$ is equivalent to $x(I) - x(J) \geq 1 - |J|$ for some disjoint subsets $I, J \subseteq [r]$.

In the first case, we see that $\lambda_i = \frac{1}{2}$. In the second case, we see that $\lambda_i = 0$ if $|I \cup J| = 2$, and that $\lambda_i \geq \frac{1}{4}$ if $|I \cup J| \geq 3$. Since $\lambda_i > 0$ for each i, and $\sum_{j=0}^k \lambda_j s_j = \frac{1}{2} \cdot \mathbf{1}$, it follows that either $k = 1$ and $\lambda_0 = \lambda_1 = \frac{1}{2}$, or $k = 3$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{4}$. After a possible twisting, we may assume that $s_0 = \mathbf{0}$. It is now easy to see that either $S = \{\mathbf{0}, \mathbf{1}\}$ or $S = \{\mathbf{0}, a+b, b+c, c+a\}$ for some nonzero points $a, b, c \in \{0,1\}^r$ whose supports partition $[r]$, thereby finishing the proof. □

5.2 Every ideal clutter has an ideal core and a cube-ideal setcore.

For a clutter $\mathcal{C}$ over ground set V, denote by $Q(\mathcal{C})$ its set covering polyhedron, that is,

$$Q(\mathcal{C}) = \left\{ x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \right\}.$$

Observe that the $0-1$ points in $Q(\mathcal{C})$ are precisely the incidence vectors of the covers of $\mathcal{C}$, while the integral vertices of $Q(\mathcal{C})$ are precisely the incidence vectors of the minimal covers of $\mathcal{C}$. We need the following two ingredients:

Lemma 5.3 ([2], Lemma 3.1)**.** *Take an integer $r \geq 1$ and let $\mathcal{C}$ be a clutter over ground set $\{u_1, v_1, \ldots, u_r, v_r\}$, where $\{u_i, v_i\}$ is a transversal of $\mathcal{C}$ for each $i \in [r]$. Then the following statements are equivalent:*

(i) *$\mathcal{C}$ is ideal,*

(ii) $\operatorname{conv}\{\chi_C : C \in \mathcal{C}\} = Q\big(b(\mathcal{C})\big) \cap \{x : x_{u_i} + x_{v_i} = 1 \ \forall i \in [r]\}$.

Lemma 5.4. *Let $\mathcal{C}$ be an ideal clutter satisfying (2CovH). Then*

$$\operatorname{core}(\mathcal{C}) = \{C \in \mathcal{C} : C \text{ is a transversal of the minimum covers of } \mathcal{C}\}.$$

Proof. ($\subseteq$) follows from Remark 1.5. ($\supseteq$) Consider the primal-dual pair of linear programs

$$(P) \quad \begin{array}{ll} \min & \mathbf{1}^\top x \\ \text{s.t.} & x(C) \geq 1 \quad C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (y_C : u \in C \in \mathcal{C}) \leq 1 \quad u \in V \\ & y \geq \mathbf{0}. \end{array}$$

Let $C^\star$ be a member of $\mathcal{C}$ that intersects every minimum cover exactly once. As $\mathcal{C}$ is an ideal clutter, it follows that $\bar{x}(C^\star) = 1$ for every optimal solution $\bar{x}$ of (P). It therefore follows from Strict Complementarity that there is an optimal solution $\bar{y}$ of (D) such that $\bar{y}_{C^\star} > 0$. As $\bar{y}$ is inevitably a fractional packing of $\mathcal{C}$ of value two, it follows that $C^\star \in \mathrm{core}(\mathcal{C})$, as required. □

For the proof of the following theorem, we will use the fact that a clutter is ideal if and only if a duplication of it is ideal.

Theorem 5.5. *Let $\mathcal{C}$ be an ideal clutter satisfying (2CovH). Then* $\mathrm{core}(\mathcal{C})$ *is an ideal clutter, and* $\mathrm{setcore}(\mathcal{C})$ *is a cube-ideal set.*

Proof. Denote by V the ground set of $\mathcal{C}$, and by $G = (V, E)$ the bipartite graph $G(\mathcal{C})$. By Lemma 5.4,

$$\{\chi_C : C \in \mathrm{core}(\mathcal{C})\} = \{\chi_C : C \in \mathcal{C}\} \cap \big\{x : x_u + x_v = 1 \ \{u, v\} \in E\big\}. \qquad (\star)$$

Claim 1. $\mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\} = Q\big(b(\mathrm{core}(\mathcal{C}))\big) \cap \big\{x : x_u + x_v = 1 \ \{u, v\} \in E\big\}$.

Proof of Claim. ($\subseteq$) follows immediately from ($\star$). ($\supseteq$) Pick a point $x^\star$ in the set on the right-hand side. As $Q\big(b(\mathrm{core}(\mathcal{C}))\big) \subseteq Q\big(b(\mathcal{C})\big)$, we have $x^\star \in Q\big(b(\mathcal{C})\big)$. Since $\mathcal{C}$ is ideal, so is $b(\mathcal{C})$, implying that for some $\lambda \in \mathbb{R}^{\mathcal{C}}_+$ with $\sum_{C\in\mathcal{C}} \lambda_C = 1$, we have that

$$x^\star \geq \sum_{C\in\mathcal{C}} \lambda_C \chi_C.$$

Since for all $\{u, v\} \in E$, we have that $x^\star_u + x^\star_v = 1$ and $\{u, v\} \in b(\mathcal{C})$, equality must hold above and by ($\star$), if $\lambda_C > 0$ then $C \in \mathrm{core}(\mathcal{C})$. Hence, $x^\star \in \mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\}$, as required. ♢

Let $r := \mathrm{rank}(\mathcal{C})$, and for each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. We know by Remark 1.6 that in $\mathrm{core}(\mathcal{C})$, the elements in each $U_i, i \in [r]$ are duplicates, and the elements in each $V_i, i \in [r]$ are duplicates. Observe further that every point $x \in \mathbb{R}^V$ in the two equal sets in Claim 1 satisfies

$$x_u = x_{u'} \quad \text{and} \quad x_v = x_{v'} \quad \forall\, i \in [r], u, u \in U_i, v, v \in V_i.$$

For each $i \in [r]$, pick $u_i \in U_i$ and $v_i \in V_i$, and let $\mathcal{C}'$ be the clutter over ground set $\{u_1, v_1, \ldots, u_r, v_r\}$ obtained from $\mathrm{core}(\mathcal{C})$ after contracting $V - \{u_1, v_1, \ldots, u_r, v_r\}$. Notice that for each $i \in [r]$, $\{u_i, v_i\}$ is a transversal of $\mathcal{C}'$. Notice further that $\mathcal{C}'$ is nothing but the cuboid of $\mathrm{setcore}(\mathcal{C})$.

Claim 2. $\mathrm{conv}\{\chi_C : C \in \mathcal{C}'\} = Q\big(b(\mathcal{C}')\big) \cap \big\{z : z_{u_i} + z_{v_i} = 1 \ i \in [r]\big\}$

Proof of Claim. We use Claim 1 to prove this equality. Observe that $\mathrm{conv}\{\chi_C : C \in \mathcal{C}'\}$ is the projection of $\mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\}$ onto the coordinates $\{u_i, v_i : i \in [r]\}$. Thus, to finish the proof, it suffices to show that $Q\big(b(\mathcal{C}')\big) \cap \big\{z : z_{u_i} + z_{v_i} = 1 \ \ i \in [r]\big\}$ is the projection of $Q\big(b(\mathrm{core}(\mathcal{C}))\big) \cap \big\{x : x_u + x_v = 1 \ \{u, v\} \in E\big\}$ onto the same coordinates. We leave this as an easy exercise for the reader. ♢

It therefore follows from Lemma 5.3 that $\mathcal{C}'$ is an ideal clutter. As $\mathrm{core}(\mathcal{C})$ is a duplication of $\mathcal{C}'$, it follows that $\mathrm{core}(\mathcal{C})$ is an ideal clutter, too. As $\mathcal{C}'$ is the cuboid of $\mathrm{setcore}(\mathcal{C})$, it follows from Theorem 5.1 that $\mathrm{setcore}(\mathcal{C})$ is a cube-ideal set, thereby finishing the proof. □

5.3 A geometric proof of Theorem 1.23

Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) where $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ is a simplex. Let $r := \mathrm{rank}(\mathcal{C})$. We know from Theorem 1.7 that $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$. We need to show that $\mathcal{C}$ is non-ideal or $\mathrm{setcore}(\mathcal{C}) \cong \{0,1\}, \{000, 110, 101, 011\}$. To this end, let us assume that $\mathcal{C}$ is an ideal clutter. It then follows from Theorem 5.5 that $\mathrm{setcore}(\mathcal{C})$ is a cube-ideal set. It therefore follows from Theorem 5.2 that $\mathrm{setcore}(\mathcal{C})$ is isomorphic to either $\{\mathbf{0}, \mathbf{1}\}$, or $\{\mathbf{0}, a+b, b+c, c+a\}$ for some nonzero points $a, b, c \in \{0,1\}^r$ whose supports partition $[r]$. As $\mathrm{setcore}(\mathcal{C})$ has no duplicated coordinates by Theorem 1.7, it follows that $r \in \{1, 3\}$ and $\mathrm{setcore}(\mathcal{C})$ is isomorphic to either $\{0,1\}$ or $\{000, 110, 101, 011\}$, thereby finishing the proof. $\square$

6 Dyadic fractional packings of value two

Let $\mathcal{C}$ be a clutter over ground set V. A *filter oracle* for $\mathcal{C}$ consists of V along with an oracle which, given any set $X \subseteq V$, decides in unit time whether or not X contains a member [23]. Observe that,

Remark 6.1. *Given a filter oracle for a clutter $\mathcal{C}$ over ground set V, one can produce all covers of cardinality two in time $\binom{|V|}{2}$.*

Proof. Given $X \subseteq V$, X is a cover if and only if $V - X$ does not contain a member of $\mathcal{C}$, so we can test in unit time whether or not X is a cover. As a result, one can produce all covers of cardinality at most two by querying the oracle accordingly for each set in $\{X \subseteq V : |X| = 2\}$. $\square$

Remark 6.2 ([23]). *Given a filter oracle for a clutter $\mathcal{C}$ over ground set V, and given disjoint $I, J \subseteq V$, one has a filter oracle for $\mathcal{C} \setminus I/J$.*

Proof. Given $X \subseteq V - (I \cup J)$, X contains a member of $\mathcal{C} \setminus I/J$ if and only if $X \cup J$ contains a member of $\mathcal{C}$, so we can test in unit time whether or not X contains a member of $\mathcal{C} \setminus I/J$. $\square$

In §6.1 we prove Theorem 1.13, and in §6.2 we prove a lemma on unique dyadic fractional packings of value two, which will be useful in the next section.

6.1 The proof of Theorem 1.13

Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH) inputted via a filter oracle. We will prove that for some integer $k \in \{1, \ldots, \mathrm{rank}(\mathcal{C})\}$ and in time at most $(2^k - 1) \cdot |V|^2$, one can find a $\frac{1}{2^{k-1}}$-integral packing of value two. We proceed by induction on the rank $r := \mathrm{rank}(\mathcal{C}) \geq 1$. Let $G := G(\mathcal{C})$, a bipartite graph where every vertex is incident with an edge, and let $\{U, U'\}$ be the bipartition of a connected component of G. Observe that G, U and U' can be found in time at most $|V|^2$ by Remark 6.1.

For the base case $r = 1$, notice that $U, U' \in \mathcal{C}$ by Lemma 2.1 (iv), so $\{U, U'\}$ gives a $\frac{1}{2^{r-1}}$-integral packing of value two found in time at most $|V|^2$, thereby proving the base case. For the induction step, assume that $r > 1$, that is, G is not connected. Then $\mathcal{C} \setminus U/U'$ and $\mathcal{C}/U \setminus U'$ are clean and satisfy (2CovH). Notice that the filter oracle $\mathcal{C}$ gives filter oracles for $\mathcal{C} \setminus U/U'$ and $\mathcal{C}/U \setminus U'$ by Remark 6.2. Let $r_1 := \mathrm{rank}(\mathcal{C} \setminus U/U')$ and $r_2 := \mathrm{rank}(\mathcal{C}/U \setminus U')$. By Lemma 3.4 (i), $r_1 \le r-1$ and $r_2 \le r-1$. Hence, by the induction hypothesis, there exist integers k_1, k_2 satisfying $1 \le k_1 \le r_1$ and $1 \le k_2 \le r_2$, and fractional packings z, z' of $\mathcal{C}\setminus U/U', \mathcal{C}/U\setminus U'$ of value two that are $\frac{1}{2^{k_1-1}}$-integral and $\frac{1}{2^{k_2-1}}$-integral, respectively. Moreover, z, z' are found in time at most

$$(2^{k_1}-1)\cdot|V-(U\cup U')|^2 + (2^{k_2}-1)\cdot|V-(U\cup U')|^2.$$

Let $y, y' \in \mathbb{R}_+^{\mathcal{C}}$ be defined as follows:

$$y_C := \begin{cases} z_{C-U'} & \text{if } C\cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y'_C := \begin{cases} z'_{C-U} & \text{if } C\cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{1}{2}y + \frac{1}{2}y'$ is a fractional packing of $\mathcal{C}$ of value two by Lemma 2.3, one that is $\frac{1}{2^{k-1}}$-integral for $k = 1 + \max\{k_1, k_2\}$ and is found in time at most

$$|V|^2 + (2^{k_1}-1)\cdot|V-(U\cup U')|^2 + (2^{k_2}-1)\cdot|V-(U\cup U')|^2 \le (2^k-1)\cdot|V|^2.$$

As $k \le 1+\max\{r_1, r_2\} \le r$, it follows that $\frac{1}{2}y+\frac{1}{2}y'$ is the desired dyadic fractional packing, thereby completing the induction step. □

6.2 A lemma on unique dyadic fractional packings

For the lemma below, recall that the support of a fractional packing is viewed as a clutter, so minor operations can be applied to it.

Lemma 6.3. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) and (UniqH), where $G := G(\mathcal{C})$ is not connected, and let $\{U, U'\}$ be the bipartition of some connected component of G. Then $\mathcal{C} \setminus U/U'$ is clean and satisfies (2CovH) and (UniqH). Moreover, if y, z are the dyadic fractional packings of $\mathcal{C}, \mathcal{C} \setminus U/U'$, respectively, then* $\mathrm{support}(z) = \mathrm{support}(y) \setminus U/U'$.

Proof. As $\mathcal{C} \setminus U/U'$ and $\mathcal{C}/U \setminus U'$ are clean and satisfy (2CovH), we may apply Theorem 1.13 and conclude that there exist dyadic fractional packings z, z' of $\mathcal{C} \setminus U/U', \mathcal{C}/U \setminus U'$ of value two, respectively. Let $t, t' \in \mathbb{R}_+^{\mathcal{C}}$ be defined as follows:

$$t_C := \begin{cases} z_{C-U'} & \text{if } C\cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t'_C := \begin{cases} z'_{C-U} & \text{if } C\cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.3, $\frac{1}{2}t + \frac{1}{2}t'$ is a fractional packing of $\mathcal{C}$ of value two, one that is dyadic. It therefore follows from (UniqH) for $\mathcal{C}$ that $\frac{1}{2}t + \frac{1}{2}t' = y$. In particular, z is the unique dyadic fractional packing of $\mathcal{C} \setminus U/U'$ of value two, z' is the unique dyadic fractional packing of $\mathcal{C}/U \setminus U'$ of value two, and

$$\mathrm{support}(z) = \mathrm{support}(y) \setminus U/U'$$
$$\mathrm{support}(z') = \mathrm{support}(y)/U \setminus U'.$$

Thus $\mathcal{C}\setminus U/U'$ is clean and satisfies (2CovH) and (UniqH), and $\mathrm{support}(z) = \mathrm{support}(y)\setminus U/U'$, as desired. □

7 Projective geometries

We start off this section with two basic and well-known results about projective geometries. In §7.1, we prove Theorem 1.16. In §7.2, we prove that every binary clutter satisfying (2CovH) embeds a projective geometry, thereby verifying Conjecture 1.17 for binary clutters. In §7.3, we prove a crucial lemma for constructing projective geometries from smaller projective geometries. Finally, in §7.4, we prove Theorem 1.19.

A *triangle* in a binary matroid is a circuit of cardinality three. Take an integer $k \geq 1$. Let A be the $k \times (2^k - 1)$ matrix whose columns are all the $0-1$ vectors of dimension k that are nonzero. Recall that $PG(k-1,2)$ is the binary matroid represented by A,

$$\mathrm{cycle}(PG(k-1,2)) = \{x : Ax \equiv \mathbf{0} \pmod 2\}$$

and $\mathrm{cocycle}(PG(k-1,2))$ is the row space of A over $GF(2)$. As A has no zero column, and no two columns of it are equal, $PG(k-1,2)$ is a simple binary matroid. We need the following two propositions throughout this section:

Proposition 7.1. *Take an integer $k \geq 2$. Then the following statements hold for $PG(k-1,2)$:*

(i) every nonempty cocycle has cardinality 2^{k-1},

(ii) every two elements appear together in a triangle,

(iii) every cycle is the symmetric difference of some triangles.

Proof. **(i)**: Let D be a nonempty cocycle. Then χ_D is nonzero and belongs to $\mathrm{cocycle}(PG(k-1,2))$. Let A' be a $k \times (2^k - 1)$ matrix with $0-1$ entries whose first row is χ_D and whose rows form a basis for $\mathrm{cocycle}(PG(k-1,2))$ over $GF(2)$. Notice that the orthogonal complement of $\mathrm{cocycle}(PG(k-1,2))$ over $GF(2)$ is equal to

$$\mathrm{cycle}(PG(k-1,2)) = \{x : A'x \equiv \mathbf{0} \pmod 2\}.$$

As $PG(k-1,2)$ is a simple binary matroid, it follows that A' has no zero column, and no two columns of it are equal. As A' has $2^k - 1$ columns and k rows, it follows that the columns of A' are all the $0-1$ vectors of dimension k that are nonzero. In particular, every row of A' has 2^{k-1} ones and $2^{k-1}-1$ zeros. In particular, $|D| = 2^{k-1}$.

(ii) Let A be the $k \times (2^k - 1)$ matrix representation of $PG(k-1,2)$ whose columns are all the $0-1$ vectors of dimension k that are nonzero. Pick distinct elements e, f of $PG(k-1,2)$, and let a, b be the corresponding columns of A. Notice that $a + b \pmod 2$ is another column of A, and let g be the corresponding element of $PG(k-1,2)$. Then $\{e,f,g\}$ is the desired triangle of $PG(k-1,2)$.

(iii) Let C be a nonempty cycle. We proceed by induction on $|C| \geq 3$. The base $|C| = 3$ holds trivially. For the induction step assume that $|C| \geq 4$. Pick distinct elements $e, f \in C$. By (ii) there is an element g such that $\{e,f,g\}$ is a triangle. Since $C \triangle \{e,f,g\}$ is a cycle of smaller cardinality than C, the induction hypothesis applies and tell us that $C \triangle \{e,f,g\}$ is the symmetric difference of some triangles, implying in turn that C is the symmetric difference of some triangles, thereby completing the induction step. □

Property (ii) in fact characterizes projective geometries:

Proposition 7.2. *Let M be a simple binary matroid where every two elements appear together in a triangle. Then M is a projective geometry.*

Proof. We may assume that $EM = [r]$ for some integer $r \geq 1$. Let A be a $0-1$ matrix with column labels $[r]$ and whose rows are linearly independent over $GF(2)$, where

$$\text{cycle}(M) = \{x : Ax \equiv \mathbf{0} \pmod 2\}.$$

After elementary row operations over $GF(2)$ and reordering the columns, if necessary, we may assume that $A = (I \mid A')$ where I is the identity matrix of appropriate dimension. As M is simple, it follows that every column of A' has at least two 1s, and no two columns of it are equal. Since every two elements of M appear together in a triangle, it follows that if a, b are distinct columns of A, then $a + b \pmod 2$ is another column. It can now be readily checked that A' consists of all $0-1$ vectors with at least two 1s, implying in turn that M is a projective geometry, as required. □

7.1 The proof of Theorem 1.16

Take an integer $k \geq 1$, let $r := 2^k - 1$, and let $S := \text{cocycle}(PG(k-1, 2))$. We know that $|S| = 2^k = r + 1$. It follows from Proposition 7.1 (i) that the inequality

$$\sum_{i=1}^{r} x_i \leq \frac{r+1}{2}$$

is valid for $\text{conv}(S)$, and that every point in S except for $\mathbf{0}$ satisfies this inequality at equality. As S is a binary space, $S \triangle p = S$ for every point $p \in S$. This transitive property implies that for each $p \in S$, the transformed inequality

$$\sum_{i:p_i=0} x_i + \sum_{j:p_j=1} (1 - x_j) \leq \frac{r+1}{2}$$

is also valid for $\text{conv}(S)$, and every point in S except for p satisfies this inequality at equality. Hence, $\text{conv}(S)$ is an r-dimensional simplex whose $r+1$ facets are as described above. As the point $\frac{1}{2} \cdot \mathbf{1}$ satisfies every inequality strictly, it lies in the interior of $\text{conv}(S)$. In fact, as S is a binary space whose points do not agree on a coordinate, it follows that $|S \cap \{x : x_i = 0\}| = |S \cap \{x : x_i = 1\}|$ for each $i \in [r]$, so

$$\sum_{p \in S} \frac{1}{r+1} \cdot p = \frac{1}{2} \cdot \mathbf{1}.$$

As $\text{conv}(S)$ is a simplex, it follows from Remark 1.8 that $\frac{2}{r+1} \cdot \mathbf{1} = \frac{1}{2^{k-1}} \cdot \mathbf{1}$ is the unique fractional packing of $\text{cuboid}(S)$ of value two, thereby finishing the proof of Theorem 1.16. □

7.2 Every binary clutter embeds a projective geometry.

An immediate consequence of Theorem 1.13 is the following:

Corollary 7.3. *Every binary clutter satisfying (2CovH) has a dyadic fractional packing of value two.*

In fact, we will show that every binary clutter satisfying (2CovH) embeds a projective geometry, thereby verifying Conjecture 1.17 for binary clutters. Given integers $n, m \geq 1$ and some vectors $a_1, \ldots, a_m \in \{0,1\}^n$, denote by $\langle a_1, \ldots, a_m \rangle$ the vector space over $GF(2)$ generated by the vectors $a_1, \ldots, a_m$.

Proposition 7.4. *Take an integer $n \geq 1$ and let $S \subseteq \{0,1\}^n$ be a binary space whose points do not agree on a coordinate. Then there exists a subset $S' \subseteq S$ whose points do not agree on a coordinate and is a duplication of the cocycle space of a projective geometry.*

Proof. Let A be a $0-1$ matrix whose rows $a_1, \ldots, a_m$ are linearly independent over $GF(2)$ such that $S = \{x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod 2\}$. Clearly $\langle a_1, \ldots, a_m \rangle$ is the orthogonal complement of S over $GF(2)$. As the points in S do not agree on a coordinate, $\langle a_1, \ldots, a_m \rangle$ does not contain the unit vectors $e_1, \ldots, e_n$. Extend $\{a_1, \ldots, a_m\}$ to a set $\{a_1, \ldots, a_m, \ldots, a_k\}$ of $0-1$ vectors such that

(i) $a_1, \ldots, a_k$ are linearly independent over $GF(2)$,

(ii) $\langle a_1, \ldots, a_k \rangle$ does not contain $e_1, \ldots, e_n$,

(iii) $\{a_1, \ldots, a_k\}$ is maximal subject to (i)-(ii).

Let M be the binary matroid over ground set $[n]$ whose cycle space is $\langle a_1, \ldots, a_k \rangle$. It follows from (ii) that M has no loop.

Claim. *For every pair of distinct elements f, g, there is a cycle of M of cardinality at most three containing f, g.*

Proof of Claim. Let $a := \chi_{\{f,g\}} \in \{0,1\}^n$. If $a \in \langle a_1, \ldots, a_k \rangle$, then $\{f, g\}$ is a cycle of M. Otherwise, $a_1, \ldots, a_k, a$ are linearly independent over $GF(2)$, so by (iii), $\langle a_1, \ldots, a_k, a \rangle$ must contain one of $e_1, \ldots, e_n$. Neither f nor g is a loop of M, so $\langle a_1, \ldots, a_k \rangle$ must contain a vector b with three 1s such that $b_f = b_g = 1$. That is, M has a triangle containing f, g, as required. $\Diamond$

Let $M' := \mathrm{si}(M)$, the simplification of M. Then the claim implies that every two elements of M' appear together in a triangle, so M' is a projective geometry by Proposition 7.2. As $\mathrm{cocycle}(M)$ is a duplication of $\mathrm{cocycle}(M')$, it follows that $\mathrm{cocycle}(M)$ is a duplication of the cocycle space of a projective geometry. Notice however that $\mathrm{cocycle}(M) = \{x \in \{0,1\}^n : Bx \equiv \mathbf{0} \pmod 2\}$ where B is the matrix whose rows are $a_1, \ldots, a_k$. As such, $S' := \mathrm{cocycle}(M) \subseteq S$. As M has no loop, the points in S' do not agree on a coordinate, so S' is the desired set. $\square$

An *affine vector space over $GF(2)$*, or simply an *affine binary space*, is a twisting of a binary space. The remark below follows immediately from our definition of binary clutters:

Remark 7.5. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If $\mathrm{cuboid}(S)$ is a binary clutter, then S is an affine binary space.*

(The converse of this remark also holds as is shown in Remark 1.15.) For the proof of the following theorem, we will use the fact that a clutter is binary if and only if a duplication of it is binary:

Theorem 7.6. *Let $\mathcal{C}$ be a binary clutter satisfying (2CovH). Then $\mathcal{C}$ is a duplication of $\mathrm{cuboid}(S)$, where S is a binary space whose points do not agree on a coordinate. In particular, every binary clutter satisfying (2CovH) embeds a projective geometry.*

Proof. As $\mathcal{C}$ is a binary clutter, we have that

$$|C \cap B| \equiv 1 \pmod 2 \quad \forall C \in \mathcal{C}, B \in b(\mathcal{C}).$$

Notice that if $|B| = 2$, then we must have that $|C \cap B| = 1$ for all $C \in \mathcal{C}$. Thus every member of $\mathcal{C}$ is a transversal of the minimum covers, implying in turn that $\mathcal{C}$ is a duplication of $\mathrm{cuboid}(S)$ for some set S. As $\mathrm{cuboid}(S)$ is also a binary clutter, we may assume by Remark 7.5 that S is a binary space, thereby proving the first part of the theorem. For the next part, notice that by Proposition 7.4, S has a subset whose points do not agree on a coordinate and is a duplication of $\mathrm{cocycle}(PG(k-1,2))$ for some integer $k \geq 1$. This implies that a subset of $\mathrm{cuboid}(S)$, and therefore of $\mathcal{C}$, is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$, so $\mathcal{C}$ embeds $PG(k-1,2)$ by definition, thereby proving the second part of the theorem. □

7.3 Constructing projective geometries

Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. The *incidence matrix of S* is the matrix whose rows are the points in S. Denote by J the all-ones matrix of appropriate dimensions. Notice that for every integer $k \geq 1$, in the incidence matrix of $\mathrm{cocycle}(PG(k-1,2))$, every column has 2^{k-1} ones and 2^{k-1} zeros. In fact,

Remark 7.7. *Take an integer $k \geq 2$. If A' is the incidence matrix of $\mathrm{cocycle}(PG(k-2,2))$, then up to permuting rows and columns,*

$$\begin{pmatrix} \mathbf{1} & A' & J - A' \\ \mathbf{0} & A' & A' \end{pmatrix}$$

is the incidence matrix of $\mathrm{cocycle}(PG(k-1,2))$. Moreover, every element of $PG(k-1,2)$ can be used as the left-most column in the incidence matrix above.

Proof. We leave this as an exercise for the reader. □

As a consequence,

Remark 7.8. *Take an integer $k \geq 2$, and let $\mathcal{C} := \mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$. Then for every minimum cover $\{u,v\}$, $\mathcal{C} \setminus u/v$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-2,2)))$.*

Let $\mathcal{C}$ be a clutter over ground set V. The *incidence matrix of* $\mathcal{C}$, denoted $M(\mathcal{C})$, is the matrix whose columns are labeled by V and whose rows are the incidence vectors of the members of $\mathcal{C}$. Two columns of a $0-1$ matrix are *complementary* if they add up to the all-ones vector. If $\mathcal{C}$ is the cuboid of $\mathrm{cocycle}(PG(k-1,2))$ for some integer $k \geq 1$, then every column has 2^{k-1} ones, and by Remark 7.7, every pair of columns of $M(\mathcal{C})$ are either complementary or have exactly 2^{k-2} ones in common.

Lemma 7.9. *Take an integer $r \geq 2$ and a clutter $\mathcal{C}$ satisfying (UniqH) whose ground set V is partitioned into nonempty parts $U_1, V_1, \ldots, U_r, V_r$ such that*

- *the elements in each part are duplicates,*
- *for each $i \in [r]$, if $u \in U_i$ and $v \in V_i$, then $\{u, v\}$ is a transversal of $\mathcal{C}$, and*
- *for each $i \in [r]$, $\mathcal{C} \setminus U_i/V_i$ (resp. $\mathcal{C}/U_i \setminus V_i$) is a duplication of the cuboid of the cocycle space of a projective geometry.*

Then there is an integer $k \geq 2$ such that $r = 2^k - 1$ and $\mathcal{C}$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$.

Proof. We may assume after contracting some duplicate elements that $U_i = \{u_i\}$ and $V_i = \{v_i\}$ for each $i \in [r]$. In particular, $\mathcal{C}$ is a cuboid. As $\mathcal{C}$ has a fractional packing of value two, it follows that $\tau(\mathcal{C}) \geq 2$, so $\mathcal{C}$ satisfies (2CovH). For each $i \in [r]$, let $f(u_i) := v_i$ and $f(v_i) := u_i$.

Claim 1. *$\mathcal{C}$ does not have duplicated elements. In particular, if $\{u, v\}$ is a transversal of $\mathcal{C}$, then $v = f(u)$.*

Proof of Claim. Suppose for a contradiction that u, u' are duplicates. Since $\tau(\mathcal{C}) = 2$, $\{u, u'\}$ is not a cover, so $u' \neq f(u)$. But then $\mathcal{C} \setminus f(u)/u$ has $\{u'\}$ as a cover, a contradiction as $\mathcal{C} \setminus f(u)/u$ is a duplication of the cuboid of the cocycle space of a projective geometry. ♢

In what follows the reader should keep in mind that our labeling of the columns of $M(\mathcal{C})$ induces a labeling for the columns of $M(\mathcal{C} \setminus f(u)/u)$, for each $u \in V$. In particular, $M(\mathcal{C} \setminus f(u)/u)$ and $M(\mathcal{C}/f(u) \setminus u)$ have the same column labels, for each $u \in V$.

Claim 2. *There is an integer $k \geq 2$ such that the following statements hold:*

(1) for each $u \in V$, $\mathcal{C} \setminus f(u)/u$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-2,2)))$,

(2) $|\mathcal{C}| = 2^k$,

(3) every column of $M(\mathcal{C})$ has exactly 2^{k-1} ones,

(4) every pair of columns of $M(\mathcal{C})$ are either complementary or have exactly 2^{k-2} ones in common.

Proof of Claim. For each $u \in V$, $\mathcal{C} \setminus f(u)/u$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k_u-2,2)))$ for some integer $k_u \geq 2$. In particular, every column u of $M(\mathcal{C})$ has exactly $2^{k_u-1} = |\mathrm{cocycle}(PG(k_u-2,2))|$ ones. Notice now that if $u \in V$ and $w \in V - \{u, f(u)\}$, then the number of ones in column w of $M(\mathcal{C})$ is equal

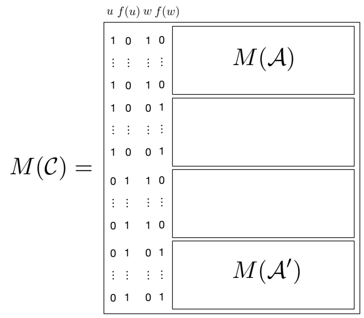

Figure 2: The incidence matrix of $\mathcal{C}$ in the proof of Lemma 7.9, Claim 4 (3)

to the sum of the number of ones in column w of $M(\mathcal{C} \setminus f(u)/u)$ and the number of ones in column w of $M(\mathcal{C}/f(u) \setminus u)$, so

$$2^{k_w - 1} = 2^{k_u - 2} + 2^{k_{f(u)} - 2}$$

implying in turn that $k_w = k_u = k_{f(u)}$. As a result, $(k_u : u \in V)$ are all equal to k for some integer $k \geq 2$. It can be readily checked that (1)-(4) hold for k, as required. $\Diamond$

If a pair of columns of $M(\mathcal{C})$ are complementary, then by Claim 1, the column labels must be $u, f(u)$ for some $u \in V$.

Claim 3. $\frac{1}{2^{k-1}} \cdot \mathbf{1} \in \mathbb{R}^{\mathcal{C}}$ *is the unique dyadic fractional packing of* $\mathcal{C}$.

Proof of Claim. This follows from Claim 2 (2)-(3) and the assumption that $\mathcal{C}$ satisfies (UniqH). $\Diamond$

Claim 4. *The following statements hold for every* $u \in V$*:*

(1) a pair of identical columns in the matrix $M(\mathcal{C} \setminus f(u)/u)$ *correspond to a complementary pair of columns in the matrix* $M(\mathcal{C}/f(u) \setminus u)$*,*

(2) $M(\mathcal{C} \setminus f(u)/u)$ *does not have three identical columns,*

(3) in $M(\mathcal{C} \setminus f(u)/u)$ *every column is identical to another one,*

(4) $r = 2^k - 1$ *and* $\mathcal{C} \setminus f(u)/u$ *is obtained from* $\text{cuboid}(\text{cocycle}(PG(k-2, 2)))$ *after duplicating every element exactly once.*

Proof of Claim. **(1)** follows from Claim 2 (4) and **(2)** follows from (1). **(3)** Pick an arbitrary $w \in V - \{u, f(u)\}$. We will prove that in $M(\mathcal{C} \setminus f(u)/u)$, column w is identical to another column. To this end, let $\mathcal{A} := \mathcal{C} \setminus \{f(u), f(w)\}/\{u, w\}$ and $\mathcal{A}' := \mathcal{C}/\{f(u), f(w)\} \setminus \{u, w\}$. See Figure 2. Notice that $|\mathcal{A}| = |\mathcal{A}'| = 2^{k-2}$ by Claim 2 (4). We claim that

$$\tau(\mathcal{A}) = 1 \quad \text{or} \quad \tau(\mathcal{A}') = 1. \qquad (\star)$$

Suppose otherwise. Then $\mathcal{A},\mathcal{A}'$ satisfy (2CovH). As $\mathcal{C}\setminus f(u)/u,\mathcal{C}/f(u)\setminus u$ are binary clutters, so are their respective minors $\mathcal{A},\mathcal{A}'$. Thus, by Corollary 7.3, $\mathcal{A},\mathcal{A}'$ have dyadic fractional packings z,z' of value two, respectively. Define $y,y'\in\mathbb{R}^{\mathcal{C}}_+$ as follows:

$$y_C := \begin{cases} z_{C-\{u,w\}} & \text{if } C\cap\{f(u),f(w)\}=\emptyset\\ 0 & \text{otherwise}\end{cases} \quad\text{and}\quad y'_C := \begin{cases} z'_{C-\{f(u),f(w)\}} & \text{if } C\cap\{u,w\}=\emptyset\\ 0 & \text{otherwise.}\end{cases}$$

It can be readily checked that $t:=\frac{1}{2}(y+y')$ is a dyadic fractional packing of $\mathcal{C}$ of value two. It therefore follows from Claim 3 that $t=\frac{1}{2^{k-1}}\cdot\mathbf{1}$. However, for any $C\in\mathcal{C}$ such that $C\cap\{u,f(u),w,f(w)\}=\{u,f(w)\}$ or $\{f(u),w\}$, of which there are 2^{k-1} many, we have $t_C=0$, a contradiction. Thus $(\star)$ holds.

Assume in the first case that $\tau(\mathcal{A})=1$, that is, there is a $B\in b(\mathcal{C})$ such that $B\cap\{u,w\}=\emptyset$ and $B-\{f(u),f(w)\}=\{v\}$ for some $v\in V$. Then column v of $M(\mathcal{A})$ is all-ones and therefore has $|\mathcal{A}|=2^{k-2}$ ones. By Claim 2 (4), in $M(\mathcal{C})$, columns v,w have 2^{k-2} ones in common and columns v,u have 2^{k-2} ones in common. As a result, in $M(\mathcal{C}\setminus f(u)/u)$, column v must be identical to column w.

Assume in the remaining case that $\tau(\mathcal{A}')=1$, that is, there is a $B\in b(\mathcal{C})$ such that $B\cap\{f(u),f(w)\}=\emptyset$ and $B-\{u,w\}=\{v\}$ for some $v\in V$. Then column v of $M(\mathcal{A}')$ is all-ones and therefore has $|\mathcal{A}'|=2^{k-2}$ ones. By Claim 2 (4), in $M(\mathcal{C})$, columns $v,f(w)$ have 2^{k-2} ones in common and columns $v,f(u)$ have 2^{k-2} ones in common. Thus, in $M(\mathcal{C}/f(u)\setminus u)$, column v is identical to column $f(w)$. It now follows from part (1) of this claim that in $M(\mathcal{C}\setminus f(u)/u)$, columns $v,f(w)$ are complementary, implying in turn that in $M(\mathcal{C}\setminus f(u)/u)$, column v is identical to column w.

In both cases we proved that in $M(\mathcal{C}\setminus f(u)/u)$, column w is identical to another column, as claimed. This finishes the proof of part (3).

(4) Claim 2 (1) together with parts (2) and (3) of this claim imply that the minor $\mathcal{C}\setminus f(u)/u$ is obtained from $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-2,2)))$ after duplicating every element exactly once. In particular,

$$2r = 2+2\cdot 2\cdot(2^{k-1}-1) = 2\cdot(2^k-1),$$

so $r=2^k-1$, as claimed. ♢

Pick $S\subseteq\{0,1\}^r$ containing $\mathbf{0}$ such that $\mathcal{C}=\mathrm{cuboid}(S)$. We will prove that $S=\mathrm{cocycle}(PG(k-1,2))$. Denote by A the incidence matrix of S. Notice that A is a column submatrix of $M(\mathcal{C})$, and the column labels of A form a subset of V and a transversal of $\{\{u,f(u)\}:u\in V\}$.

Claim 5. *In A the sum of every two columns modulo 2 is equal to another column.*

Proof of Claim. Pick two columns of A with column labels $u,w\in V$. By Claim 4 (3), in $M(\mathcal{C}\setminus f(u)/u)$, column w is identical to another column v. Notice that $v\in V-\{u,f(u),w,f(w)\}$. By Claim 4 (1), in $M(\mathcal{C}/f(u)\setminus u)$, columns w,v are complementary. Thus, in $M(\mathcal{C})$, columns u,w,v add up to $\mathbf{1}$ modulo 2, implying in turn that columns $u,w,f(v)$ add up to $\mathbf{0}$ modulo 2. We know that columns u,w of $M(\mathcal{C})$ are also present in A, and that exactly one of the columns $v,f(v)$ of $M(\mathcal{C})$ is present in A. As $\mathbf{0}\in S$, A has a zero row, so no three of its columns can add up to $\mathbf{1}$ modulo 2, implying in turn that $f(v)$ must be a column of A instead of v. As a result, in A, columns u,w add up to column $f(v)$ modulo 2, as required. ♢

We next use Remark 7.7 to argue that up to permuting rows and columns, A is the incidence matrix of $\mathrm{cocycle}(PG(k-1,2))$. To this end, denote by $v_0 \in V$ the label of the first column of A. For $j \in \{0,1\}$, denote by I_j the rows of A corresponding to $\{x \in S : x_{v_0} = j\}$. By Claim 2 (3), $|I_0| = |I_1| = 2^{k-1}$. Notice that $\frac{r-1}{2} = 2^{k-1} - 1$ by Claim 4 (4). Label the columns of A other than v_0 as $v_1, u_1, v_2, u_2, \ldots, v_{\frac{r-1}{2}}, u_{\frac{r-1}{2}}$ where for each $i \in \left[\frac{r-1}{2}\right]$, the sum of columns v_0 and v_i modulo 2 is equal to column u_i – such a labeling exists because of Claim 5. Define matrices A_1, A_2, A_3, A_4:

- A_1 is the $I_1 \times \{v_1, \ldots, v_{\frac{r-1}{2}}\}$ submatrix of A,
- A_2 is the $I_1 \times \{u_1, \ldots, u_{\frac{r-1}{2}}\}$ submatrix of A,
- A_3 is the $I_0 \times \{v_1, \ldots, v_{\frac{r-1}{2}}\}$ submatrix of A,
- A_4 is the $I_0 \times \{u_1, \ldots, u_{\frac{r-1}{2}}\}$ submatrix of A.

Then $A_3 = A_4$ and $A_1 + A_2 = J$. After swapping the labels v_i and $u_i, i \in \left[\frac{r-1}{2}\right]$, if necessary, we may assume that A_1 has a zero row. Notice further that as $\mathbf{0} \in S$ and $A_3 = A_4$, the matrix A_3 also has a zero row. As a result, by Claim 4 (4), up to permuting rows and columns, the following three matrices are equal: A_1, A_3, and the incidence matrix of $\mathrm{cocycle}(PG(k-2,2))$.

For the rest of the proof, we work with the projective geometry $PG(k-2,2)$ whose labeling agrees with the column labels of A_3, that is, the cocycles of the labeled $PG(k-2,2)$ are the rows of A_3.

Claim 6. *Up to permuting rows, A_1 and A_3 are equal.*

Proof of Claim. This is obviously true if $k = 2$. We may therefore assume that $k \geq 3$. It suffices to show that every row of A_1 is equal to some row of A_3, because the two matrices are already equal up to permuting rows and columns. Pick a row χ_D of A_1 for some $D \subseteq \{v_1, \ldots, v_{\frac{n-1}{2}}\}$. We need to show that D is a cocycle of (the labeled) $PG(k-2,2)$. Pick a triangle $\{v_i, v_j, v_k\}$ of $PG(k-2,2)$, that is, the corresponding columns of A_3 add up to zero modulo 2. Consider now the columns v_i, v_j of A. By Claim 5, the sum of these two columns modulo 2 is another column of A. This column is either v_k or u_k, and in fact since A_1 has a zero row, it must be v_k. As a result, columns v_i, v_j, v_k of A_1 also add up to zero modulo 2, implying in turn that $|D \cap \{v_i, v_j, v_k\}|$ is even. Thus, D intersects every triangle of $PG(k-2,2)$ an even number of times, so by Proposition 7.1 (iii), D intersects every cycle of $PG(k-2,2)$ an even number of times, implying in turn that D is a cocycle of $PG(k-2,2)$, as required. $\Diamond$

We may therefore assume that $A_1 = A_3$, implying in turn that $A_1 = A_3 = A_4$ and $A_2 = J - A_1$. As A_1 is the incidence matrix of $\mathrm{cocycle}(PG(k-2,2))$, it follows from Remark 7.7 that A is the incidence matrix of $\mathrm{cocycle}(PG(k-1,2))$, so $S = \mathrm{cocycle}(PG(k-1,2))$. As $\mathcal{C} = \mathrm{cuboid}(S)$, and as $r = 2^k - 1$ by Claim 4 (4), we have finished the proof of Lemma 7.9. $\square$

7.4 The proof of Theorem 1.19

Let $\mathcal{C}$ be a clean clutter over ground set V satisfying (2CovH) and (UniqH), and let y be the dyadic fractional packing of $\mathcal{C}$ of value two. We prove by induction on $|V| \geq 2$ that

> $(\star)$ there is an integer $k \geq 1$ such that y is $\frac{1}{2^{k-1}}$-integral, $\mathrm{rank}(\mathcal{C}) = 2^k - 1$ and $\mathrm{support}(y)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1, 2)))$.

For the base case $|V| = 2$, as $\mathcal{C}$ satisfies (2CovH), it must consists of two members of size one each, so $(\star)$ holds for $k = 1$. For the induction step, assume that $|V| \geq 3$. Let $r := \mathrm{rank}(\mathcal{C})$, $G := G(\mathcal{C})$, and for each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. As $\mathrm{support}(y) \subseteq \mathrm{core}(\mathcal{C})$, Remark 1.6 implies Claim 1 below:

Claim 1. *For each $C \in \mathrm{support}(y)$ and $i \in [r]$, $C \cap (U_i \cup V_i)$ is either U_i or V_i.*

Claim 2. *If $r = 1$, then $(\star)$ holds for $k = 1$.*

Proof of Claim. Assume that $r = 1$. Then $\mathrm{support}(y) \subseteq \{U_1, V_1\}$ by Claim 1, and as $\mathrm{support}(y)$ contains a fractional packing of $\mathcal{C}$ of value two, we must have that $\mathrm{support}(y) = \{U_1, V_1\}$, and the claim follows. $\Diamond$

We may therefore assume that $r \geq 2$.

Claim 3. *The following statements hold:*

(1) $\mathrm{support}(y)$ *satisfies (UniqH),*

(2) the elements in each of $U_1, V_1, \ldots, U_r, V_r$ are duplicates in $\mathrm{support}(y)$*,*

(3) for each $i \in [r]$, if $u \in U_i$ and $v \in V_i$, then $\{u, v\}$ is a transversal of $\mathrm{support}(y)$*, and*

(4) for each $i \in [r]$, $\mathrm{support}(y) \setminus U_i / V_i$ *(resp.* $\mathrm{support}(y)/U_i \setminus V_i$*) is a duplication of the cuboid of the cocycle space of a projective geometry.*

Proof of Claim. **(1)** is obvious, and **(2)** and **(3)** follow from Claim 1. **(4)** By Lemma 6.3, the minor $\mathcal{C} \setminus U_i/V_i$ is clean and satisfies (2CovH) and (UniqH), and given that z is the unique dyadic fractional packing of $\mathcal{C} \setminus U_i/V_i$, we have that $\mathrm{support}(z) = \mathrm{support}(y) \setminus U_i/V_i$. Our induction hypothesis applied to $\mathcal{C} \setminus U_i/V_i$ implies that $\mathrm{support}(z)$, which is equal to $\mathrm{support}(y) \setminus U_i/V_i$, is a duplication of the cuboid of the cocycle space of a projective geometry, as required. $\Diamond$

We may therefore apply Lemma 7.9 to $\mathrm{support}(y)$ to conclude that for some integer $k \geq 2$, $r = 2^k - 1$ and $\mathrm{support}(y)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1, 2)))$. It follows from Theorem 1.16 that y assigns $\frac{1}{2^{k-1}}$ to the members of $\mathrm{support}(y)$, so $(\star)$ holds. This completes the induction step, thereby finishing the proof of Theorem 1.19. $\square$

8 Finding the Fano plane as a minor

Recall that

$$\mathbb{L}_7 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,5,6\},\{2,4,7\},\{3,4,6\},\{3,5,7\}\}$$

and $b(\mathbb{L}_7) = \mathbb{L}_7$. This clutter enjoys a lot of symmetries. $\mathbb{L}_7$ has an automorphism mapping every element to every other element, and an automorphism mapping every member to every other member. These facts are crucial throughout this section. In §8.1 we prove a lemma for finding an $\mathbb{L}_7$ minor, and in §8.2 we prove Theorem 1.20. We need the following ingredient:

Remark 8.1. *Let $G = (V,E)$ be a connected, bipartite graph with bipartition $\{U,U'\}$ where $U,U' \neq \emptyset$. Assume that there exists a subset $X \subseteq U'$ such that $2 \leq |X| \leq 3$, and there is no proper vertex-induced subgraph that is connected and contains X. Then G is a tree whose leaves are in X.*

Proof. By our minimality assumption, every vertex in $V - X$ is a cut-vertex of G separating at least two vertices in X. We claim that G is a tree. Suppose otherwise. Then there is a circuit $C \subseteq V$. For every vertex $v \in C$, there is a vertex $g(v) \in X$ such that

- if $v \in X$, then $g(v) = v$,
- otherwise, $g(v)$ is a vertex of X such that every path between it and $C - \{v\}$ includes v.

Notice that if v, v' are distinct vertices of C, then $g(v) \neq g(v')$. In particular, $|X| \geq |C|$, implying in turn that $|C| = 3$, a contradiction as G is bipartite. Thus G is a tree. It is immediate from our minimality assumption that every leaf of G belongs to X. □

8.1 A lemma for finding an $\mathbb{L}_7$ minor

Lemma 8.2. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH), where the following statements hold:*

(a) $\mathcal{C}$ has rank 7, and for each $i \in [7]$, the i^{th} connected component of $G(\mathcal{C})$ has bipartition $\{U_i, V_i\}$.

(b) For each $L \in \mathbb{L}_7$, $\bigcup_{i \notin L} U_i \cup \bigcup_{j \in L} V_j$ contains a member of $\mathcal{C}$.

(c) For all $L \in \mathbb{L}_7$ but at most one, $\bigcup_{j \in L} V_j$ is a cover of $\mathcal{C}$.

Then $\mathcal{C}$ has an $\mathbb{L}_7$ minor.

(In (b), $\bigcup_{i \notin L} U_i \cup \bigcup_{j \in L} V_j$ for each $L \in \mathbb{L}_7$ must in fact be a member by Lemma 2.1 (ii); but the proof is easier to follow given the current version of (b).)

Proof. Let $G := G(\mathcal{C})$.

Claim 1. *Take a subset $L \subseteq [7]$ such that $|L| \leq 3$ and $\bigcup_{i \in L} V_i$ is a cover. Then $L \in \mathbb{L}_7$ and $\bigcup_{i \in L} V_i$ is an irreducible monochromatic cover.*

Proof of Claim. As (b) holds, $\bigcup_{i\in L} V_i$ intersects each $\bigcup_{i\notin L} U_i \cup \bigcup_{j\in L} V_j$, $L \in \mathbb{L}_7$, implying in turn that L is a cover of $\mathbb{L}_7$. As $b(\mathbb{L}_7) = \mathbb{L}_7$ and $|L| \leq 3$, it follows that $L \in \mathbb{L}_7$. After a possible relabeling, we may assume that $L = \{1,2,3\}$. As $\bigcup_{i\notin\{1,4,5\}} U_i \cup \bigcup_{j\in\{1,4,5\}} V_j$ contains a member by (b), it follows that $U_1 \cup V_2 \cup V_3$ is not a cover. Similarly, $V_1 \cup U_2 \cup V_3$ and $V_1 \cup V_2 \cup U_3$ are not covers either. As a result, $V_1 \cup V_2 \cup V_3$ is an irreducible monochromatic cover, as claimed. ♢

Claim 2. *Take an $L \in \mathbb{L}_7$ such that $\bigcup_{i\in L} V_i$ is a cover. Then $\bigcup_{i\in L} V_i$ contains a minimal cover of cardinality 3 picking one element from each $V_i, i \in L$.*

Proof of Claim. This follows from Claim 1 and Lemma 3.2 (ii). ♢

Claim 3. *For each $L \in \mathbb{L}_7$, $\bigcup_{i\in L} V_i$ is a cover.*

Proof of Claim. We may assume because of (c) that for each $L \in \mathbb{L}_7 - \{\{3,5,7\}\}$, $\bigcup_{i\in L} V_i$ contains a minimal cover B_L; we may assume by Claim 2 that B_L has cardinality 3 and picks one element from each $V_i, i \in L$. It remains to prove that $V_3 \cup V_5 \cup V_7$ is a cover. Suppose otherwise. Let $\mathcal{C}' := \mathcal{C} \setminus (V_5 \cup V_7)/(U_5 \cup U_7)$.

Assume in the first case that $\tau(\mathcal{C}') \leq 1$. That is, there is a minimal cover $D \in b(\mathcal{C})$ such that $D \cap (U_5 \cup U_7) = \emptyset$ and $|D - (V_5 \cup V_7)| \leq 1$. It follows from Claim 1 that $D - (V_5 \cup V_7) = \{u\}$ for some $u \in V_3 \cup U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_6$. Our contrary assumption tells us that $u \notin V_3$. But then D is disjoint from one of

$$\bigcup_{i\notin L} U_i \cup \bigcup_{j\in L} V_j \quad L = \{1,2,3\}, \{3,4,6\},$$

a contradiction to (b).

Assume in the remaining case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and satisfies (2CovH), and $G' := G(\mathcal{C}')$ has $G \setminus (U_5 \cup V_5 \cup U_7 \cup V_7)$ as a subgraph. Then G' is a bipartite graph where for each $i \in \{1,2,3,4,6\}$, $G'[U_i \cup V_i]$ is connected and has bipartition $\{U_i, V_i\}$. Observe that for $L = \{1,4,5\}, \{2,4,7\}, \{2,5,6\}, \{1,6,7\}$, $B_L - (V_5 \cup V_7)$ is a cardinality two cover, and therefore a minimum cover, of $\mathcal{C}'$. As a consequence, G' has an edge between V_1, V_4, an edge between V_4, V_2, an edge between V_2, V_6, and an edge between V_6, V_1. Let $U := U_1 \cup U_2 \cup V_4 \cup V_6$ and $U' := V_1 \cup V_2 \cup U_4 \cup U_6$. Then $G'[U \cup U']$ is connected and has bipartition $\{U, U'\}$. Since $G'[U_3 \cup V_3]$ is also connected, G' has at most two connected components. It therefore follows from Lemma 2.2 (i)-(ii) that either

$$U \cup U_3, U' \cup V_3 \in \mathcal{C}' \quad \text{or} \quad U \cup V_3, U' \cup U_3 \in \mathcal{C}'.$$

Observe that $B_L - (V_5 \cup V_7) = B_L$ is a cover of $\mathcal{C}'$ for $L = \{1,2,3\}, \{3,4,6\}$. However, $B_{\{1,2,3\}} \cap (U \cup U_3) = \emptyset$ and $B_{\{3,4,6\}} \cap (U' \cup U_3) = \emptyset$, a contradiction.

As a result, $V_3 \cup V_5 \cup V_7$ is a cover, as claimed. ♢

We may assume that $\mathcal{C}$ is contraction minimal clean clutter with respect to satisfying (2CovH) and (a)-(c). By Claims 2 and 3, for each $L \in \mathbb{L}_7$, there exists a minimal cover $B_L \in b(\mathcal{C})$ of cardinality 3 picking one element from each $V_i, i \in L$. For each $i \in [7]$, let

$$X_i := V_i \cap \left(\bigcup_{L \in \mathbb{L}_7} B_L \right);$$

notice that $1 \le |X_i| \le 3$.

Claim 4. *For each $i \in [7]$, either $|X_i| = 1$ and $|U_i| = |V_i| = 1$, or $2 \le |X_i| \le 3$ and $G[U_i \cup V_i]$ is a tree whose leaves are contained in X_i.*

Proof of Claim. Let W be a subset of V such that (1) $W \subseteq U_i \cup V_i$, (2) $X_i \subseteq W$, (3) $|W| \ge 2$, (4) $G[W]$ is connected, and (5) W is minimal subject to (1)-(4). Let $\{U_i', V_i'\}$ be the bipartition of $G[W]$ where $U_i' \subseteq U_i$ and $X_i \subseteq V_i' \subseteq V_i$. Notice that if $|X_i| = 1$ then $|U_i'| = |V_i'| = 1$, and if $2 \le |X_i| \le 3$ then $G[W]$ must be a tree whose leaves are contained in X_i by Remark 8.1. Let $I := (U_i \cup V_i) - (U_i' \cup V_i')$. Notice that $\mathcal{C}/I$ is clean and satisfies (2CovH), (a) and (b). Moreover, since $B_L \cap I = \emptyset$ for each $L \in \mathbb{L}_7$, $\mathcal{C}/I$ also satisfies (c). Our minimal choice of $\mathcal{C}$ implies that $I = \emptyset$, so $U_i' = U_i$ and $V_i' = V_i$, thereby finishing the proof of the claim. $\Diamond$

Claim 5. *For each $i \in [7]$, $|X_i| = 1$ and $|U_i| = |V_i| = 1$.*

Proof of Claim. Suppose otherwise. We may assume that $G[U_1 \cup V_1]$ is not an edge. It then follows from Claim 4 that $2 \le |X_1| \le 3$ and $G[U_1 \cup V_1]$ is a tree whose leaves are contained in X_1. Pick a leaf u of the tree $G[U_1 \cup V_1]$ that belongs to exactly one of $B_{\{1,2,3\}}, B_{\{1,4,5\}}, B_{\{1,6,7\}}$, and let $\mathcal{C}' := \mathcal{C}/u$. Since u is a leaf of $G[U_1 \cup V_1]$, $\mathcal{C}'$ is clean and satisfies (2CovH), (a) and (b). Moreover, as u belongs to exactly one of $(B_L : L \in \mathbb{L}_7)$, $\mathcal{C}'$ also satisfies (c), a contradiction to the minimality of $\mathcal{C}$. $\Diamond$

Let $\mathcal{C}' := \mathcal{C}/(U_1 \cup \cdots \cup U_7)$.

Claim 6. $\mathcal{C}' \cong \mathbb{L}_7$.

Proof of Claim. We know that $B_L \in b(\mathcal{C}')$ for each $L \in \mathbb{L}_7$, and that by Claim 1, these are the only minimal covers of $\mathcal{C}'$ of cardinality at most 3. After a possible relabeling of its elements, we may assume that $\mathcal{C}'$ has ground set [7], and that $B_L = L$ for each $L \in \mathbb{L}_7$. We claim that $b(\mathcal{C}') = \mathbb{L}_7$. Suppose otherwise. Then $b(\mathcal{C}')$ has a member B of cardinality at least 4. As $\mathbb{L}_7 \subseteq b(\mathcal{C}')$, it follows that $|B| = 4$ and $B = [7] - L$ for some $L \in \mathbb{L}_7$. However, B is also a minimal cover of $\mathcal{C}$ that is disjoint from $\bigcup_{i \notin L} U_i \cup \bigcup_{j \in L} V_j$, a contradiction to (b). As a result, $b(\mathcal{C}') = \mathbb{L}_7$, so $\mathcal{C}' = b(\mathbb{L}_7) = \mathbb{L}_7$, as claimed. $\Diamond$

As a result, $\mathcal{C}'$ has an $\mathbb{L}_7$ minor, thereby finishing the proof of Lemma 8.2. $\Box$

8.2 The proof of Theorem 1.20

Remark 8.3. $\mathrm{cuboid}(\mathrm{cocycle}(PG(2,2)))$ *is, after a possible relabeling, a clutter over ground set* $\{1, 2, \ldots, 7, \bar{1}, \bar{2}, \ldots, \bar{7}\}$ *satisfying the following statements:*

- *the members are* $\{\bar{i} : i \in [7]\}$ *and* $\{i : i \notin L\} \cup \{\bar{j} : j \in L\}$ *for all* $L \in \mathbb{L}_7$,

- *the cardinality* 3 *minimal covers are* $\{\bar{i},\bar{j},\bar{k}\},\{\bar{i},j,k\},\{i,\bar{j},k\},\{i,j,\bar{k}\}$ *for all* $\{i,j,k\}\in\mathbb{L}_7$,
- *every cardinality* 3 *minimal cover is contained in exactly two members.*

Let $S:=\mathrm{cocycle}(PG(2,2))$. As S is a binary space, it follows that $S\triangle p=S$ for every point $p\in S$. In particular, every member of $\mathrm{cuboid}(S)$ can be treated as the first member $\{\bar{i}:i\in[7]\}$ above.

Proposition 8.4. *Let* $\mathcal{C}$ *be a clean clutter over ground set* V *satisfying (2CovH) and (UniqH), and of rank* 7*. Then* $\mathcal{C}$ *has an* $\mathbb{L}_7$ *minor.*

Proof. Let $G:=G(\mathcal{C})$, and for each $i\in[7]$, let $\{U_i,V_i\}$ be the bipartition of the i^{th} connected component of G. Let y be the dyadic fractional packing of $\mathcal{C}$ of value two. As $\mathcal{C}$ has rank 7, it follows from Theorem 1.19 that $\mathrm{support}(y)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(2,2)))$. As $\mathrm{support}(y)\subseteq\mathrm{core}(\mathcal{C})$, it follows from Remarks 1.6 and 8.3 that, after possibly relabeling and swapping $U_i,V_i,i\in[7]$, the following sets are the members of $\mathrm{support}(y)$:

$$\bigcup_{j=1}^{7}V_j \quad\text{and}\quad \bigcup_{i\notin L}U_i\cup\bigcup_{j\in L}V_j \quad \forall L\in\mathbb{L}_7.$$

A subset $B\subseteq V$ is a *special cover of* $\mathcal{C}$ if it is a monochromatic minimal cover intersecting at most 3 connected components of G.

Claim 1. *If* B *is a special cover of* $\mathcal{C}$*, then*

- *there is a unique* $L\in\mathbb{L}_7$ *such that* $B\cap(U_i\cup V_i)\neq\emptyset$ *for each* $i\in L$,
- $\{i\in L:B\cap U_i\neq\emptyset\}$ *has even cardinality, and*
- B *is contained in exactly two members of* $\mathrm{support}(y)$.

Proof of Claim. This follows immediately from Remark 8.3. ◊

Given a special cover B, we refer to L from Claim 1 as the *Fano line corresponding to* B, and to $\{i\in L: B\cap U_i\neq\emptyset\}$ as the *trace of* B.

Claim 2. *For every Fano line* $L\in\mathbb{L}_7$*, there are* 3 *corresponding special covers with pairwise different traces.*

Proof of Claim. Suppose otherwise. We may assume by symmetry between the members of $\mathbb{L}_7$ that $L=\{1,2,3\}$. By Claim 1, every special cover corresponding to L has trace $\emptyset,\{1,2\},\{1,3\}$ or $\{2,3\}$. We may assume by symmetry between the members of $\mathrm{cuboid}(\mathrm{cocycle}(PG(2,2)))$ that every special cover corresponding to line L, if any, has trace $\{1,2\}$ or $\{1,3\}$. Let $\mathcal{C}':=\mathcal{C}\setminus V_1/U_1$ and $G':=G(\mathcal{C}')$. By Lemma 6.3, $\mathcal{C}'$ is clean and satisfies (2CovH) and (UniqH), and given that z is the dyadic fractional packing of $\mathcal{C}'$ of value two, $\mathrm{support}(z)=\mathrm{support}(y)\setminus V_1/U_1$. In particular, $\mathrm{support}(z)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(1,2)))$ by Remark 7.8. Theorem 1.19 applied to $\mathcal{C}'$ now tells us that

$$\mathrm{rank}(\mathcal{C}')=2^2-1=3.$$

Observe that for $i \in [7] - \{1\}$, $G[U_i \cup V_i] \subseteq G'[U_i \cup V_i]$, so $G'[U_i \cup V_i]$ is connected. Let us refer to the edges of G' not contained in any $G'[U_i \cup V_i], i \in [7] - \{1\}$ as *crossing* edges. We claim that

($\star$) for every crossing edge $\{u, v\}$, either $\{u, v\} \subseteq U_4 \cup U_5$, $\{u, v\} \subseteq V_4 \cup V_5$, $\{u, v\} \subseteq U_6 \cup U_7$ or $\{u, v\} \subseteq V_6 \cup V_7$.

To this end, pick distinct $i, j \in [7] - \{1\}$ such that $u \in U_i \cup V_i$ and $v \in U_j \cup V_j$. Then $V_1 \cup \{u, v\}$ contains a minimal cover of $\mathcal{C}$, which is inevitably special. It therefore follows from Claim 1 that $\{1, i, j\} \in \mathbb{L}_7$, and either $\{u, v\} \subseteq U_i \cup U_j$ or $\{u, v\} \subseteq V_i \cup V_j$. Since there is no special cover corresponding to line $\{1, 2, 3\}$ and trace either $\emptyset, \{2, 3\}$, it follows that $\{i, j\} = \{4, 5\}$ or $\{6, 7\}$, so ($\star$) holds. However, ($\star$) implies that G' has at least 4 connected components, so $\mathrm{rank}(\mathcal{C}') \geq 4$, a contradiction. ♢

Claim 3. *There is a member of* $\mathrm{support}(y)$ *that contains* 6 *special covers corresponding to different Fano lines.*

Proof of Claim. By Claim 2, there are $21 = 7 \times 3$ special covers $B_1, \ldots, B_{21}$ such that for distinct $i, j \in [21]$, if B_i and B_j correspond to the same Fano line, then they have different traces. By Claim 1, each $B_i, i \in [21]$ is contained in exactly two members of $\mathrm{support}(y)$. As a result, there is a member of $\mathrm{support}(y)$ containing at least $\frac{21 \times 2}{8} > 5$ special covers among $B_1, \ldots, B_{21}$, as required. ♢

We may assume that $\bigcup_{j=1}^{7} V_j$ contains 6 special covers corresponding to different Fano lines. As $\mathcal{C}$ satisfies conditions (a)-(c), we may apply Lemma 8.2 to conclude that $\mathcal{C}$ has an $\mathbb{L}_7$ minor, as required. □

We are now ready for the main result of this section:

Proof of Theorem 1.20. Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) and (UniqH) and of rank more than 3. Let y be the dyadic fractional packing of $\mathcal{C}$ of value two. It then follows from Theorem 1.19 that for some integer $k \geq 3$, $\mathcal{C}$ has rank $2^k - 1$, and $\mathrm{support}(y)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1, 2)))$. We prove by induction on $k \geq 3$ that $\mathcal{C}$ has an $\mathbb{L}_7$ minor. The base case $k = 3$ follows from Proposition 8.4. For the induction step, assume that $k \geq 4$. Let $\{U, U'\}$ be a connected component of $G(\mathcal{C})$, and let $\mathcal{C}' := \mathcal{C} \backslash U / U'$. By Lemma 6.3, $\mathcal{C}'$ is clean and satisfies (2CovH) and (UniqH), and if z is the dyadic fractional packing of $\mathcal{C}'$ of value two, then $\mathrm{support}(z) = \mathrm{support}(y) \backslash U / U'$. In particular, $\mathrm{support}(z)$ is a duplication of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-2, 2)))$ by Remark 7.8. Thus $\mathcal{C}'$ has rank $2^{k-1} - 1$ by Theorem 1.19, so by the induction hypothesis, $\mathcal{C}'$ and therefore $\mathcal{C}$ has an $\mathbb{L}_7$ minor, thereby completing the induction step. This finishes the proof of Theorem 1.20. □

9 Ideal minimally non-packing clutters of rank three

In this section we prove Theorem 1.29. Recall from Remark 1.27 that every ideal minimally non-packing clutter with covering number two is clean and satisfies (2CovH). In [8] the authors provided, up to isomorphism, 13 instances of such clutters of rank three. Let us analyze these examples and their blockers. Our analysis will lead to a guide for generating more examples of such clutters.

Twelve of these instances are denoted $Q_6 \otimes X$ where X is one of $\emptyset, \{6\}, \{5,6\}, \{4,6\}, \{4,5,6\}, \{3,4,5,6\}, \{2,4,6\}, \{2,4,5\}, \{2,4,5,6\}, \{2,3,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}$, where $Q_6 \otimes \emptyset = Q_6$, $Q_6 \otimes \{2,4,6\}$ is Schrijver's counterexample [21] to a conjecture of Edmonds and Giles on dijoins [12], and in general, $Q_6 \otimes X$ is a clutter over ground set $[6] \cup \{a' : a \in X\}$ obtained from Q_6 following a certain procedure. For each X, $b(Q_6 \otimes X)$ consists of some cardinality two members and exactly 4 cardinality three members; see Figure 3 for an illustration of $b(Q_6 \otimes X)$ where the cardinality two members are represented as edges of a graph, and the cardinality three members are specified below every graph.

The thirteenth instance was a "one-off" example not conforming to the construction and has the following incidence matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Label the columns of this matrix $1, 1', 2, 2', 3, 4, 5, 6$ from left to right. Then the blocker has some cardinality two members, represented as edges of the graph in Figure 4, and exactly 5 cardinality three members, specified below the graph.

Let us step back. All of these 13 examples satisfy the hypotheses of the following lemma:

Lemma 9.1. *Let $\mathcal{C}$ be a clutter satisfying (2CovH). Assume that $G(\mathcal{C})$ is bipartite and has exactly three connected components, where for $i \in [3]$, $\{U_i, V_i\}$ is the bipartition of the i^{th} connected component. Assume further that $V_1 \cup V_2 \cup V_3, V_1 \cup U_2 \cup U_3, U_1 \cup V_2 \cup U_3, U_1 \cup U_2 \cup V_3$ are covers. Then*

(i) $\mathcal{C}$ does not pack, and

(ii) if $\mathcal{C}$ is clean, then $\mathrm{setcore}(\mathcal{C}) \cong \{000, 110, 101, 011\}$.

Proof. **(i)** Suppose for a contradiction that $\mathcal{C}$ packs. Then $\mathcal{C}$ has disjoint members C_1, C_2. Observe that $C_i \cap (U_j \cup V_j) \in \{U_j, V_j\}$ for $i \in [2]$ and $j \in [3]$. We may assume that $C_1 \cap (U_1 \cup V_1) = U_1$ and so $C_2 \cap (U_1 \cup V_1) = V_1$. Assume in the first case that $C_1 \cap (U_2 \cup V_2) = U_2$ and so $C_2 \cap (U_2 \cup V_2) = V_2$. As $V_1 \cup V_2 \cup V_3$ is a cover, it follows that $C_1 \cap (U_3 \cup V_3) = V_3$ and so $C_2 \cap (U_3 \cup V_3) = U_3$. But then C_2 is disjoint from the cover $U_1 \cup U_2 \cup V_3$, a contradiction. Assume in the remaining case that $C_1 \cap (U_2 \cup V_2) = V_2$ and so $C_2 \cap (U_2 \cup V_2) = U_2$. As $V_1 \cup U_2 \cup U_3$ is a cover, it follows that $C_1 \cap (U_3 \cup V_3) = U_3$ and so $C_2 \cap (U_3 \cup V_3) = V_3$. But then C_2 is disjoint from the cover $U_1 \cup V_2 \cup U_3$, a contradiction.

(ii) As $\mathrm{rank}(\mathcal{C}) = 3$ we have that $\mathrm{setcore}(\mathcal{C}) \subseteq \{0,1\}^3$. As $\mathcal{C}$ does not pack by (i), $\mathrm{core}(\mathcal{C})$ does not have disjoint members, implying in turn that $\mathrm{setcore}(\mathcal{C})$ does not have antipodal points. As $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ contains $\frac{1}{2} \cdot \mathbf{1}$ by Theorem 1.7, it follows that $\mathrm{setcore}(\mathcal{C}) \cong \{000, 110, 101, 011\}$, as required. □

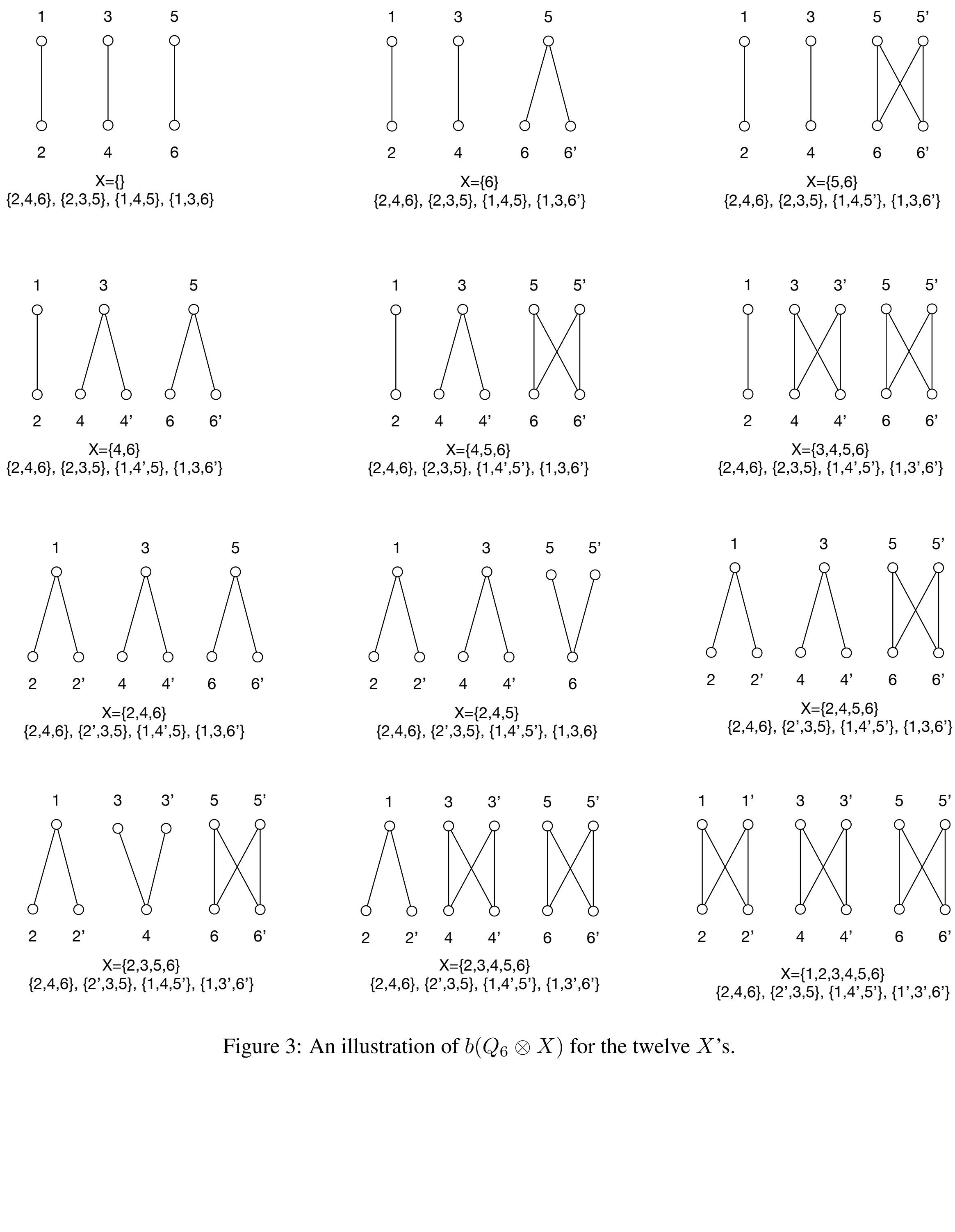

Figure 3: An illustration of $b(Q_6 \otimes X)$ for the twelve X's.

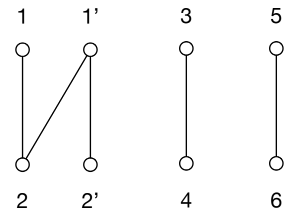

Figure 4: An illustration of the blocker of the 13th ideal minimally non-packing clutter of rank 3 from [8].

In particular, the 13 instances have rank three and setcore $\{000, 110, 101, 011\}$. They also satisfy the hypotheses of the following lemma:

Lemma 9.2. *Let $\mathcal{C}$ be a clean clutter satisfying (2CovH) and of rank three, let $G := G(\mathcal{C})$, and for $i \in [3]$ let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. Assume that*

$$C_1 := V_1 \cup V_2 \cup V_3 \quad C_2 := V_1 \cup U_2 \cup U_3 \quad C_3 := U_1 \cup V_2 \cup U_3 \quad C_4 := U_1 \cup U_2 \cup V_3$$

are members. Then,

(i) if C_i is a cover for some $i \in [4]$, then it contains a minimal cover of cardinality 3 picking one element from each connected component.

Assume now that $\{u_1, u_2, v_3'\}$ and $\{v_1, v_2, v_3\}$ are minimal covers for some $u_1 \in U_1, u_2 \in U_2, v_1 \in V_1, v_2 \in V_2$ and $v_3, v_3' \in V_3$. Then

(ii) there is a minimal cover B such that $B \cap (U_1 \cup V_1) = \{u_1, v_1\}$ and $|B| \leq 3$.

Proof. **(i)** It suffices to prove this for $i = 1$. Observe that $U_1 \cup V_2 \cup V_3, V_1 \cup U_2 \cup V_3, V_1 \cup V_2 \cup U_3$ are not covers as they are respectively disjoint from the members C_2, C_3, C_4. As a result, if C_1 is a cover, then it is an irreducible monochromatic cover, so (i) follows from Lemma 3.2 (ii). **(ii)** Suppose otherwise. Let $J := (U_1 \cup V_1) - \{u_1, v_1\}$ and $\mathcal{C}' := \mathcal{C} \setminus \{u_1, v_1\}/J$. Our contrary assumption implies that $\tau(\mathcal{C}') \geq 2$, implying in turn that $\mathcal{C}'$ is clean and satisfies (2CovH). Consider the bipartite graph $G' := G(\mathcal{C}')$. Notice that for $i \in \{2, 3\}$, $G[U_i \cup V_i] \subseteq G'[U_i \cup V_i]$, so $G'[U_i \cup V_i]$ is a connected, bipartite graph whose bipartition is inevitably $\{U_i, V_i\}$. Since $\{u_1, u_2, v_3'\}$ and $\{v_1, v_2, v_3\}$ are minimal covers of $\mathcal{C}$, $\{u_2, v_3'\}, \{v_2, v_3\}$ are covers of $\mathcal{C}'$, implying that G' has an edge between U_2, V_3 and an edge between V_2, V_3, so G' is not bipartite, a contradiction. □

In particular, for each of the 13 instances, Lemma 9.2 (i) justifies the existence of the four cardinality 3 minimal covers picking exactly one element from each of the 3 connected components of the graph, and Lemma 9.2 (ii) justifies the existence of the minimal covers of the form $\{i, j\}, \{i, j'\}, \{i', j\}, \{i', j'\}, i \neq j$, and for the 13th instance the existence of the minimal cover $\{1, 2', 6\}$.

We may use Lemma 9.2 as a guide for constructing more examples of ideal minimally non-packing clutters of covering number two, rank three and setcore $\{000, 110, 101, 011\}$. Let us provide one such infinite family.

9.1 The proof of Theorem 1.29

Recall that a clutter has the packing property if every minor of it, including the clutter itself, packs. Recall further that given an arbitrary clutter $\mathcal{C}$ over ground set V, we denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimal covers of cardinality two. Notice that $G(\mathcal{C})$ may have no edges.

Lemma 9.3. *Let $\mathcal{C}$ be a clutter over ground set V, and let $G := G(\mathcal{C})$. Assume that*

(i) G is a bipartite graph with bipartition $\{U, U'\}$,

(ii) $|\{B \in b(\mathcal{C}) : |B| > 2\}| \leq 1$, and

(iii) if $B \in b(\mathcal{C})$ satisfies $|B| > 2$, then $B = \{u, v, w\}$ where

- *$u \in U$ and $\{v, w\} \subseteq U'$, and*
- *in G, either v, w belong to different connected components, or some neighbor of u is a cut-vertex separating v, w.*

Then $\mathcal{C}$ has the packing property.

Notice that in Lemma 9.3, the graph G may have isolated vertices.

Proof. Let us proceed by induction on $|V| \geq 2$. The base case $|V| = 2$ holds trivially. For the induction step, assume that $|V| \geq 3$. We may assume that $\tau(\mathcal{C}) \geq 2$.

Claim 1. *$\mathcal{C}$ packs.*

Proof of Claim. If $\tau(\mathcal{C}) \geq 3$, then every minimal cover of $\mathcal{C}$ has cardinality at least 3, implying in turn that $b(\mathcal{C}) = \{\{u, v, w\}\}$, so $\mathcal{C} = \{\{u\}, \{v\}, \{w\}\}$, which clearly packs. Otherwise, $\tau(\mathcal{C}) = 2$. Notice that both U, U' are covers of $b(\mathcal{C})$, so they contain members of $\mathcal{C}$, implying in turn that $\mathcal{C}$ has two disjoint members, so $\mathcal{C}$ packs. $\diamondsuit$

Claim 2. *For each $x \in V$, $\mathcal{C}/x$ has the packing property.*

Proof of Claim. Notice that $\mathcal{C}/x$ satisfies (i)-(iii), so the claim follows from the induction hypothesis. $\diamondsuit$

Claim 3. *For each $x \in V$ and $N := \{y \in V : \{x, y\} \in b(\mathcal{C})\}$, $\mathcal{C} \setminus x/N$ has the packing property.*

Proof of Claim. Let $\mathcal{C}' := \mathcal{C} \setminus x/N$. If $N \neq \emptyset$, then $\mathcal{C}'$ has the packing property by Claim 2. We may therefore assume that $N = \emptyset$. By the induction hypothesis, it suffices to prove that $\mathcal{C}'$ satisfies (i)-(iii). If $x \notin \{u, v, w\}$, then $\mathcal{C}'$ clearly satisfies (i)-(iii). If $x \in \{v, w\}$, then as $u \in U$ and $\{v, w\} \subseteq U'$, it follows that $\mathcal{C}'$ satisfies (i)-(iii). Otherwise $x = u$. As $N = \emptyset$, we see that v, w belong to different connected components of G, so $\mathcal{C}'$ satisfies (i)-(iii), as required. $\diamondsuit$

These three claims imply that $\mathcal{C}$ has the packing property, thereby completing the induction step. $\square$

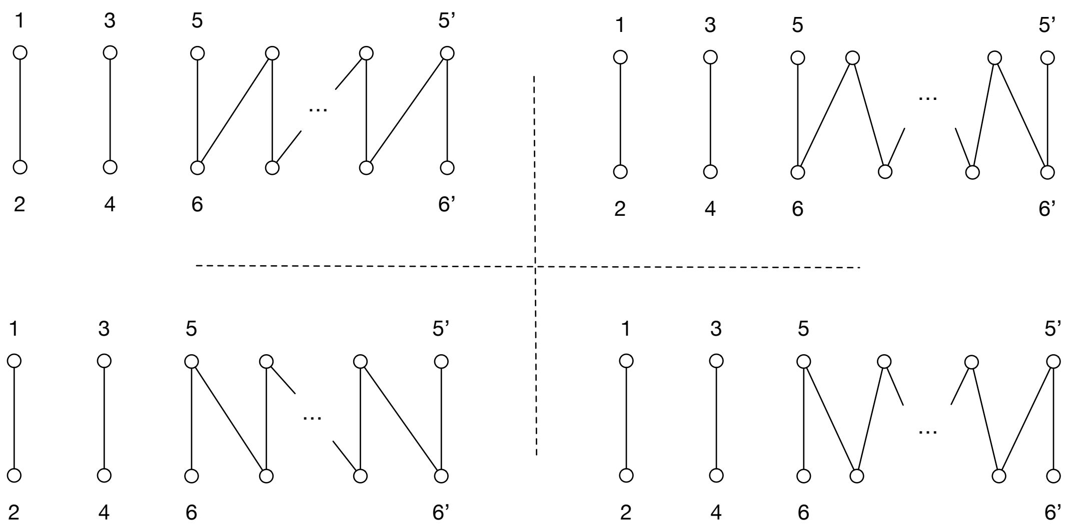

Figure 5: The four possibilities for G from Proposition 9.4.

We are now ready to provide an infinite class of ideal minimally non-packing clutters of covering number two and of rank three:

Proposition 9.4. *Let $\mathcal{C}$ be a clutter over ground set V, and let $G := G(\mathcal{C})$. Assume that*

- *G is bipartite and has exactly 3 connected components,*
- *the first connected component of G has two vertices $1, 2$ and an edge between them,*
- *the second connected component of G has two vertices $3, 4$ and an edge between them,*
- *the third connected component of G is a path on at least four edges, where the first edge is $\{5, 6\}$, the last edge is $\{5', 6'\}$, $5, 5'$ belong to the same part of the bipartition, and $6, 6'$ belong to the other part of the bipartition, and*
- *the minimal covers of $\mathcal{C}$ of cardinality different from two are precisely*

$$\{2,4,6\},\{2,3,5\},\{1,4,5'\},\{1,3,6'\} \quad \textit{and} \quad \{3,5,6'\},\{4,5',6\}.$$

Then $\mathcal{C}$ is an ideal minimally non-packing clutter.

See Figure 5 for an illustration of the graph G from Proposition 9.4.

Proof. Observe that $\mathcal{C}$ satisfies (2CovH). Let $\{U, U'\}$ be the bipartition of the third connected component of G where $\{5, 5'\} \subseteq U$ and $\{6, 6'\} \subseteq U'$.

Claim 1. *$\mathcal{C}$ does not pack.*

Proof of Claim. As $\{2,4,6\},\{2,3,5\},\{1,4,5'\},\{1,3,6'\}$ are minimal covers, $\{2,4\}\cup U'$, $\{2,3\}\cup U$, $\{1,4\}\cup U$ and $\{1,3\}\cup U'$ are covers, so $\mathcal{C}$ does not pack by Lemma 9.1 (i). ♢

In what follows, notice that in our setup, there is symmetry between $1,2$, between $3,4$, between $5,6$, between $5',6'$, and between $\{5,6\},\{5',6'\}$.

Claim 2. *Every proper contraction minor of $\mathcal{C}$ packs.*

Proof of Claim. Pick $I\subseteq V$ such that $\tau(\mathcal{C}/I)\geq 2$. Let $\mathcal{C}':=\mathcal{C}/I$ and notice that $b(\mathcal{C}')=b(\mathcal{C})\setminus I$. As a result, $\tau(\mathcal{C}')\in\{2,3\}$. If $\tau(\mathcal{C}')=3$, then it can be readily checked that $b(\mathcal{C}')$ has at most two members, each of cardinality 3, implying in turn that $\mathcal{C}'$ packs. Otherwise, $\tau(\mathcal{C}')=2$, in which case we need to look for disjoint members in $\mathcal{C}'$. As disjoint members remain disjoint in contraction minors, we may assume that $I=\{x\}$. In what follows, we find disjoint covers in $b(\mathcal{C}')$. By symmetry, we may assume that $x\notin\{2,4,6,5',6'\}$.

If $x=1$, then $\{2,3\}\cup U',\{4\}\cup U$ are disjoint covers of $b(\mathcal{C}')$. If $x=3$, then $\{1,4\}\cup U',\{2\}\cup U$ are disjoint covers of $b(\mathcal{C}')$. If $x=5$, then $\{2,3\}\cup(U-\{5\}),\{1,4\}\cup U'$ are disjoint covers of $b(\mathcal{C}')$. Otherwise, $x\in V-\{1,2,3,4,5,6,5',6'\}=(U\cup U')-\{5,6,5',6'\}$. Notice that deleting x from G disconnects the third connected component, that is, $G\setminus x$ has four connected components with bipartitions $\{\{1\},\{2\}\},\{\{3\},\{4\}\},\{U_1,U_1'\},\{U_2,U_2'\}$, where $U_1\cup U_2=U-\{x\}$, $U_1'\cup U_2'=U'-\{x\}$, $5\in U_1, 6\in U_1', 5'\in U_2$ and $6'\in U_2'$. Observe now that $\{1,3\}\cup(U_1'\cup U_2),\{2,4\}\cup(U_1\cup U_2')$ are disjoint covers of $b(\mathcal{C}')$.

In each case, we proved that $b(\mathcal{C}')$ has disjoint covers, giving disjoint members of $\mathcal{C}'$ in turn, as desired. ♢

Claim 3. *Let I be a nonempty subset of V that is disjoint from $\{1,2,3,4,5,6,5',6'\}$ and is not a cover of $\mathcal{C}$, and let*

$$N:=\big\{y\in V-I:\{x,y\}\in b(\mathcal{C})\text{ for some } x\in I\big\}.$$

Then $\mathcal{C}\setminus I/N$ packs.

Proof of Claim. Let $\mathcal{C}':=\mathcal{C}\setminus I/N$, and note that $b(\mathcal{C}')=b(\mathcal{C})/I\setminus N$. As $I\cap\{1,2,3,4,5,6,5',6'\}=\emptyset$, it follows that $b(\mathcal{C})/I\setminus N=b(\mathcal{C})\setminus(I\cup N)$, implying that $\mathcal{C}'=\mathcal{C}/(I\cup N)$, so $\mathcal{C}'$ packs by Claim 2. ♢

Claim 4. *Let $x\in\{1,2,3,4,5,6,5',6'\}$ and let $N:=\{y\in V:\{x,y\}\in b(\mathcal{C})\}$. Then $\mathcal{C}\setminus x/N$ has the packing property.*

Proof of Claim. By symmetry, we may assume that $x\notin\{2,4,6,5',6'\}$. To prove the claim, it suffices to show that $\mathcal{C}\setminus 1/2,\mathcal{C}\setminus 3/4$ and $\mathcal{C}\setminus 5/6$ have the packing property.

Every minimal cover of $\mathcal{C}\setminus 1/2$ has cardinality two, and the graph over vertex set $V-\{1,2\}$ of the minimal covers of the minor is bipartite with bipartition $\{3\}\cup U,\{4\}\cup U'$, so $\mathcal{C}\setminus 1/2$ has the packing property by Lemma 9.3.

Every minimal cover of $\mathcal{C}\setminus 3/4$ also has cardinality two, and the graph over vertex set $V-\{3,4\}$ of the minimal covers of the minor is bipartite with bipartition $\{1\}\cup U,\{2\}\cup U'$, so once again $\mathcal{C}\setminus 3/4$ has the packing property by Lemma 9.3.

Finally, let $\mathcal{C}' := \mathcal{C} \setminus 5/6$, and let G' be the graph over vertex set $V - \{5, 6\}$ whose edges correspond to the cardinality two minimal covers of $\mathcal{C}'$. Notice that G' is a bipartite graph with bipartition $\{1, 3\} \cup (U - \{5\}), \{2, 4\} \cup (U - \{6\})$. Moreover, there is only one minimal cover with cardinality greater than two, namely $\{1, 4, 5'\}$. Furthermore, the neighbor 3 of 4 in G' is a cut-vertex separating $1, 5'$. Thus $\mathcal{C}'$ has the packing property by Lemma 9.3. $\Diamond$

These four claims imply that $\mathcal{C}$ is a minimally non-packing clutter, so by Corollary 1.26, $\mathcal{C}$ is either ideal or a delta or the blocker of an extended odd hole. However, as G is a bipartite graph with at least two connected components, $\mathcal{C}$ must be an ideal clutter, thereby finishing the proof of Proposition 9.4. $\square$

We are now ready to prove Theorem 1.29:

Proof of Theorem 1.29. Pick one of the infinitely many clutters $\mathcal{C}$ over ground set V from Proposition 9.4. Notice that $|V| \geq 9$. We proved that $\mathcal{C}$ is an ideal minimally non-packing clutter of covering number two. As $G(\mathcal{C})$ has exactly three connected components, it follows that $\mathrm{rank}(\mathcal{C}) = 3$. As $\{2, 4, 6\}, \{2, 3, 5\}, \{1, 4, 5'\}, \{1, 3, 6'\}$ are covers, it follows from Lemma 9.1 (ii) that $\mathrm{setcore}(\mathcal{C}) \cong \{000, 110, 101, 011\}$. It can be readily checked that $\mathcal{C}$ has a member of cardinality greater than $\frac{|V|}{2}$, thereby showing that $\mathcal{C}$ is not a cuboid. This finishes the proof of Theorem 1.29. $\square$

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Notions at a glance

.

Delta of dimension n: a clutter over ground set $[n]$ whose members are $\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}$. This clutter is denoted Δ_n. Its blocker is itself.

Extended odd hole of dimension n: a clutter over ground set $[n]$ whose minimum cardinality members are $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$. Every cover has cardinality at least $\frac{n+1}{2}$.

$\mathbb{L}_7$: a clutter whose ground set consists of the points of the Fano plane, and whose members are the lines of the Fano plane. Its blocker is itself.

$PG(k-1,2)$: the rank-k projective geometry over $GF(2)$. It is the binary matroid represented by the k by 2^k-1 matrix whose columns consist of all k-dimensional vectors that are nonzero and have $0-1$ entries.

.

Ideal clutter: a clutter whose set covering polyhedron is integral. If a clutter is ideal, then so is the blocker and every minor of it.

Binary clutter: a clutter where the symmetric difference of any odd number of members contains a member. If a clutter is binary, then is the blocker and every minor of it.

Pack: a clutter packs if the packing number is equal to the covering number.

The packing property: a clutter has the packing property if every minor of it packs.

Minimally non-packing: a clutter is minimally non-packing if it does not pack but every proper minor does.

.

Clean clutter: a clutter without delta or the blocker of an extended odd hole minor. If a clutter is clean, then so is every minor of it.

(2CovH) the covering number is two and every element is in a minimum cover

(UniqH) there is a unique dyadic fractional packing of value two

.

$G(\mathcal{C})$: A graph defined for clutters $\mathcal{C}$ over ground set V. The vertex set is V, and the edges correspond to the minimal covers of $\mathcal{C}$ of cardinality two.

.

The following are defined for clean clutters $\mathcal{C}$ satisfying (2CovH):

Monochromatic cover: a cover of $\mathcal{C}$ that is monochromatic in some proper 2-vertex-coloring of $G(\mathcal{C})$.

The core: the clutter of the members of $\mathcal{C}$ that appear in the support of a fractional packing of $\mathcal{C}$ of value two.

The setcore: the core is the duplication of the cuboid of a set without duplicated coordinates. This set is the setcore of $\mathcal{C}$

.

$\mathrm{rank}(\mathcal{C})$: the number of connected components of $G(\mathcal{C})$.

$\mathrm{girth}(\mathcal{C})$: the minimum cardinality of a monochromatic cover of $\mathcal{C}$.

$\mathrm{depth}(\mathcal{C})$: the maximum dimension of an infeasible hypercube of $\mathrm{setcore}(\mathcal{C})$.

.

Weak duality: $\mathrm{rank}(\mathcal{C}) - \mathrm{girth}(\mathcal{C}) \leq \mathrm{depth}(\mathcal{C})$

Strong duality: $\mathrm{rank}(\mathcal{C}) - \mathrm{girth}(\mathcal{C}) = \mathrm{depth}(\mathcal{C})$