On Some Polytopes Contained in the 0,1 Hypercube that Have a Small Chvátal Rank *

Gérard Cornuéjols and Dabeen Lee

Tepper School of Business, Carnegie Mellon University
gc0v@andrew.cmu.edu dabeenl@andrew.cmu.edu

Abstract. In this paper, we consider polytopes P that are contained in the unit hypercube. We provide conditions on the set of infeasible 0,1 vectors that guarantee that P has a small Chvátal rank. Our conditions are in terms of the subgraph induced by these infeasible 0,1 vertices in the skeleton graph of the unit hypercube. In particular, we show that when this subgraph contains no 4-cycle, the Chvátal rank is at most 3; and when it has tree width 2, the Chvátal rank is at most 4. We also give polyhedral decomposition theorems when this graph has a vertex cutset of size one or two.

1 Introduction

Let $H_n := [0,1]^n$ denote the 0,1 hypercube in $\mathbb{R}^n$. Let $P \subseteq H_n$ be a polytope. Let $S := P \cap \{0,1\}^n$ denote the set of 0,1 vectors in P. If an inequality $cx \geq d$ is valid for P for some $c \in \mathbb{Z}^n$, then $cx \geq \lceil d \rceil$ is valid for conv(S) since it holds for any $x \in P \cap \mathbb{Z}^n$. Chvátal [4] introduced an elegant notion of closure as follows.

$$P' = \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \geq \lceil \max\{cx : \ x \in P\} \rceil \}$$

is the *Chvátal closure* of P. Chvátal [4] proved that the closure of a rational polyhedron is, again, a rational polyhedron. Recently, Dadush et al. [7] showed that the Chvátal closure of any convex compact set is a rational polytope. Let $P^{(0)}$ denote P and $P^{(t)}$ denote $(P^{(t-1)})'$ for $t \geq 1$. Then $P^{(t)}$ is the tth Chvátal closure of P, and the smallest k such that $P^{(k)} = \text{conv}(S)$ is called the *Chvátal rank* of P. Chvátal [4] proved that the Chvátal rank of every rational polytope is finite, and Schrijver [11] later proved that the Chvátal rank of every rational polyhedron is also finite.

Eisenbrand and Schulz [8] proved that the Chvátal rank of any polytope $P \subseteq H_n$ is $O(n^2 \log n)$. Rothvoss and Sanitá [10] constructed a polytope $P \subseteq H_n$ whose Chvátal rank is $\Omega(n^2)$. However, some special polytopes arising in

* This work was supported in part by NSF grant CMMI1263239 and ONR grant N00014-12-10032.

combinatorial optimization problems have small Chvátal rank; for example, the fractional matching polytope has Chvátal rank 1. Hartmann, Queyranne and Wang [9] gave a necessary and sufficient condition for a facet-defining inequality of conv(S) to have rank 1. In this paper, we investigate 0,1 polytopes with a Chvátal rank that is a small constant or grows slowly with n.

The *skeleton* of H_n is the graph $G := (V, E)$ whose vertices correspond to the 2^n extreme points of H_n and whose edges correspond to the 1-dimensional faces of H_n, namely the $n2^{n-1}$ line segments joining two extreme points of H_n that differ in exactly one coordinate. Let $\bar{S} := \{0,1\}^n \setminus S$ denote the set of 0,1 vectors that are not in P. Consider the subgraph $G(\bar{S})$ of G induced by the vertices in $\bar{S}$. In this paper, we give conditions on $G(\bar{S})$ that guarantee a small Chvátal rank. For example, we show that when $\bar{S}$ is a stable set in G, the Chvátal rank of P is at most 1; when each connected component of $G(\bar{S})$ is a cycle of length greater than 4 or a path, the Chvátal rank is at most 2; when $G(\bar{S})$ contains no 4-cycle, the Chvátal rank is at most 3; in particular when $G(\bar{S})$ is a forest, the Chvátal rank is at most 3; when the tree width of $G(\bar{S})$ is 2, the Chvátal rank is at most 4. In Section 4, we give polyhedral decomposition theorems for conv(S) when $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2. These decomposition theorems are used to prove the results on forests and on graphs of tree width two mentioned above. In Section 5, we give an upper bound on the Chvátal rank of P that depends on the cardinality of $\bar{S}$. In particular, we show that if only a constant number of 0,1 vectors are infeasible, then the Chvátal rank of P is also a constant. We also give a superpolynomial range on the number of infeasible 0,1 vectors where the upper bound of $O(n^2 \log n)$ on the Chvátal rank obtained by Eisenbrand and Schulz can be slightly improved to $O(n^2 \log\log n)$. Finally, in Section 6, we show that optimizing a linear function over S is polynomially solvable when the Chvátal rank of a canonical polytope for S is constant.

Although our results are mostly of theoretical interest, we mention two applications. The first is to the theory of clutters with the packing property. Abdi, Cornuéjols and Pashkovich [1] constructed a class of minimal nonpacking clutters from 0,1 polytopes with Chvátal rank at most 2. In particular, a 0,1 polytope in $[0,1]^5$ where the infeasible 0,1 vectors induce 2 cycles of length 8 and the remaining 16 points are feasible lead to the discovery of a new minimally non-packing clutter on 10 elements. Another application occurs when S is the set of 0,1 vectors whose sum of entries is congruent to i modulo k. The cases $k = 2$ and $k = 3$ are discussed in Sections 2.1 and 3.

2 Some polytopes with small Chvátal rank

To prove results on a polytope $P \subset [0,1]^n$, we will work with a canonical polytope Q_S that has exactly the same set S of feasible $0,1$ vectors. The description of Q_S is as follows.

$$Q_S := \{x \in [0,1]^n : \sum_{j=1}^n (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1/2 \text{ for } \bar{x} \in \bar{S}\}.$$

The reason for working with Q_S is that the Chvátal rank of P is always less than or equal to the Chvátal rank of Q_S. Furthermore, we have a good handle on the kth Chvátal closure $Q_S^{(k)}$ because of the following lemma.

Lemma 1 (CCH [5]). *The middle points of all $k+1$ dimensional faces of H_n belong to $Q_S^{(k)}$ for $0 \leq k \leq n-1$.*

Chvátal, Cook and Hartmann proved this result when $S = \emptyset$. The result clearly follows for general $S \subseteq \{0,1\}^n$ since $Q_\emptyset \subseteq Q_S$ implies $Q_\emptyset^{(k)} \subseteq Q_S^{(k)}$. We also make repeated use of the two following results in our proofs.

Lemma 2. *Consider a half-space $D := \{x \in \mathbb{R}^n : \ dx \geq d_0\}$. Let $T := D \cap \{0,1\}^n$ and $\bar{T} := \{0,1\}^n \setminus T$. For every face F of H_n, the graph $G(F \cap \bar{T})$ is connected. In particular $G(\bar{T})$ is a connected graph.*

Theorem 1 (AADK [2]). *Let P be a polytope and let $G = (V,E)$ be its skeleton. Let $S \subset V$, $\bar{S} = V \setminus S$, and $\bar{S}_1, \ldots, \bar{S}_m$ be a partition of $\bar{S}$ such that there are no edges of G connecting $\bar{S}_i, \bar{S}_j$ for all $1 \leq i < j \leq m$. Then $conv(S) = \bigcap_{i=1}^m conv(V \setminus \bar{S}_i)$.*

Theorem 1, due to Angulo, Ahmed, Dey and Kaibel [2], shows that we can consider each connected component of $G(\bar{S})$ separately when studying $\text{conv}(S)$. In Section 4, we give similar theorems in the case where $P \subset [0,1]^n$ and $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2. In this section, we provide the descriptions for $Q_S^{(1)}$, $Q_S^{(2)}$, $Q_S^{(3)}$.

2.1 Chvátal rank 1

Theorem 2. *The polytope P has Chvátal rank at most 1 when $\bar{S}$ is a stable set in G.*

In particular, if S contains all the 0,1 vertices of H_n with an even (odd resp.) number of 1s, then P has Chvátal rank at most 1. Theorem 2 is obtained by characterizing $Q_S^{(1)}$. For each $\bar{x} \in \bar{S}$, we call

$$\sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{1}$$

the *vertex inequality* corresponding to $\bar{x}$. For example, when $\bar{x} = 0$, the corresponding vertex inequality is $x_1 + x_2 + \ldots + x_n \geq 1$. Note that each vertex inequality cuts off exactly the vertex $\bar{x}$ and it goes through all the neighbors of $\bar{x}$ on H_n.

Theorem 3. *$Q_S^{(1)}$ is the intersection of $[0,1]^n$ with the half-spaces defined by the vertex inequalities (1) for $\bar{x} \in \bar{S}$.*

2.2 Chvátal rank 2

Theorem 4. *For $n \geq 3$, the Chvátal rank of Q_S is 2 if and only if $G(\bar{S})$ contains a connected component of cardinality at least 2, and each connected component of $G(\bar{S})$ is either a cycle of length greater than 4 or a path.*

To prove this theorem, we provide an explicit characterization of $Q_S^{(2)}$.

Let $N := \{1, \ldots, n\}$. Throughout the paper, we will use the following notation. For a 0,1 vector $\bar{x}$, we denote by $\bar{x}^i$ the 0,1 vector that differs from $\bar{x}$ only in coordinate $i \in N$, and more generally, for $J \subseteq N$, we denote by $\bar{x}^J$ the 0,1 vector that differs from $\bar{x}$ exactly in the coordinates J. We denote by e^i the ith unit vector for $i \in N$.

Let $\bar{x}, \bar{y} \in \bar{S}$ be two vertices of $G(\bar{S})$ such that they differ in exactly one coordinate, say $\bar{y} = \bar{x}^i$. The inequality

$$\sum_{j \in N \setminus \{i\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \tag{2}$$

is called the *edge inequality* corresponding to edge $\bar{x}\bar{y}$. For example, when $\bar{x} = 0$ and $\bar{y} = e^1$, the corresponding edge inequality is $x_2 + x_3 + \ldots + x_n \geq 1$. The inequality (2) is the strongest inequality that cuts off $\bar{x}$ and $\bar{y}$ but no other vertex of H_n. Indeed, its boundary contains all $2(n-1)$ neighbors of $\bar{x}$ or $\bar{y}$ on H_n (other than $\bar{x}$ and $\bar{y}$ themselves). The next theorem states that vertex and edge inequalities are sufficient to describe the second Chvátal closure of Q_S.

Theorem 5. *$Q_S^{(2)}$ is the intersection of $Q_S^{(1)}$ with the half-spaces defined by the edge inequalities (2) for $\bar{x}, \bar{y} \in \bar{S}$ such that $\bar{x}\bar{y}$ is an edge of H_n.*

Note that the edge inequality (2) dominates the vertex inequalities for $\bar{x} \in \bar{S}$ and for $\bar{y} \in \bar{S}$. Thus vertex inequalities are only needed for the isolated vertices of $G(\bar{S})$.

2.3 Chvátal rank 3

Theorem 6 below is the main result of this section. It characterizes $Q_S^{(3)}$.

4-cycles of $G(\bar{S})$ correspond to 2-dimensional faces of H_n that are squares. Using our notation, if $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell} \in \bar{S}$, we say that $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$ is a square. Note that

$$\sum_{j \in N \setminus \{i, \ell\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \tag{3}$$

is the strongest inequality cutting off exactly the four points of the square $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$. Indeed, the $4(n-2)$ neighbors of $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}$ in H_n (other than $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}$ themselves) all satisfy (3) at equality. We call (3) a *square inequality*. As an example, if $(0, e^1, e^2, e^1+e^2)$ is a square contained in $G(\bar{S})$, the corresponding square inequality is $x_3 + x_4 + \ldots + x_n \geq 1$.

If $\bar{x}$ and $t \geq 3$ of its neighbors $\bar{x}^{i_1} := \bar{x} + (1 - 2\bar{x}_{i_1})e^{i_1}, \ldots, \bar{x}^{i_t} := \bar{x} + (1 - 2\bar{x}_{i_t})e^{i_t}$ all belong to $\bar{S}$, then we say that $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ is a *star*. The following *star inequality* is valid for conv(S)

$$\sum_{r=1}^{t}(\bar{x}_{i_t}(1 - x_{i_t}) + (1 - \bar{x}_{i_t})x_{i_t}) + 2 \sum_{j \neq i_1, \ldots, i_t} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 2. \quad (4)$$

It cuts off the vertices of the star, and goes through the other $n - t$ neighbors of $\bar{x}$ on H_n and the $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$. For example, if $(0, e^1, \ldots, e^t)$ is a star, then (4) is $x_1 + \ldots + x_t + 2(x_{t+1} + \ldots + x_n) \geq 2$.

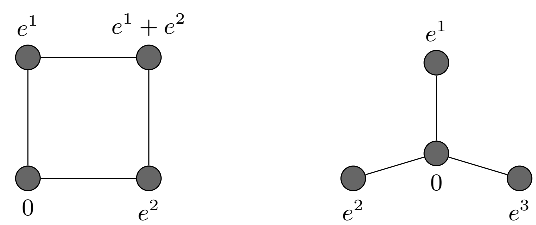

Fig. 1. Square and star with $\bar{x} = 0$

Theorem 6. *$Q_S^{(3)}$ is the intersection of $Q_S^{(2)}$ with the half-spaces defined by the square inequalities* (3) *and the star inequalities* (4).

To illustrate our proof techniques, we will prove Theorem 6 in this extended abstract. The proof uses the following lemma, which gives the linear description of conv(S) when $\bar{S}$ is a star.

Lemma 3. *Let $n \geq 3$. If $\bar{S}$ is a star, then conv(S) is completely defined by the corresponding star inequality together with the edge inequalities and the bounds $0 \leq x \leq 1$.*

Proof. We may assume that $\bar{x} = 0$, $\bar{S} = \{0, e^1, \ldots, e^t\}$ and $I := \{1, \ldots, t\}$.

If $t = n$, then S is the set of 0,1 vectors satisfying the system $\sum_{j=1}^{n} x_j \geq 2$ with $0 \leq x \leq 1$. This constraint matrix is totally unimodular. Therefore it defines an integral polytope, which must be conv(S).

If $t = 2$, we observe similarly that $\{x \in [0,1]^n : \sum_{j \in N \setminus \{r\}}(\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \text{ for } r = 1, 2\}$ is an integral polytope. Indeed, the corresponding constraint matrix is also totally unimodular.

If $3 \leq t < n$, it is sufficient to show that $A := \{x \in [0,1]^n : \sum_{i \in I} x_i + 2\sum_{j \in N \setminus I} x_j \geq 2, \sum_{j \in N \setminus \{r\}} x_j \geq 1 \text{ for } 1 \leq r \leq t\}$ is an integral polytope. Let v be an extreme point of A. We will show that v is an integral vector. Since we assumed $n \geq 3$, A has dimension n and there exist n linearly independent inequalities active at v.

First, consider the case when the star inequality is active at v. If no edge inequality is active at v, then $n-1$ inequalities among $0 \leq x \leq 1$ are active at v. Since $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j = 2$, it follows that all coordinates of v are integral. Thus we may assume that an edge inequality $\sum_{j \in N \setminus \{1\}} x_j \geq 1$ is active at v. Consider the face F of A defined by setting this edge inequality and the star inequality as equalities. Clearly v is a vertex of F. Observe that the two equations defining F can be written equivalently as $\sum_{j \in N \setminus \{1\}} x_j = 1$ and $x_1 + \sum_{j \in N \setminus I} x_j = 1$. Furthermore, any other edge inequality $\sum_{j \in N \setminus \{r\}} x_j \geq 1$ is implied by $x \geq 0$ since it can be rewritten as $\sum_{j \in I \setminus \{1,r\}} x_j \geq 0$ using $x_1 + \sum_{j \in N \setminus I} x_j = 1$. This means that F is entirely defined by $0 \leq x \leq 1$ and the two equations $x_1 + \sum_{j \in N \setminus I} x_j = 1$ and $\sum_{j \in N \setminus \{1\}} x_j = 1$. This constraint matrix is totally unimodular, showing that v is an integral vertex.

Assume now that the star inequality is not active at v, namely $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j > 2$. If at most one edge inequality is tight at v, then v is obviously integral. Thus, we may assume that $k \geq 2$ edge inequalities are tight at v, say $\sum_{j \in N \setminus \{r\}} x_j \geq 1$ for $1 \leq r \leq k$. Then $v_1 = \ldots = v_k$. If v_1 is fractional, v has at least k fractional coordinates. We assumed that only k inequalities other than $0 \leq x \leq 1$ are active at v, so the other coordinates are integral. Hence, $v_j = 0$ for $j \notin \{1, \ldots, k\}$ and $v_1 = \ldots = v_k = \frac{1}{k-1}$. Then $\sum_{r=1}^{t} v_r + 2\sum_{j \in N \setminus I} v_j = \frac{k}{k-1} \leq 2$. However, this contradicts the assumption that $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j > 2$. □

Proof of Theorem 6:

Applying the Chvátal procedure to inequalities defining $Q_S^{(2)}$, it is straightforward to show the validity of the inequalities (3) and (4) for $Q_S^{(3)}$.

To complete the proof of the theorem, we need to show that all other valid inequalities for $Q_S^{(3)}$ are implied by those defining $Q_S^{(2)}$, (3) and (4).

Consider a valid inequality for $Q_S^{(3)}$ and let $\bar{T}$ denote the set of 0,1 vectors cut off by this inequality. If $\bar{T} = \emptyset$, then the inequality is implied by $0 \leq x \leq 1$. Thus, we assume that $\bar{T} \neq \emptyset$. Let $T := \{0,1\}^n \setminus \bar{T}$. By the definition of a Chvátal inequality, there exists an inequality $ax \geq b$ valid for $Q_S^{(2)}$ that cuts off exactly the vertices in $\bar{T}$. By Lemma 1, the center points of the cubes of H_n all belong to $Q_S^{(2)}$. This means $ax \geq b$ does not cut off any of them. By Lemma 2, $G(\bar{T})$ is a connected graph. We claim that the distance between any 2 vertices in $G(\bar{T})$ is at most 2. Indeed, otherwise $G(\bar{T})$ contains two opposite vertices of a cube, and therefore its center satisfies $ax < b$, a contradiction.

We consider 3 cases: $|\bar{T}| \leq 3$, $G(\bar{T})$ contains a square, and $G(\bar{T})$ contains no square.

If $|\bar{T}| \leq 3$, then $G(\bar{T})$ is either an isolated vertex, an edge, or a path of length two. Then vertex and edge inequalities with the bounds $0 \leq x \leq 1$ are sufficient to describe $\text{conv}(T)$ by Theorem 4.

If $G(\bar{T})$ contains a square $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$, it cannot cut off any other vertex of H_n (otherwise, by Lemma 2 there would be another vertex of $\bar{T}$ adjacent to the square, and thus in a cube, and cut off by the inequality, a contradiction). Thus, $\bar{T} = \{\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}\}$. Since

$$\text{conv}(T) = \{x \in [0,1]^n : \sum_{j \in N \setminus \{i,\ell\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1\},$$

a Chvátal inequality derived from $ax \geq b$ will therefore be implied by the square inequality that corresponds to $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$ and the bounds $0 \leq x \leq 1$.

Assume that $G(\bar{T})$ contains no square and $|\bar{T}| \geq 4$. Note that a cycle of H_n that is not a square has length at least six. Since the distance between any two vertices in $G(\bar{T})$ is at most two, $G(\bar{T})$ contains no cycle of H_n. Thus, $G(\bar{T})$ is a tree. In fact, $G(\bar{T})$ is a star since the distance between any two of its vertices is at most two. Thus $\bar{T} = \{\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t}\}$ for some $t \geq 3$. By Lemma 3, conv(T) is described by edge and star inequalities with the bounds $0 \leq x \leq 1$. Any Chvátal inequality that one can obtain from $ax \geq b$ is therefore implied by the edge inequalities corresponding to the edges $(\bar{x}, \bar{x}^{i_1}), \ldots, (\bar{x}, \bar{x}^{i_t})$ and the star inequality that corresponds to the star $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$. □

Note that, if an edge $\bar{x}\bar{y}$ of $G(\bar{S})$ belongs to a square of $G(\bar{S})$, the corresponding inequality is not needed in the description of $Q_S^{(3)}$ since it is dominated by the square inequality. On the other hand, if an edge belongs to a star $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ of $G(\bar{S})$ with $t < n$, there is no domination relationship between the corresponding edge inequality and star inequality by Lemma 3.

3 Chvátal rank 4

In this section, we give the characterization of $Q_S^{(4)}$. It is somewhat more involved than the results for $Q_S^{(1)}$, $Q_S^{(2)}$ and $Q_S^{(3)}$, but it is in the same spirit.

Consider any cube with vertices in $G(\bar{S})$. Specifically, for $\bar{x} \in \{0,1\}^n$, recall that we use the notation $\bar{x}^i$ to denote the 0,1 vertex that differs from $\bar{x}$ only in coordinate i, and more generally, for $J \subseteq N$, let $\bar{x}^J$ denote the 0,1 vector that differs from $\bar{x}$ exactly in the coordinates J. If the 8 points $\bar{x}, \bar{x}^i, \bar{x}^k, \bar{x}^\ell, \bar{x}^{ik}, \bar{x}^{i\ell}, \bar{x}^{k\ell}, \bar{x}^{ik\ell}$ all belong to $\bar{S}$, then we say that these points form a *cube*. Note that

$$\sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \tag{5}$$

is a valid inequality for conv(S) and that it cuts off exactly 8 vertices of H_n, namely the 8 corners of the cube. In fact, it is the strongest such inequality since it is satisfied at equality by all $8(n-3)$ of their neighbors in H_n. We call (5) a *cube inequality.*

If $\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \bar{x}^{i_3}, \bar{x}^{i_1 i_2}, \bar{x}^{i_2 i_3}, \bar{x}^{i_3 i_1}, \bar{x}^{i_4}, \ldots, \bar{x}^{i_t}$ all belong to $\bar{S}$ for some $t \geq 4$, then we say that these points form a *tulip.* Let $I_T := \{i_1, \ldots, i_t\}$. Note that

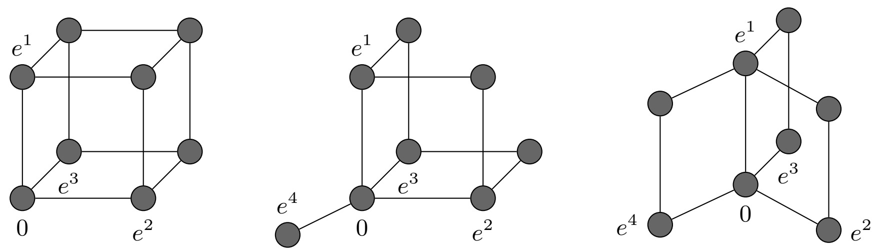

Fig. 2. Cube, tulip, and propeller with $\bar{x} = 0$

$$\sum_{k=1}^{3}(\bar{x}_{i_k}(1 - x_{i_k}) + (1 - \bar{x}_{i_k})x_{i_k}) + 2\sum_{r=4}^{t}(\bar{x}_{i_r}(1 - x_{i_r}) + (1 - \bar{x}_{i_r})x_{i_r}) \\ +3\sum_{j \notin I_T}(\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 3 \qquad (6)$$

is a valid inequality for conv(S) that cuts off exactly these points. We call it a *tulip inequality*. For example, if $\bar{x} = 0$, and $\bar{x}^{i_k} = e^k$ for $k = 1, 2, 3$, (6) is $x_1 + x_2 + x_3 + 2(x_4 + \ldots + x_t) + 3(x_{t+1} + \ldots + x_n) \geq 3$.

If $\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \ldots, \bar{x}^{i_t}, \bar{x}^{i_{t+1}}, \bar{x}^{i_1 i_{t+1}}, \bar{x}^{i_2 i_{t+1}}, \ldots, \bar{x}^{i_t i_{t+1}}$ all belong to $\bar{S}$ for some $t \geq 3$, then we say that these points form a *propeller*. Let $I_P := \{i_1, \ldots, i_{t+1}\}$. Note that

$$\sum_{r=1}^{t}(\bar{x}_{i_r}(1 - x_{i_r}) + (1 - \bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P}(\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 2 \qquad (7)$$

is a valid inequality that cuts off exactly the above points. We call it a *propeller inequality*. For example, if $\bar{x} = 0$, $\bar{x}^{i_{t+1}} = e^1$ and $\bar{x}^{i_k} = e^{k+1}$ for $k = 1, \ldots, t$, the propeller inequality is $x_2 + \ldots + x_{t+1} + 2(x_{t+2} + \ldots + x_n) \geq 2$.

Theorem 7. *$Q_S^{(4)}$ is the intersection of $Q_S^{(3)}$ and the half spaces defined by all cube, tulip, and propeller inequalities.*

Corollary 1. *Let $P \subseteq [0, 1]^n$ be a polytope, $S = P \cap \{0, 1\}^n$ and $\bar{S} = \{0, 1\}^n \setminus S$. If $G(\bar{S})$ contains no 4-cycle, then P has Chvátal rank at most 3.*

The set of vertices $\bar{T}$ cut off by a linear inequality induces a connected graph by Lemma 2. One can show that if $G(\bar{T})$ contains vertices at distance greater than 2, then it contains a 4-cycle. Therefore, if $G(\bar{T})$ contains no 4-cycle, it is a star in the bipartite graph $G(H_n)$ with one vertex on one side and at most n on the other.

Remark 1. Let $P \subseteq [0,1]^n$ be given by a system of k inequalities. If $G(\bar{S})$ contains no 4-cycle, then $|\bar{S}| \leq k(n+1)$. It follows that optimizing a linear function over S can be solved in polynomial time in this case.

Corollary 2. *Let $n \geq 3$ and $i = 0, 1$ or 2. For $S \supseteq \{x \in \{0,1\}^n : \sum_{j=1}^n x_j = i \ (mod\ 3)\}$, the set conv$(S)$ is entirely described by vertex, edge, star inequalities and bounds $0 \leq x \leq 1$.*

We note that, for $n \geq 5$, $i = 0, 1, 2, 3$ and $S \supseteq \{x \in \{0,1\}^n : \sum_{j=1}^n x_j = i \ (mod\ 4)\}$, conv$(S)$ might contain an inequality with Chvátal rank 5 in its linear description.

4 Vertex cutsets

Corollary 1 implies that if $G(\bar{S})$ induces a forest, the Chvátal rank of P is at most 3. This can also be proved directly using a vertex cutset decomposition theorem in the spirit of Theorem 1. We present it below in Section 4.1.

Trees can be generalized using the notion of tree width. A connected graph has tree width one if and only if it is a tree. Next, we focus our attention on the case when $G(\bar{S})$ has tree width two. Instead of working directly with the definition of tree width, we will use the following characterization: A graph has tree width at most two if and only if it contains no K_4-minor; furthermore a graph with no K_4-minor and at least four vertices always has a vertex cut of size two.

The main result of this section is that P has Chvátal rank at most 4 when $G(\bar{S})$ has tree width 2.

Theorem 8. *Let $P \subseteq [0,1]^n$, $S = P \cap \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. If $G(\bar{S})$ has tree width 2, the Chvátal rank of P is at most 4.*

The proof follows from a 2-vertex cutset decomposition theorem, which we state below in Section 4.2.

4.1 1-vertex cutset

The next theorem shows that conv(S) can be decomposed when $G(\bar{S})$ contains a vertex cut. This result is in the spirit of the theorem of Angulo, Ahmed, Dey and Kaibel (Theorem 1) but it is specific to polytopes contained in the unit hypercube.

Let $G = (V, E)$ be a graph and let $X \subseteq V$. For $v \in X$, let $N_X[v]$ denote the closed neighborhood of v in the graph $G(X)$. That is $N_X[v] := \{v\} \cup \{u \in X : uv \in E\}$.

Theorem 9. *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let v be a cut vertex in $G(\bar{S})$ and let $\bar{S}_1, \ldots, \bar{S}_\ell$ denote the connected components of $G(\bar{S} \setminus \{v\})$. Then conv$(S) = \bigcap_{i=1}^{\ell}$ conv$(\{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i))$.*

Furthermore, if v does not belong to any 4-cycle in $G(\bar{S})$, then conv(S) = conv$(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \bigcap_{i=1}^{\ell}$ conv$(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$.

Theorem 9 cannot be extended to general polytopes, as shown in the following example.

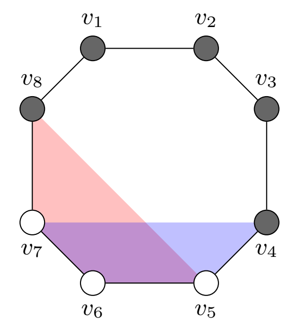

Fig. 3. An example in $\mathbb{R}^2$

Example 1. Let P be the polytope in $\mathbb{R}^2$ shown in Figure 3. Let $V := \{v_1, \ldots, v_8\}$ denote its vertex set and let $G = (V, E)$ be its skeleton graph. Let $S := \{v_5, v_6, v_7\}$ and $\bar{S} := V \setminus S$. In the figure the set of white vertices is S, while the set of black vertices is $\bar{S}$. Note that v_2 is a cut vertex of $G(\bar{S})$, and $N_{\bar{S}}[v_2] = \{v_1, v_2, v_3\}$. Therefore, $\bar{S}_1 := \{v_1, v_8\}$ and $\bar{S}_2 := \{v_3, v_4\}$ induce two distinct connected components of $G(\bar{S} \setminus \{v_2\})$.

Note that conv(S) is a triangle, but the intersection of conv$(V \setminus \{v_1, v_2, v_3, v_4\})$ and conv$(V \setminus \{v_1, v_2, v_3, v_8\})$ is a parallelogram. Therefore, we get that

$$\text{conv}(S) \neq \text{conv}(V \setminus (N_{\bar{S}}[v_2] \cup \bar{S}_1)) \cap \text{conv}(V \setminus (N_{\bar{S}}[v_2] \cup \bar{S}_2)).$$

□

4.2 2-vertex cut

A key step in proving Theorem 8 is the next theorem.

Theorem 10. *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let $\{v_1, v_2\}$ be a vertex cut of size two in $G(\bar{S})$. Let $\bar{S}_1, \ldots, \bar{S}_k$ denote the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. Then conv$(S) = \bigcap_{i=1}^k$ conv$(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] \cup \bar{S}_i))$.*

It is natural to ask whether this theorem can be extended to vertex cuts of larger sizes. The 3-vertex cut case is open, but it turns out that Theorem 10 cannot be generalized to 4-vertex cutsets as shown by the following example.

Example 2. Consider $\bar{S} = ((\{0,1\}^4 \times \{0\}) \setminus \{e^1 + e^2 + e^3 + e^4\}) \cup \{e^5\}$. Then $x_1 + x_2 + x_3 + x_4 + 3x_5 \geq 4$ is a facet-defining inequality for conv(S). Note that it cuts off all points in $\bar{S}$. In addition, $\bar{C} := \{e^1, e^2, e^3, e^4\}$ is a vertex cut of cardinality four in $\bar{S}$. Then $\bar{S}_1 := \{0, e^5\}$ and $\bar{S}_2 := \{e^1+e^2+e^3, e^1+e^2+e^4, e^1+e^3+e^4, e^2+e^3+e^4, e^1+e^2, e^1+e^3, e^1+e^4, e^2+e^3, e^2+e^4, e^3+e^4\}$ induce two connected components of $G(\bar{S} \setminus \bar{C})$. However,

$$\text{conv}(S) \neq \bigcap_{i=1}^{2} \text{conv}(\{0,1\}^5 \setminus (N_{\bar{S}}[e^1] \cup \ldots \cup N_{\bar{S}}[e^4] \cup \bar{S}_i))$$

since $x_1 + x_2 + x_3 + x_4 + 3x_5 \geq 4$ is not valid for $\text{conv}(\{0,1\}^5 \setminus (N_{\bar{S}}[e^1] \cup \ldots \cup N_{\bar{S}}[e^4] \cup \bar{S}_i))$ for $i = 1, 2$. □

4.3 Implication for the Chvátal rank

Theorems 9 and 10 imply bounds on the Chvátal rank of P when $G(\bar{S})$ has a vertex cutset of size one or two.

Corollary 3. *Let $P = \cap_{i=1}^{t} P_i$, where $P_i \subseteq [0,1]^n$ are polytopes. Let $V_i = P_i \cap \{0,1\}^n$, $S = P \cap \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$.*

(i) Let v be a cut vertex in $G(\bar{S})$, let $\bar{S}_1, \ldots, \bar{S}_t$ induce the connected components of $G(\bar{S} \setminus \{v\})$. Assume $V_i = \{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i)$. Then the Chvátal rank of P is no greater than the maximum Chvátal rank of P_i, $i = 1, \ldots, t$.

(ii) Let $\{v_1, v_2\}$ be a vertex cut of size two in $G(\bar{S})$. Let $\bar{S}_1, \ldots, \bar{S}_t$ induce the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. Assume $V_i = \{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] \cup \bar{S}_i)$. Then the Chvátal rank of P is no greater than the maximum Chvátal rank of P_i, $i = 1, \ldots, t$.

5 Dependency on the cardinality of the infeasible set

One can derive an upper bound on the Chvátal rank as a function of $|\bar{S}|$ using the result of Eisenbrand and Schulz [8] showing that the Chvátal rank of a 0,1 polytope is at most $n^2(1 + \log_2 n)$.

Theorem 11. *If $|\bar{S}| = k$ for some $k \leq n$, then the Chvátal rank of P is at most $k^2(1 + \log_2 k)$.*

This theorem implies that if the number of infeasible 0,1 vectors is a constant, then P is of constant Chvátal rank.

The next theorem shows that the Chvátal rank of P can be guaranteed to be smaller than the upper bound of $O(n^2 \log n)$ when the cardinality of $\bar{S}$ is bounded above by a subexponential but superpolynomial function of n. The proof uses a result of Eisenbrand and Schulz [8] stating that, if $cx \geq c_0$ is a valid inequality for $\text{conv}(S)$, where the c_js are relatively prime integers, then the Chvátal rank of P is at most $n^2 + 2n \log_2 \|c\|_\infty$.

Theorem 12. *If $|\bar{S}| < n^{f_k(n)}$ where $f_k(n) \leq (\log_2 n)^k$ for some positive constant k, then the Chvátal rank of P is $O(n^2 \log \log n)$.*

6 Optimization problem under small Chvátal rank

Let $P \subseteq [0,1]^n$ and $S = P \cap \{0,1\}^n$. Even when the Chvátal rank of P is just 1, it is still an open question whether optimizing a linear function over S is polynomially solvable or not [6]. In this section, we prove a weaker result.

Theorem 13. *Let $P \subseteq [0,1]^n$ and $S = P \cap \{0,1\}^n$. If the Chvátal rank of Q_S is constant, then there is a polynomial algorithm to optimize a linear function over S.*

Proof. The optimization problem is of the form $\min\{cx : x \in S\}$ where $c \in \mathbb{R}^n$. By complementing variables, we may assume $c \geq 0$. By hypothesis, $\text{conv}(S) = Q_S^{(k)}$ for some constant k. We claim that an optimal solution can be found among the 0,1 vectors with at most $k+1$ nonzero components. This will prove the theorem since there are only polynomially many such vectors. Indeed, if an optimal solution $\bar{x}$ has more than $k+1$ nonzero components, any 0,1 vector $\bar{z} \leq \bar{x}$ with exactly $k+1$ nonzero components satisfies $c\bar{z} \leq c\bar{x}$. Because $\text{conv}(S) = Q_S^{(k)}$ Lemma 1 implies that the face of H_n of dimension $k+1$ that contains 0 and $\bar{z}$ contains a feasible point $\bar{y} \in S$. Since $c\bar{y} \leq c\bar{z} \leq c\bar{x}$, the solution $\bar{y}$ is an optimal solution. □

References

1. A. Abdi, G. Cornuéjols, and K. Pashkovich, Delta minors in clutters, work in progress.
2. G. Angulo, S. Ahmed, S. S. Dey, and V. Kaibel, Forbidden vertices, *Mathematics of Operations Research 40* (2015) 350-360.
3. A. Bockmayr, F. Eisenbrand, M. Hartmann, and A.S. Schulz, On the Chvátal rank of polytopes in the 0/1 cube, *Discrete Applied Mathematics 98* (1999) 21-27.
4. V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, *Discrete Mathematics 4* (1973) 305-337.
5. V. Chvátal, W. Cook, and M. Hartmann, On cutting-plane proofs in combinatorial optimization, *Linear Algebra and its Applications 114/115* (1989) 455-499.
6. G. Cornuéjols and Y. Li, Deciding emptiness of the Gomory-Chvátal closure is NP-complete, even for a rational polyhedron containing no integer point, IPCO, Q. Louveaux and M. Skutella (Eds.), *LNCS* (2016).
7. D. Dadush, S. S. Dey, and J. P. Vielma, On the Chvátal-Gomory closure of a compact convex set, *Mathematical Programming Ser. A 145* (2014) 327-348.
8. F. Eisenbrand and A. S. Schulz, Bounds on the Chvátal rank of polytopes in the 0/1 cube, *Combinatorica 23* (2003) 245-261.
9. M. E. Hartmann, M. Queyranne, and Y. Wang, On the Chvátal rank of certain inequalities, IPCO, G. Cornuéjols, R. E. Burkard, and G. J. Woeginger (Eds.), *LNCS 1610* (1999) 218-233.
10. T. Rothvoss and L. Sanitá, 0/1 polytopes with quadratic Chvatal rank, IPCO, M. Goemans and J. Correa (Eds.), *LNCS 7801* (2013) 349-361.
11. A. Schrijver, On cutting planes, *Annals of Discrete Mathematics 9* (1980) 291-296.