LIST OF OPEN PROBLEMS
MATRIX-IBS WORKSHOP
STRUCTURAL GRAPH THEORY DOWNUNDER III

MAINTAINED BY LINDA COOK AND DAVID WOOD

Contents

Date: Last updated on April 26, 2023.

Problems Related to Graph Product Structure Theory

1. Polynomial Growth (David Wood)

The *growth* of a graph G is the function $f_G\colon \mathbb{N} \to \mathbb{N}$ where $f_G(r)$ is the maximum number of vertices in a subgraph of G with radius at most r. Campbell, Distel, Gollin, Harvey, Hendrey, Hickingbotham, Mohar, Wood conjectured the following rough characterisation of graphs with polynomial growth.

Conjecture 1.1. *There exist functions $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for any $c \geqslant 1$ and $d \in \mathbb{N}$ every graph G with growth $f_G(r) \leqslant cr^d$ is isomorphic to a subgraph of $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$, where each T_i is a tree of growth $f_{T_i}(r) \leqslant h(c,d)r$.*

The conjecture is even open for $d = 1$ (linear growth). In this case, Campbell et al. proved that G is isomorphic to a subgraph of $T \boxtimes K_{c'}$, where T is a tree of maximum degree less than $6c$. For general d, it is even open whether every graph G with growth $f_G(r) \leqslant cr^d$ is isomorphic to a subgraph of $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$, where each T_i is a tree (regardless of its growth). This conjecture is closely related to the work of Krauthgamer and Lee, who proved that the graph G in the conjecture is a subgraph of $P_1 \boxtimes \cdots \boxtimes P_k$, where each P_i is a path, and $k \in O(d \log d)$.

2. $O(\sqrt{n})$ Blow-Ups in Minor-Closed Classes (David Wood)

In what follows, $\mathcal{G}$ is an arbitrary minor-closed graph class excluding at least one graph. Alon, Seymour & Thomas proved that n-vertex graphs in $\mathcal{G}$ have treewidth $O(\sqrt{n})$. The following definition, implicitly introduced by Illingworth, Scott & Wood, naturally arises. Let $f(\mathcal{G})$ be the minimum integer k such that for some c, every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$, for some graph H with treewidth at most k, where $m \leqslant c\sqrt{n}$. Here $H \boxtimes K_m$ is the graph obtained from H by replacing each vertex by a copy of K_m, and replacing each edge by the complete join between the corresponding copies of K_m. Illingworth, Scott & Wood showed that $f(\mathcal{G})$ is well-defined; in particular, if $\mathcal{G}_t$ is the class of K_t-minor-free graphs, then $f(\mathcal{G}_t) \leqslant t-2$, and if $\mathcal{G}_{s,t}$ is the class of $K_{s,t}$-minor-free graphs, then $f(\mathcal{G}_{s,t}) \leqslant s$. Improving on these results, Distel, Dujmović, Eppstein, Hickingbotham, Joret, Morin, Seweryn & Wood showed that $f(\mathcal{G}) \leqslant 4$ for every minor-closed class $\mathcal{G}$. Results of Wood imply that for a minor-closed class $\mathcal{G}$, $f(\mathcal{G}) \leqslant 1$ if and only if $\mathcal{G}$ has bounded treewidth, where the lower bound follows from the Grid Minor Theorem together with a lower bound for planar graphs by Linial, Matoušek, Sheffet & Tardos. Recently, Dujmović, Eppstein, Hickingbotham, Joret, Micek, Seweryn & Wood showed that if $\mathcal{P}$ is the class of planar graphs, then $f(\mathcal{P}) = 2$. More generally, for $t \geqslant 3$, they showed that if $\mathcal{G}_t$ is the class of $K_{3,t}$-minor-free graphs, then $f(\mathcal{G}_t) = 2$. It is an intriguing open problem to determine $f(\mathcal{G})$. It is possible that $f(\mathcal{G}) \leqslant 2$ for every minor-closed class $\mathcal{G}$. This is open even when $\mathcal{G}$ is the class of K_5-minor-free graphs [Illingworth et al.]. Let $\mathcal{A}$ be the class of apex graphs[1], which is minor-closed. It is open whether $f(\mathcal{A}) \leqslant 2$. This is equivalent to the following open problem (which would strengthen the above result of Dujmović, Eppstein, Hickingbotham, Joret, Micek, Seweryn & Wood: for every n-vertex planar graph G, does there exist an apex-forest[2] H such that G is contained in $H \boxtimes K_m$ where $m \in O(\sqrt{n})$?

3. $O(n^{1-\epsilon})$ Blow-Ups in Classes with Sublinear Separators (David Wood)

Problem 3.1 (Wood)**.** *Let $\mathcal{G}$ be an hereditary graph class such that every n-vertex graph in $\mathcal{G}$ has a balanced separator of of order $O(n^{1-\epsilon})$. Does there exist a constant $c = c(\mathcal{G})$ such that every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$, where $\operatorname{tw}(H) \leqslant c$ and $m \in O(n^{1-\epsilon})$.*

Illingworth, Scott & Wood solved this question for minor-closed classes with $\epsilon = \frac{1}{2}$ (see Problem 2). It may even be true that c is a function of ϵ only, which has been proved for minor-closed classes [Distel et al.]. Wood proved it with $O(n^{1-\epsilon})$ replaced by $O(n^{1-\epsilon+\delta})$ for any fixed $\delta > 0$, where $c = c(\epsilon, \delta)$; and also with $\operatorname{tw}(H) \leqslant c$ replaced by $\operatorname{tw}(H) \leqslant O(\log \log n)$.

Here are some particular graph classes, where this problem is unsolved and interesting:

- touching graphs of 3-D spheres, which have $O(n^{2/3})$ balanced separators [MTTV '97]?
- *k-crossing-degenerate* graphs (i.e., have a drawing in the plane such that the associated crossing graph is k-degenerate), which have $O(k^{3/4}n^{1/2})$ separators [EG '17]
- string graphs on m edges, which have $O(m^{1/2})$ balanced separators [Lee '16]?

4. Product Structure of Apex-Minor-Free Graphs (David Wood)

A graph X is *apex* if $X - v$ is planar for some vertex v of X. Dujmović, Joret, Micek, Morin, Ueckerdt & Wood proved that for any apex graph X there is an integer c such that every X-minor-free graph is contained in $H \boxtimes P$, where $\operatorname{tw}(H) \leqslant c$ and P is a path.

The proof uses a version of the Graph Minor Structure Theorem by Dvorak & Norin. Is there a direct proof of this result that does not use the Graph Minor Structure Theorem? Note this result implies the Grid Minor Theorem, so we either re-prove the Grid-Minor Theorem or use it somehow.

[1]A graph H is *apex* if $H - v$ is planar for some vertex v of H.

[2]A graph H is an *apex forest* if $H - v$ is a forest for some vertex v of H.

Now consider the following version of the above result.

Theorem 4.1 (Dujmović et al.). *There are functions f and g such that for every apex graph X, every X-minor-free graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_{g(X)}$, where $\operatorname{tw}(H) \leqslant f(X)$ and P is a path.*

The goal now is to minimise f, not caring so much about g. Illingworth, Scott & Wood proved this with $f(X) = \tau(X)$, the vertex-cover number of X. Can we prove this with $f(X) \approx \operatorname{td}(X)$, the treedepth of X (which I expect is tight by the standard examples).

5. Planar products (Freddie Illingworth)

The seminal result in graph product structure theory is the planar graph product structure theorem. Throughout P is a path.

Theorem 5.1 (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood). *Every planar graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_3$ for some planar graph H of treewidth* 3.

A similar result for graphs of genus g has also been proved [3].

Theorem 5.2 (Distel, Hickingbotham, Huynh, and Wood). *Every graph G of genus g is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_{\max\{3,2g\}}$ for some planar graph H of treewidth* 3.

These are very similar results with the only difference being the order of the clique. Perhaps there is a direct reason for this.

Question 5.3. *Is every graph G of genus g isomorphic to a subgraph of $H \boxtimes K_{f(g)}$ where H is a planar graph?*

One could ask a similar question for k-planar graphs and for $K_{3,t}$-minor-free graphs.

NOTE (David Wood) The answer is 'no' for K_5-minor-free graphs. For every integer c there is a treewidth 3 graph (and thus K_5-minor-free graph) that is not a subgraph of $H \boxtimes K_c$ where H is planar.

6. Product Structure of Intersecting Balls (David Wood)

The intersection graph of a collection of balls in $\mathbb{R}^d$, such that each point is contained in a bounded number of balls, is a classical geometric example generalising planar graphs (via Koebe circle packing). Such graphs have $O(n^{1-1/d})$-separators [Miller, Teng, Thurston, Vavasis]. I conjecture the following product structure theorem for this class.

[3] A graph has genus at most g if it can be embedded without crossing on a surface of genus g.

Conjecture 6.1. *The intersection graph of a collection of balls in $\mathbb{R}^d$, such that each point of $\mathbb{R}^d$ is contained in at most c balls, is obtained by clique-sums of graphs of the form $(H \boxtimes P_1 \boxtimes \cdots \boxtimes P_{d-1}) + K_a$ for some graph H of treewidth at most some function $f(d,c)$ and integer $a = a(d,c)$, where each P_i is a path.*

The case where the balls are touching is also of interest. In the $d = 2$ case, this is the Planar Graph Product Structure Theorem (with $a = 0$). Maybe $a = d - 2$ in general?

7. Nearest Neighbour Graphs (David Wood)

Nearest neighbour graphs are of interest in computational geometry. For a finite point set X in $\mathbb{R}^d$, the *k-nearest-neighbour graph* of X has vertex set X, where two vertices v, w are adjacent if v is among the k nearest neighbours of w in X, or vice versa. Dujmović, Morin, Wood conjectured the following product structure, and proved it in the $d = 2$ case.

Conjecture 7.1 (Dujmović, Morin, Wood)**.** *Every k-nearest neighbour graph in $\mathbb{R}^d$ is a subgraph of $H \boxtimes P_1 \boxtimes \cdots \boxtimes P_{d-1}$ for some graph H of treewidth at most some function $f(k,d)$, where each P_i is a path.*

χ-boundedness

8. Gyárfás-Sumner Conjecture for directed graphs (Linda Cook, James Davies)

Question 8.1. *Is there a function f such that every digraph D not containing a digon or a directed P_5 has dichromatic number at most $f(\omega(D))$? Here $\omega(D)$ is the size of a maximum clique in the graph underlying D.*

An oriented graph is a digraph that does not contain a directed cycle of length two. An (oriented) graph D is *H-free* if D does not contain H as an induced sub(di)graph. Aboulker, Charbit, and Naserasr [Extension of Gyárfás-Sumner Conjecture to Digraphs; E-JC 2021] proposed an analogue of the Gyárfás-Sumner Conjecture conjecture to the dichromatic number of oriented graphs: for every oriented forest F, there is some function f such that every F-free oriented graph D has dichromatic number at most $f(\omega(D))$.

Note the converse is true: Harutyunyan and Mohar proved that there exist directed graphs of arbitrarily large undirected girth and dichromatic number (Two results on the digraph chromatic number, Discrete Mathematics, 2012) The conjecture is widely open. Chudnovsky, Scott and Seymour showed that all oriented stars and P_4 with orientation $\leftarrow\leftarrow\rightarrow$ or $\rightarrow\rightarrow\leftarrow$ all satisfy the conjecture, by proving a stronger result Induced subgraphs of graphs with large chromatic number. XI. Orientations

Tomáš Masařík, Marcin Pilipczuk, Amadeus Reinald and Uéverton S. Souza and I extended this to say that it all holds for the other orientations of P_4 at a Sparse Graphs Coallition workshop last year. `https://arxiv.org/pdf/2209.06171.pdf` It is open for any orientation of a tree that is not a star or a path of length at most three. Recently, Aboulker, Aubian, Charbit, and Thomassé showed that every $\overrightarrow{P_6}$-free oriented graph D with $\omega(D) \leqslant 2$ has dichromatic number at most 382. (P6, triangle)-free digraphs have bounded dichromatic number (Here $\overrightarrow{P_6}$ is the path on 5 edges with orientation $\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$.)

Problem 8.2. *Can we show the conjecture for another tree? maybe $\overrightarrow{P_5}$? Easier question: Can we show that for some oriented tree T and integer k that every T-free oriented graph D with $\omega(D) \leqslant k$ has bounded dichromatic number.*

Problem 8.3. *Is it true that for every oriented tree $\overrightarrow{T}$, and $t \geqslant 2$, the class of $K_{t,t}$-free graphs with a $\overrightarrow{T}$-free orientation has bounded chromatic number?*

8.1. **Update (Alex Scott).** April 17. Alex Scott provided a proof of Problem 8.3 in the affirmative.

9. Square and diamond-free graphs and χ-boundedness (James Davies)

Not all χ-bounded classes are polynomially χ-bounded, but perhaps there are general conditions that would imply certain χ-bounded classes are polynomially χ-bounded. As a starting point, I conjecture that this is the case for hereditary classes containing no induced square or diamond subgraph.

Conjecture 9.1. *Every hereditary χ-bounded class of graphs containing no induced C_4 or $K_4 - e$ is polynomially χ-bounded.*

10. χ-certificates (António Girão)

Recently, Briański, Davies, and Walczak (developing the ideas of Carbonero, Hompe, Moore, and Spirkl) showed that for every prime p there is $c(p)$ such that for every k there is a graph G which is K_{p+1}-free, with chromatic number is at least k and with the property that every induced subgraph $G' \subset G$ with $\chi(G') \geqslant c(p)$ must contain a K_p. This has been extended for all graphs H by Girao, Illingworth, Powierski, Savery, Scott, Tamitegama and, Tan. Namely, we showed that for every H, there is $c(H)$ such that for every k, there is a graph G with chromatic number at least k, with the same clique number as H and with the property that for every $G' \subset G$ with $\chi(G') \geqslant c(H)$ must contain an induced copy of H.

All these constructions rely on the use of congruence classes and hence they have very large complete bipartite graphs (depending on k). The problem I suggest is to find for every graph H (perhaps cliques, first) a graph G as above for which the bi-clique number is bounded by a function of H.

11. Linearly χ-boundedness for vertex-minors (Sang-il Oum)

Can we identify classes of graphs closed under taking vertex-minors that are linearly χ-bounded? Or, can we determine graphs H such that the class of H-vertex-minor-free graphs is linearly χ-bounded?

Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz showed that every class of graphs of bounded linear rank-width is linearly χ-bounded. This implies that if H is a path, then the class of H-vertex-minor-free graphs is linearly χ-bounded, because they are proven to have bouned rank-depth, implying bounded linear rank-witdh.

A conjecture would say that if H is a tree, then the class of H-vertex-minor-free graphs will have bounded linear rank-width. If so, then for a tree H (or a distance-hereditary graph H), the class of H-vertex-minor-free graphs will be linearly χ-bounded.

Davies proved that the class of circle graphs is not linearly χ-bounded. This would imply that if the class of H-vertex-minor-free graphs is linearly χ-bounded, then H is necessarily a circle graph.

If $H = C_5$, then H-vertex-minor-free graphs are distance-hereditary and so perfect, implying that they are linearly χ-bounded.

12. Polynomial χ-boundedness of graphs of bounded mim-width (from O-joung Kwon)

Mim-width is a width parameter introduced by Martin Vatshelle in his Ph.D thesis. For a vertex partition (A, B) of a graph G, let $G[A, B]$ denote the bipartite graph on $V(G)$ where the edge set of $G[A, B]$ is the set of all edges of G incident with both A and B. A branch-decomposition of a graph G is a pair of a subcubic tree T and a bijection from $V(G)$ to the set of leaves of T. For each edge e of T, let (A_e, B_e) be the corresponding vertex partition, and the width of e is defined as the maximum induced matching of $G[A_e, B_e]$. Then the width of the decomposition is the maximum width among all edges e of T, and the mim-width of G is the minimum width over all decompositions of G. Graphs of bounded clique-width are bounded mim-width, but interval graphs and permutation graphs have mim-width 1 and unbounded clique-width. Mim-width and twin-width are incomparable.

For every fixed k, is the class of graphs of mim-width at most k is polynomially χ-bounded? A positive answer to this question would generalize a result for bounded clique-width graphs by Bonamy and Pilipczuk.

Induced Subgraphs

13. Anticomplete Subgraphs (Alex Scott)

Does there exist a constant c such that every triangle-free graph G with $\chi(G) \geqslant c$ contains two disjoint anti-complete induced subgraphs G_1 and G_2 with $\chi(G_1) \geqslant 4$ and $\chi(G_2) \geqslant 4$. Here 'anticomplete' means there is no edge between G_1 and G_2. See `https://arxiv.org/abs/2303.13449` for background.

14. Erdős-Posa Property for induced cycles (Linda Cook)

This problem was told to me by Jihna Kim. Her motivation to study this problem comes from topological combinatorics, but it can be phrased entirely as a problem about induced subgraphs. See Jihna Kim's paper for the topological motivation.

A class of graphs $\mathcal{H}$ has the induced Erdős-Posa property if there is a function f such that for every $k > 0$ and graph G either G has k vertex disjoint induced copies of graphs in $\mathcal{H}$ or there is a $S \subseteq V(G)$ of cardinality at most $f(k)$ such that $G \setminus S$ does not contain a graph in $\mathcal{H}$ (as an induced subgraph).

Problem 14.1. *Do cycles of length* $0 \mod 3$ *have the induced Erdős-Posa property?*

In fact, the following easier version of Problem 14.1 would already have nice applications in topological combinatorics.

Problem 14.2. *Let* $\mathcal{G}$ *be the class of graphs satisfying the following condition: There are no two non-adjacent* $u, v \in V(G)$ *such that* $N(u) \subseteq N(v)$*. Does Problem 14.1 hold if we restrict ourselves to* $\mathcal{G}$*?*

In other words, is there a function f *such that for integer* $k > 0$ *and graph* $G \in \mathcal{G}$ *either* G *has* k *vertex disjoint ternary cycles or there is a* $S \subseteq V(G)$ *of cardinality at most* $f(k)$ *such that* $G \setminus S$ *does not contain a ternary cycle.*

Note we may also restrict ourselves of graphs that have no two adjacent vertices of degree 2.

Some related results:

- The structure of graphs without cycles of length 0 mod 3 has been examined by Marthe Bonamy, Stéphan Thomassé, Pierre Charbit and Maria Chudnovsky, Alex Scott, Paul Seymour, Sophie Spirkl.
- It is a nice result of Eunjung Kim and O-joung Kwon that the set of all induced cycles of length at least 4 has the induced Erdos-Posa property.
- This has since been strengthened to show that the set of all induced cycles of length at least 6 has the induced Erdős-Posa property when restricted to C_4-free graphs. (Tony Huynh, O-joung Kwon)
- Moreover Kim and Kwon provide a construction showing that for any set S of cycles, if S has the induced Erdős-Posa property then $C_3 \in S$ or $C_4 \in S$.

15. Unavoidable induced subgraphs of large treewidth graphs (Robert Hickingbotham)

Problem 15.1 (Hickingbotham, Illingworth, Mohar & Wood)**.** *Let $\mathcal{G}$ be a proper vertex-minor closed class. Describe the unavoidable induced subgraphs of graphs in $\mathcal{G}$ with large treewidth.*

There has been significant interest in understanding the induced subgraphs of graphs with large treewidth. To date, most of the results in this area have focused on graph classes where the unavoidable induced subgraphs are the following *usual suspects*: a complete graph K_t, a complete bipartite graph $K_{t,t}$, a subdivision of the $(t \times t)$-wall, or the line graph of a subdivision of the $(t \times t)$-wall. Recently, Hickingbotham, Illingworth, Mohar & Wood described the unavoidable induced subgraphs of circle graphs with large treewidth. Here, a *circle graph* is an intersection graph of a set of chords of a circle.

Theorem 15.2 (Hickingbotham, Illingworth, Mohar & Wood)**.** *Let $t \in \mathbb{N}$ and let G be a circle graph with treewidth at least $12t + 2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \ldots, C_t)$ such that for all $i < j$ every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of G has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

It would be of interest to describe the unavoidable induced subgraphs for other graph classes where the candidates for the unavoidable induced subgraphs include graphs that are not the usual suspect. One natural extension of circle graphs is to arbitrary (proper) vector-minor closed classes. For a vertex v of a graph G, to *locally complement at v* means to replace the induced subgraph on the neighbourhood of v by its complement. A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of vertex deletions and local complementations. It is easy to see that circle graphs are closed under vertex-minors. Moreover, every vertex-minor closed class with unbounded rank-width contains all circle graphs (see The Grid Theorem for Vertex-Minors).

One useful property of proper vertex-minor-closed graph classes is that treewidth and Hadwiger number are tied in such classes.

Theorem 15.3 (Unpublished)**.** *For any proper vertex-minor closed class $\mathcal{G}$, there is a function f such that every graph $G \in \mathcal{G}$ with treewidth at least $f(t)$ contains a K_t-minor.*

Theorem 15.3 can easily be deduced from an observation about vertex-minors by James Davies (personal communication) together with a result in On the Tree-Width of Even-Hole-Free Graphs. The proof, however, goes through the Graph Minor Structure Theorem so it isn't very informative.

Problem 15.4. *Can we find a direct proof of Theorem 15.3 that does not use the Graph Minor Structure Theorem?*

I suspect a solution to Problem 15.4 should help solve Problem 15.1.

16. Wheels and Tree-Width (James Davies)

A wheel is a graph consisting of an induced cycle of length at least 4, and a single additional vertex adjacent to at least 3 vertices of the cycle. A graph is wheel-free if it contains no induced wheel.

Problem 16.1. *Describe the unavoidable induced subgraphs of wheel-free graphs with large tree-width.*

17. α-treewidth of graphs excluding a fixed induced minor (Sebastian Wiederrecht)

The α-treewidth of a graph is a variant of treewidth that measures the distance of a graph to being chordal rather than to being a tree as "regular" treewidth does. Given a tree-decomposition (T, β) of a graph G, $\alpha(T, \beta)$ is defined to be $\max_{t \in V(T)} \alpha(G[\beta(t)])$. The *$\alpha$-treewidth* of G is the minimum α over all tree-decompositions for G. This parameter was defined independently by Yolov and Dallard, Milanič, and Štorgel to obtain better algorithms for the Maximum Independent Set problem.

Two nice properties of α-treewidth are:

(1) It is closed under taking induced minors, and
(2) given a hereditary class $\mathcal{G}$ of graphs such that α-$tw(G) \leqslant c \in \mathbb{N}$ for all $G \in \mathcal{G}$, then $tw(G) \leqslant \omega(G)^c$ for all $G \in \mathcal{G}$.

Since walls and their linegraphs have arbitrary large α-treewidth, excluding a non-planar graph as an induced minor will never lead to a class of bounded α-degeneracy. Moreover, Trotignon, Korhonen, Hatzel, and myself (unpublished) recently proved that induced $K_{3,4}$-minor-free graphs still contain a class of graphs found by Sintari and Trotignon which is triangle-free and of unbounded treewidth, also implying unbounded α-treewidth. This class avoids all of the usual suspects as induced minors. On the other side, Dallard, Milanič, and Štorgel showed that induced $K_{2,3}$-minor-free graphs have α-treewidth at most 3.

This leaves the following open problem.

Problem 17.1. *For which planar graphs H does the class of graphs excluding $K_{3,3}$ and H as induced minors have bounded α-treewidth?*

An interesting subproblem is a generalisation of a recent result of Kwon (personal communication) who proved that for all integers s, t, every graph excluding P_t and the star with s leaves as induced subgraphs has bounded α-treewidth.

Problem 17.2. *For which choices of r, s, t does the class of graphs excluding P_r and $K_{s,t}$ as induced subgraphs have bounded α-treewidth?*

18. α-degeneracy of hereditary graph classes (Linda Cook, Sebastian Wiederrecht)

We say that a hereditary graph class $\mathcal{G}$ is (tw, ω)-*bounded* if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $tw(G) \leqslant f(\omega(G))$ for all $G \in \mathcal{G}$.

Using the notion of α-treewidth from Problem 17 one can observe that every hereditary graph class of bounded α-treewidth is (tw, ω)-bounded. Dallard, Milanič, and Štorgel conjectured the reverse.

Conjecture 18.1 (Dallard, Milanič, and Štorgel)**.** *Every (tw, ω)-bounded graph class has bounded α-treewidth.*

Approaching this conjecture appears to be not easy. Instead one might consider the following seemingly simpler question.

Given a linear order $\leqslant_\lambda$ of the vertices of a graph G, we define the *α-weight* of $\leqslant_\lambda$ has the value $\max_{v \in V(G)} \alpha(G[\{x \leqslant_\lambda v \mid x \in N_G(v)\}])$. The *$\alpha$-degeneracy* of G is then defined to be the minimum α-weight over all linear orders of the vertices of G. This notion was first investigated in the context of approximation algorithms for the Maximum Independent Set problem by Borodin and Ye.

As for α-treewidth one can observe that for every hereditary graph class $\mathcal{G}$ where all $G \in \mathcal{G}$ have α-degeneracy at most c it holds that every $G \in \mathcal{G}$ has degeneracy at most $\omega(G)^c$. We say that a hereditary class of graphs $\mathcal{G}$ is *$(degeneracy, \omega)$-bounded* of there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{G}$ it holds that G has degeneracy at most $f(\omega(G))$. Clearly, every class of bounded α-degeneracy is $(degeneracy, \omega)$-bounded by the observation above. We can now formulate our problem as the degeneracy-variant of Conjecture 18.1.

Problem 18.2. *Is every $(degeneracy, \omega)$-bounded graph class also of bounded α-degeneracy?*

18.1. **UPDATE.** Question 18.2 was answered in the negative by Rose McCarty, António Girão), and Raphael Steiner (and others?).

19. Well-quasi-ordering of permutation graphs (Rutger Campbell)

Given a permutation π on $1, \dots, n$, we can define a graph on vertices $v_1, \dots, v_n$ by taking an edge $v_i v_j$ when $i < j$ yet $\pi(i) > \pi(j)$. Graphs that are isomorphic to such a construction are called *permutation graphs*. Permutation graphs are closed under induced subgraphs. Permutation graphs are not well-quasi-ordered under induced subgraphs; we can construct an infinite set of permutation graphs where no two are induced subgraphs of one another. However, such infinite anti-chains seem to have common features.

Conjecture 19.1. *For any positive integers s, t, permutation graphs with no induced K_s or P_t are well-quasi-ordered under induced subgraphs.*

20. δ-boundedness (Rose McCarty)

A class of graphs $\mathcal{F}$ is *δ-bounded* if there exists a function f so that for each integer t, every graph $G \in \mathcal{F}$ without $K_{t,t}$ as a subgraph (even as a non-induced subgraph) has $\delta(G) \leqslant f(t)$. Such a function f is called a *δ-bounding function* for $\mathcal{F}$. So δ-boundedness is like χ-boundedness but for minimum degree instead of chromatic number.

Briański, Davies, and Walczak [5] recently proved that optimal χ-bounding functions can grow arbitrarily quickly, even for hereditary classes. This disproved Esperet's Conjecture in a strong way. However, for δ-boundedness, the analogous question is still open:

Question 20.1 (Bonamy, Bousquet, Pilipczuk, Rzażewski, Thomassé, and Walczak [3])**.** *Does every hereditary δ-bounded class have a δ-bounding function that is a polynomial?*

Very, very recently, Xiying Du and I have been able to prove the following theorem, which we are currently writing up.

Theorem 20.2 (Du and McCarty, in preparation)**.** *For every hereditary δ-bounded class, there exists an integer c so that $2^{2^{2^{ct}}}$ is a δ-bounding function.*

In fact, we think we can prove that there is a doubly-exponential δ-bounding function.

The intuition for why there is such a huge difference with χ-boundedness can be seen in the following theorem; the analogous statement for χ-boundedness is false [6].

Theorem 20.3 (Kwan, Letzter, Sudakov, and Tran [9] plus McCarty [10])**.** *A hereditary class of graphs $\mathcal{F}$ is δ-bounded if and only if there exists an integer c so that every bipartite graph $G \in \mathcal{F}$ with no 4-cycles has $\delta(G) \leqslant c$.*

This area has many natural open questions. Let me suggest a few.

Question 20.4. *Let $\mathcal{F}$ be a hereditary δ-bounded class so that $\mathcal{F}_{bip} = \{G \in \mathcal{F} : G \text{ is bipartite}\}$ has a polynomial δ-bounding function. Then does $\mathcal{F}$ have a polynomial δ-bounding function?*

We say that a class $\mathcal{F}$ is *almost δ-bounded* if, for each $\epsilon > 0$, there exists a function f so that any n-vertex graph $G \in \mathcal{F}$ with no $K_{t,t}$-subgraph has $|E(G)| \leqslant f(t)n^{1+\epsilon}$. Note that for hereditary classes, this would be exactly δ-boundedness if we could take $\epsilon = 0$. There are some connections to the Zarankiewicz problem, to nowhere denseness, and to model theory. In particular, semilinear graphs are almost δ-bounded [2].

Question 20.5. *Do the analogs of Theorems 20.2 and 20.3 hold for classes which are almost δ-bounded?*

The other main problems I would suggest involve particular examples of δ-bounded classes. Such classes include:

(1) for each tree T, the class of all graphs without T as an induced subgraph [12],
(2) for each graph H, the class of all graphs without H as an induced subdivision [8],
(3) any class of bounded twin-width [4] or bounded flip-width [13] (essentially by [7]),
(4) for each integer s, the class of all graphs with no independent set of size s (see [1]),
(5) for each integer d, the class of all intersection graphs of balls in $\mathbb{R}^d$ (see [11]).

It is open whether *(ii)* is polynomially δ-bounded, and we have not looked into *(iii)* yet. The others are known to be polynomially δ-bounded. It seems interesting to know whether the bounds for *(iv)* are the same as for off-diagonal Ramsey numbers.

References from Rose's problem

1. Miklós Ajtai, János Komlós, and Endre Szemerédi, *A note on ramsey numbers*, Journal of Combinatorial Theory, Series A **29** (1980), no. 3, 354–360. 14
2. Abdul Basit, Artem Chernikov, Sergei Starchenko, Terence Tao, and Chieu-Minh Tran, *Zarankiewicz's problem for semilinear hypergraphs*, Forum of Mathematics, Sigma **9** (2021), e59. 14
3. Marthe Bonamy, Nicolas Bousquet, Michał Pilipczuk, Paweł Rzażewski, Stéphan Thomassé, and Bartosz Walczak, *Degeneracy of pt-free and $c_{\geqslant t}$-free graphs with no large complete bipartite subgraphs*, Journal of Combinatorial Theory, Series B **152** (2022), 353–378. 13
4. Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant, *Twin-width I: tractable FO model checking*, J. ACM **69** (2022), no. 1, A3:1–A3:46. MR 4402362 14
5. M. Briański, J. Davies, and B. Walczak, *Separating polynomial χ-boundedness from χ-boundedness*, arXiv (2022), 2201.08814. 13
6. Alvaro Carbonero, Patrick Hompe, Benjamin Moore, and Sophie Spirkl, *A counterexample to a conjecture about triangle-free induced subgraphs of graphs with large chromatic number*, Journal of Combinatorial Theory, Series B **158** (2023), 63–69. 14
7. Z. Dvořák, *Induced subdivisions and bounded expansion*, European J. Combin. **69** (2018), 143–148. 14
8. D. Kühn and D. Osthus, *Induced subdivisions in $K_{s,s}$-free graphs of large average degree*, Combinatorica **24** (2004), no. 2, 287–304. 14
9. M. Kwan, S. Letzter, B. Sudakov, and T. Tran, *Dense induced bipartite subgraphs in triangle-free graphs*, Combinatorica **40** (2020), no. 2, 283–305. 14
10. Rose McCarty, *Dense induced subgraphs of dense bipartite graphs*, SIAM Journal on Discrete Mathematics **35** (2021), no. 2, 661–667. 14
11. Gary L. Miller, Shang-Hua Teng, William Thurston, and Stephen A. Vavasis, *Separators for sphere-packings and nearest neighbor graphs*, J. ACM **44** (1997), no. 1, 1–29. 14
12. Alex Scott, Paul Seymour, and Sophie Spirkl, *Polynomial bounds for chromatic number. i. excluding a biclique and an induced tree*, Journal of Graph Theory **102** (2023), no. 3, 458–471. 14
13. Szymon Toruńczyk, *Flip-width: cops and robber on dense graphs*, Electronic preprint arxiv:2302.00352, 2023. 14

Graph Colouring

21. Attacking Hadwiger's Conjecture via Chordal Partitions (David Wood)

Reed and Seymour asked the following question: For any graph G, is there a partition $\mathcal{P}$ of $V(G)$ such that the quotient of $\mathcal{P}$ is chordal, and for each part $X \in \mathcal{P}$, the induced subgraph $G[X]$ is connected and bipartite? A positive answer to this question would imply that K_{t+1}-minor-free graphs are $2t$-colourable, which would be a major breakthrough on Hadwiger's Conjecture. Scott, Seymour and Wood showed that the answer is "no", and remains "no" with "chordal" replaced by by "perfect" (and under various other weakenings). However, their construction replies heavily on clique-separators, which one can assume do not exist when colouring in a hereditary family. The following question arises:

Problem 21.1. *For any graph G with no clique separator, is there a partition $\mathcal{P}$ of $V(G)$ such that the quotient of $\mathcal{P}$ is perfect, and for each part $X \in \mathcal{P}$, the induced subgraph $G[X]$ is connected and bipartite?*

A positive answer to this question would still imply that K_{t+1}-minor-free graphs are $2t$-colourable. It would still be interesting with "bipartite" replaced by "c-colourable" for some constant c.

22. Clustered Colouring of Planar Graphs (David Wood)

A (non-proper) vertex colouring of a graph has *clustering* c if each monochromatic component has at most c vertices. The 4-Color Theorem is best possible, even in the setting of clustered colouring. That is, for all c there are planar graphs for which every 3-coloring has a monochromatic component of size greater than c; see my survey on this topic. These examples have unbounded maximum degree. This is necessary since Esperet and Joret proved that every planar graph with bounded maximum degree is 3-colorable with bounded clustering. In fact, these example contain large $K_{2,t}$ subgraphs. The following question naturally arises:

Problem 22.1 (Liu & Wood)**.** *Does every planar graph with no $K_{2,t}$ subgraph have a 3-coloring with clustering at most some function $f(t)$?*

22.1. **UPDATE (Zdeněk Dvořák).** I think a positive answer to Problem 22.1 follows using the method of Dvořák & Norin [Weak diameter coloring of graphs on surfaces] even if a path + two universal vertices is forbidden instead of $K_{2,t}$.

23. Odd colourings of planar graphs (Michael Savery)

An *odd colouring* of a graph G is a proper colouring of its vertex set such that for every non-isolated vertex of G, there exists a colour that appears an odd number of times in its neighbourhood. The *odd chromatic number* of G, denoted $\chi_o(G)$, is the minimum number of colours required in an odd colouring of G. These definitions, which are motivated by certain hypergraph colouring problems, were introduced recently by Petruševski and Škrekovski in a paper which focussed on the odd chromatic number of planar graphs. In particular, they sought an analogue of the four colour theorem in this setting. Observing that $\chi_o(C_5) = 5$, they conjectured that five colours suffice for all planar graphs.

Conjecture 23.1. *For every planar graph G we have $\chi_o(G) \leqslant 5$.*

They showed that $\chi_o(G) \leqslant 9$ for all planar G using a discharging argument. Shortly afterwards Caro, Petruševski, and Škrekovski improved the bound to 8 for large classes of planar graphs, before Petr and Portier extended this to all planar graphs, again via a discharging argument. The former group also proved Conjecture 23.1 for outerplanar graphs.

One direction of study related to this conjecture which has been particularly fruitful has been bounding the odd chromatic number of planar graphs under a minimum girth assumption. The state of the art here is that $\chi_o(G) \leqslant 4$ for planar G of girth at least 11, $\chi_o(G) \leqslant 5$ if the girth is at least 7, and $\chi_o(G) \leqslant 6$ if the girth is at least 5 (the first and third are due to Cho, Choi, Kwon, and Park, the second is due to Cranston). It was conjectured by Petruševski and Škrekovski that in fact $\chi_o(G) \leqslant 4$ for planar G of girth at least 6 (note that the four colour theorem would follow easily from this by subdividing all edges).

At the workshop we could try to make progress towards Conjectrue 23.1 or to make some improvements to the girth conditions above. There are also many other related questions we could think about. For example, there are various interesting results and conjectures on bounding odd chromatic number in terms of maximum degree, and on the minimum number of colours needed in the stronger notion of a proper vertex-colouring in which for every non-isolated vertex there is a colour which appears exactly once in its neighbourhood (the so-called *proper conflict-free chromatic number*).

24. Low treewidth colouring of bounded degree graphs (Robert Hickingbotham)

Problem 24.1 (Wood). *What is the maximum Δ such that every graph with maximum degree Δ can be 2-coloured such that each colour class has bounded treewidth? It is known that $\Delta \in \{5, \dots, 15\}$.*

Berger, Dvořák, & Norin showed that if a graph G is a 'triangulation' of a sufficiently large $n \times n \times n$ grid, then for every 2-colouring of G, there is a colour class with large treewidth. Eppstein, Hickingbotham, Merker, Norin, Seweryn, & Wood introduced a family of graphs $(G_n : n \in \mathbb{N})$ based on a tessellation of $\mathbb{R}^3$ by truncated octahedra that have similar structural properties to triangulations of 3D-grids but with maximum degree 7.

Problem 24.2. *For sufficiently large n, does every 2-colouring of G_n have a colour class with large treewidth?*

It seems feasible that the proof technique in Treewidth of Grid Subsets could be adapted to resolve Problem 24.2. In which case, this would imply that $\Delta \in \{5, 6\}$ for Problem 24.1. The remaining case would then be $\Delta = 6$ which I don't have any ideas for. See Section 10 of Defective and Clustered Graph Colouring for further background on this question.

24.1. **UPDATE (Zdeněk Dvořák).** I chatted about this with Sergey Norin in February; he developed a theory of higher-dimensional brambles which among other things implies a positive answer to Problem 24.2.

24.2. **UPDATE (Zdeněk Dvořák et al.)** (Zdeněk Dvořák et al.) The following theorem shows that $\Delta \leqslant 6$ in Problem 24.1, and in fact proves a stronger result. For $c > 0$ and $\beta \in (0, 1)$, say a graph G is (c, β)-*separable* if every subgraph G' of G has a balanced separator of size at most $c|V(G')|^{1-\beta}$. The following folklore lemma is proved in Wood's survey.

Lemma 24.3. *Fix $c > 0$, $\beta \in (0, 1)$ and $p \geqslant 1$. Let G be a (c, β)-separable graph with n vertices. Then there exists $S \subseteq V(G)$ of size at most $\frac{c2^\beta n}{(2^\beta - 1)p^\beta}$ such that each component of $G - S$ has at most p vertices.*

Theorem 24.4. *For all $c > 0$ and $\beta \in (0, 1)$ there is a 7-regular graph that has no vertex-partition into two induced (c, β)-separable subgraphs.*

Proof. By ??? for infinitely many integers n there is an n-vertex bipartite graph G_0 with bipartition (A, B) where every vertex in A has degree 4 and every vertex in B has degree 5, and G_0 has girth at least some function $f(n)$ with $\lim_{n\to\infty} f(n) = \infty$ (for fixed β, c). Note that $|E(G_0)| = 4|A| = 5|B|$ and $n = |A| + |B| = |A| + \frac{4}{5}|A| = \frac{9}{5}|A|$. Let $G := L(G_0)$. So G is 7-regular. Suppose for the sake of contradiction that G has a vertex-partition V_1, V_2 such that each $G[V_i]$ is (c,β)-separable. Without loss of generality, $|V_1| \geqslant |V(G)|/2$. Thus $|V_1| \geqslant |E(G_0)|/2 = 2|A| = \frac{10}{9}n$. On the other hand, $|V_1| \leqslant |V(G)| = |E(G_0)| = 4|A| = \frac{20}{9}n$. Define $p := (\frac{c2^\beta 20}{2^\beta-1})^{1/\beta}$. By the lemma applied to $G[V_1]$, there exists $S \subseteq V_1$ of size at most $\frac{c2^\beta|V_1|}{(2^\beta-1)p^\beta} \leqslant \frac{c2^\beta 20n}{(2^\beta-1)p^\beta 9} = \frac{n}{9}$ such that each component of $G[V_1] - S$ has at most p vertices. Let G_1 be the spanning subgraph of G_0 corresponding to V_1. Let S_1 be the set of edges of G_1 corresponding to S. So G_1 has n vertices and at least $\frac{10}{9}n$ edges. Thus $G_1 - S_1$ has at least n edges. Hence $G_1 - S_1$ has a cycle C, which corresponds to a cycle in $G[V_1] - S$. Thus $f(n) \leqslant |C| \leqslant p$, which contradicts that $f(n) \to \infty$. This contradiction proves the theorem. $\square$

The following generalisation is proved by the same method.

Theorem 24.5. *For all $c > 0$ and $\beta \in (0,1)$ and $k \in \mathbb{N}$ there is a $(4k-1)$-regular graph that has no vertex-partition into k induced (c,β)-separable subgraphs.*

25. Clustered coloring for bounded degree graphs (Chun-Hung Liu)

Problem 25.1. *For every integer d, let $f(d)$ be the smallest integer k such that there exists a constant c such that every graph with maximum degree at most d can be partitioned into $f(d)$ induced subgraphs with no component on more than c vertices. Determine $f(d)$ for $d \geqslant 9$.*

It is known that $\lfloor \frac{d+6}{4} \rfloor \leqslant f(d) \leqslant \lceil \frac{d+1}{3} \rceil$ for every $d \geqslant 2$, and $f(d) = \lfloor \frac{d+6}{4} \rfloor$ when $d \leqslant 8$. See Wood for a survey. See Liu for the relation to the clustered chromatic number of the class of H-immersion free graphs for every graph H.

Other Problems

26. Monochromatic path covers of hypergraphs (Jane Tan)

Let $K_n^{(r)}$ be the r-uniform complete hypergraph on n vertices. A *tight path* in such a graph is a sequence of (at least r, say) vertices such that every k consecutive vertices form an edge. We also allow the empty path, which contains no vertices or edges. The following question is due to Maya Stein.

Question: Is it true that for every r, n and any 2-colouring of the edges of $K_n^{(r)}$, there exist two monochromatic vertex-disjoint tight paths of different colours that together cover all vertices in the graph?

The $r = 3$ case was originally asked by Gyárfás and Sárközy, and Stein gave a lovely short proof of this (see arXiv:2204.12464). In the same paper, Stein conjectured that the answer to the above question is yes for higher uniformities as well, but it is open for $r \geqslant 4$. It is worth noting that the $r = 3$ result is not true when paths are replaced by cycles, although it becomes true again if we additionally allow the paths to have the same colour. There is also a good deal of surrounding literature that uses loose or Berge paths and cycles.

27. Does edge expansion force a large minor? (Anita Liebenau)

For a graph G on a n vertices and a subset $S \subseteq V(G)$ let

$$h(S) = \frac{e(S, V \setminus S)}{|S|}$$

and let $h(G)$ be the minimum of $h(S)$ taken over all nonempty subsets of size at most $n/2$. So $h(S)$ can be thought of as the average number of edges that a vertex of S sends to $V(G) \setminus S$.

In 2019, Krivelevich and Nenadov asked the following question.

Question 27.1. *Let G be a graph on $n \geqslant n_0$ vertices, maximum degree at most $d = d(n) \geqslant 3$, and $h(G) \geqslant \varepsilon d$, for some constant $\varepsilon > 0$. Is it true that G has a clique minor on $\Omega(\sqrt{nd/\log d})$ vertices?*

They proved this question under the additional assumption that

$$h(S) \geqslant (1/2 + \varepsilon)\, d$$

for every set $S \subseteq V(G)$ of size at most εn. They also proved that any graph as in the question has a clique minor on $\Omega(\sqrt{n})$ vertices, that is, the answer is yes for constant d.

Reference Complete minors in graphs without sparse cuts, Krivelevich and Nenadov, International Mathematics Research Notices 2021, no. 12 (2021): 8996-9015, arXiv:1812.01961.

28. Is domination number a local property? (António Girão)

The domination number of a tournament T is the smallest size $S \subset V(T)$ for which for every $z \in V(T) \setminus S$ there a directed edge wz for some $w \in S$.

In a very nice paper Harutyunyan, Tien-Nam Le, Thomassé, Wu showed that for every every k there is $g(k), f(k)$ such that every tournament with domination number at least $f(k)$ must contain a sub-tournament on at most $g(k)$ vertices with chromatic number at least k.

They asked whether the same phenomenon must happen with domination number. Namely, whether there is $f'(k), g'(k)$ such that every tournament with domination number at least $f'(k)$ must contain a subtournament on at most $g'(k)$ vertices which has domination number at least k.

29. Covering with a Linear Number of Rainbow Paths (Lukas Michel)

A rainbow path in an edge-coloured graph is a path whose edges all have distinct colours. The following problem was stated in a paper by Bonamy, Botler, Dross, Naia, and Skokan.

Problem 29.1. *Is it true that for all properly edge-coloured graphs G on n vertices there exists a covering of all edges of G with $O(n)$ rainbow paths?*

By greedily selecting a longest rainbow path and deleting it, it is possible to show that $O(n \log n)$ rainbow paths suffice. It can also be proved that $O(n)$ rainbow trails suffice to cover all edges of G, where a trail is allowed to repeat vertices. Both of these constructions produce partitions of the edges of G, but a covering with $O(n)$ rainbow paths where edges are allowed to be in multiple paths is already interesting.

30. Number of Perfect Matchings (Liana Yepremyan)

This is a problem of Maria Chudnovsky, from Barbados 2014 Graph Theory workshop. I learned about it from Cosmin Pohoata recently. I do not know if any progress has been made on this conjecture so if you do please let me know.

For a graph H, let pm(H) be the number of perfect matchings in H. A k-lift G^k of a graph G is obtained by substituting a stable set S_x of size k for every vertex x of G, and then joining S_x with S_y by a perfect matching whenever xy is an edge of G. Is it true that if G is bipartite then

$$\mathrm{pm(G^k)} \leqslant \mathrm{(pm(G))^k}.$$

It is not hard to see that this bound does not hold for non-bipartite G. What if G is regular?

One can try to use some techniques that have been used for similar problems in the following papers for upper bounding the number of independent sets or the number of colourings, but I have not really tried any of these.

1. The number of independent sets in an irregular graph, A. Sah, M. Sawhney, D. Stoner, Y. Zhao `https://dx.doi.org/10.1016/j.jctb.2019.01.007`
2. The Number of Independent Sets in a Regular Graph, Y. Zhao, `https://dx.doi.org/10.1017/S0963548309990538`
3. An entropy approach to the Hard-Core Model on Bipartite Graphs, J. Kahn, `https://dx.doi.org/10.1017/S0963548301004631`
4. On weighted graph homomorphisms, D. Galvin, P. Tetali, `https://arxiv.org/pdf/1206.3160.pdf`

31. Rooted K_4-subdivisions (Raphael Steiner)

Problem 31.1. *Given four distinct vertices a, b, c, d, characterize the edge-maximal graphs G that do not have a K_4-subdivision rooted at $\{a, b, c, d\}$, i.e., a subdivision of K_4 where a, b, c, d constitute the four branch-vertices.*

This problem came up in email discussion with David Wood. It is of high interest, as a full answer to it could be very useful for resolving several other problems relating to subdivision containment, such as a proof of the still open case of Hajos' conjecture for K_5-subdivisions (stating that every graph with no K_5-subdivision is 4-colorable). Also, see the recent JCTB-paper by Hayashi and Kawarabayashi which partially resolves a weaker form of the problem, essentially looking at $K_4 - e$ (the diamond graph) instead of K_4, and even this appears to be non-trivial.

A generalization of rooted K_4-subdivisions are rooted K_4-minors, defined as follows: Given four distinct vertices a, b, c, d in a graph G, we say that G has a K_4-minor rooted at $\{a, b, c, d\}$ if G contains 4 disjoint sets A, B, C, D of vertices such that $a \in A, b \in B, c \in C, d \in D$, each of $G[A], G[B], G[C], G[D]$ is connected and there is at least one edge between any pair in $\{A, B, C, D\}$. Fabila-Monroy and Wood gave a precise characterization of the edge-maximal graphs with no K_4-minor rooted at four given vertices. The characterization exhibits 6 different classes of such graphs, the most interesting of them being a planar graph with all four nominated vertices on a common face, plus some attachments along separations that form a facial triangle in the planar graph. While all these classes are also obstructions for rooted K_4-subdivisions, this is not everything, and indeed the answer for rooted K_4-subdivisions is expected to be a lot more complex. A simple example that has a rooted K_4-minor but no rooted K_4-subdivision is if one take the graph K_4 and adds one additional degree 1-vertex connected to one of the four original vertices, and declares this vertex to be a and the three non-neighbors of it to be b, c, d.

Given that the above example is loosely connected and of small minimum degree, it may be natural to hope that for sufficiently highly connected graphs, such as 4-connected graphs, say, the notions of rooted K_4-minors and K_4-subdivisions coincide. However, also this is not the case, as obstructions to K_4-subdivisions can also be in terms of space requirements for the 6 necessary subdivision paths: Consider the complete bipartite graph $K_{4,5}$ with all four nominated vertices in the small color class. Then this does not have a rooted K_4-subdivision due to lack of space, is a 4-connected graph, and also in fact is easily seen to have a rooted K_4-minor at the nominated vertices. More generally, if a graph G with vertices a, b, c, d has a set S of at most 5 vertices such that a, b, c, d are in distinct components of $G-S$, then a roted K_4-subdivision cannot exist. Thus, even stronger assumptions are required to make the two notions of rooted K_4-minors and -subdivisions coincide. In this direction, attempting to get around the previously mentioned space-obstacle, David Wood asked the following question.

Definition 31.2. A set $R \subseteq V(G)$ is *well-connected* in a graph G if for every set of vertices S in $G - R$, if $X_1, \ldots, X_c$ are the components of $G - S$, and $r_i := |R \cap X_i|$, then $|S| \geqslant \sum_{1 \leqslant i < j \leqslant c} r_i r_j$.

Note that if a graph has a K_t-subdivision rooted at the vertices in R (so necessarily $|R| = t$) then R is well-connected in G, since every subdivision path between two vertices in $R - S$ that are in distinct components of $G - S$ must traverse S.

Problem 31.3 (Wood)**.** *Is it true that if a graph G has a K_4-minor rooted at $\{a, b, c, d\}$ and $R := \{a, b, c, d\}$ is well-connected in G, then G also has a K_4-subdivision rooted at $\{a, b, c, d\}$?*

If this were true, it would be a big step forward in understanding the essential obstacles to having a rooted K_4-subdivision.

32. Graph classes admitting weak guidance systems (Zdeněk Dvořák)

Question 32.1. *Which graph classes admit weak guidance systems of bounded maximum outdegree?*

A *partial orientation* of an undirected graph G assigns orientation to a subset of edges of G (it is possible for an edge to be directed in both ways at the same time). A path P in G

is *weakly inwards-directed* in a partial orientation of G if there exists an edge $e \in E(P)$ such all edges of P (except for e) are directed towards e (the edge e may or may not be directed in the partial orientation).

For a positive integer r, a *weak r-guidance system* in a graph G is a partial orientation of G such that for any distinct vertices u and v at distance at most r in G, there exists a shortest path between u and v in G that is weakly inwards-directed. An algorithmic motivation for this notion is as follows: If G has a weak r-guidance system of maximum outdegree at most Δ (for some $\Delta \geqslant 2$), then we can test whether input vertices u and v are at distance at most r in time $O(\Delta^r)$, by enumerating the outgoing paths from u and v and checking where they meet or become adjacent.

For a function f, we say that a class of graphs $\mathcal{G}$ *admits f-bounded weak guidance systems* if for every positive integer r, every graph $G \in \mathcal{G}$ has a weak r-guidance system of outdegree at most $f(r, |V(G)|)$. A few examples:

- Graph classes with bounded expansion, as well as their first-order transductions, admit f-bounded weak guidance systems for $f(r,n) = O_r(1)$.
- Hereditary graph classes of girth at least five admit f-bounded weak guidance systems for $f(r,n) = O_r(1)$ if and only if they have bounded expansion.
- Interval graphs admit f-bounded weak guidance systems for $f(r,n) = 2$.
- Graphs of bounded cliquewidth admit f-bounded weak guidance systems for $f(r,n) = O_r(\log n)$, but not for $f(r,n) = o_r(\log n/\log\log n)$.
- Chordal graphs do not admit f-bounded weak guidance systems for any $f(r,n) = o_r(n^{1/2})$.

For more background, see https://arxiv.org/abs/2204.09113.

Question 32.2. *Which graph classes admit f-bounded weak guidance systems for $f(r,n) = O_r(1)$? Or for $f(r,n) = n^{o_r(1)}$?*

33. Erdős-Gallai Conjectures for binary matroids (Bryce Frederickson)

Erdős and Gallai conjectured that the edge set of any Eulerian graph on n vertices can be decomposed into $O(n)$ vertex-disjoint cycles. We consider a handful of analogues of this conjecture in the setting of matroids. A *(simple) binary matroid* is a subset $M \subseteq \mathbb{F}_2^n \setminus \{0\}$ for some $n \geqslant 1$. We say that M is *Eulerian* if $\sum_{x\in M} x = 0$, and we say that M is a *circuit* if M is Eulerian, but no nontrivial proper subset of M is. We define $\operatorname{rank}(M)$ to be the cardinality of the largest linearly independent subset of M.

Problem 33.1. *For some binary matroid parameter $f(M)$, which of the following statements hold?*

i Every Eulerian binary matroid M can be expressed as the union of disjoint circuits $C_1, \ldots, C_t \subseteq M$, with $t \leqslant f(M)$.

ii Every Eulerian binary matroid M can be expressed as the union of (not necessarily disjoint) circuits $C_1, \ldots, C_t \subseteq M$, with $t \leqslant f(M)$.

iii Every Eulerian binary matroid M can be expressed as the symmetric difference of circuits $C_1, \ldots, C_t \subseteq M$, with $t \leqslant f(M)$.

iv Every Eulerian binary matroid $M \subseteq \mathbb{F}_2^n \setminus \{0\}$ can be expressed as the symmetric difference of circuits $C_1, \ldots, C_t \subseteq \mathbb{F}_2^n \setminus \{0\}$.

In the Erdős-Gallai setting, the conjectured value $O(n)$ is quite natural since an Eulerian graph can have $\Omega(n^2)$ edges, and each cycle in a decomposition covers at most n edges. By the same token, a natural choice of $f(M)$ to consider for our context might be something along the lines of

$$f(M) = O\left(\frac{2^{\mathrm{rank}(M)} - 1}{\mathrm{rank}(M) + 1}\right).$$

In fact, since (iv) provides some extra flexibility compared to (i), we might even be able to get away with something like

$$f(M) = \max_{N \subseteq M} \left\lceil \frac{|N|}{\mathrm{rank}(N) + 1} \right\rceil$$

for (iv). This would give an analogue to the Nash-Williams Theorem on arboricity, which was generalized to matroids by Edmonds.

34. Universal Graphs for Planar Graphs (Linda Cook)

Tony Huynh, Bojan Mohar, Robert Šámal, Carsten Thomassen and David Wood [Universality in minor-closed graph classes] asked several nice questions. In particular:

Problem 34.1. *Is there a minimal graph U (under subgraph relation) that contains every planar graph as a subgraph?*

Some necessary criteria for U and further nice quesetions are found in Section 7 of Universality in minor-closed graph classes. Stanisław Ulam originally asked whether there is a countable planar graph that contains every planar graph as a subgraph. János Pach answered this question in the negative in 1981. Then Huynh, Mohar, Šámal, Thomassen and Wood strengthed this to show that a countable graph that contains all countable planar graphs as subgraphs must contain an infinite complete graph as a minor, and a subdivision of the complete graph K_t with multiplicity t, for every finite t. Last year, Thilo Krill showed that there is no countable planar graph that contains every planar graph as a subdivision answering a question of Diestel and Kuhn in the negative.

35. Universal graphs with bounded maximum degree (Florian Lehner)

Let us call a graph H *universal* for a graph class $\mathcal{G}$, if every member of $\mathcal{G}$ is a subgraph of H. Huynh, Mohar, Šámal, Thomassen, and Wood asked the following question:

Question 35.1. *Does the class of planar graphs of maximal degree* 3 *admit a universal graph with bounded maximal degree?*

Question 35.2. *Does the class of graphs of maximal degree* 3 *(without the requirement of planarity) admit a universal graph with bounded maximal degree?*

I was able to prove [A note on classes of subgraphs of locally finite graphs] that the second question has a negative answer. The proof is based on the fact that there are 'too many' graphs of maximum degree 3. More precisely, let $B_G(v, r)$ denote the ball in G with centre v and radius r, then the size of the set $\{B_G(v, r) \mid \Delta(G) \leqslant 3, v \in V(G)\}$ grows superexponentially fast in r.

Let us call a graph class *small*, if it does not satisfy this property, that is, the number of non-isomorphic balls of radius r appearing in members of the class is bounded above by some exponential function.

Problem 35.3. *Is there a small graph class $\mathcal{G}$ such that $\Delta(G) \leqslant 3$ for every $G \in \mathcal{G}$ which does not admit a universal graph of finite maximum degree?*

36. Treewidth in perturbations of circle graphs (Rutger Campbell, Pascal Gollin, O-joung Kwon, Sebastian Wiederrecht)

A *circle graph* is the intersection graph of chords of a circle, that is lines (curves) drawn in the disk whose endpoints are the points intersecting the boundary.

Geelen conjectured that for every H, every sufficiently rank-connected graph G excluding H as a vertex-minor is a "pertubation" of a circle graph.

We want to look at a class of graphs of graphs obtained from a simple type of pertubation defined by some geometrtic condition.

If we draw chords on a surface Σ with boundary and looking at their intersection graphs, we get other classes of graphs, which we call Σ-*diagram graphs*. If Σ is a disk, we get circle graphs. Looking at chord diagrams in the crosscap / Möbius strip gives us *crosscap-diagram graphs*. This class of graphs is closed under pivot-minors and each crosscap-diagram graph can be obtained from some circle graph by performing a rank 1 pertubation. Looking at the chord diagrams in the annulus gives us *annulus-diagram graphs*. We can switch between the classes of crosscap-diagram graphs and annulus diagram graphs by performing a single local complementation (which again is a rank 1 pertubation). So the union of crosscap-diagram graphs and annulus-diagram graphs is closed under vertex-minors. Moreover, each annulus-diagram graphs can be obtained from circle graphs by performing a rank 2 pertubation.

Building on 15 we want to study the unavoidable induced subgraphs of graphs with large treewidth for graphs that are the symmetric difference of a circle graph and a clique on a subset of vertices.

As a starting point, we want to consider the crosscap-diagram graphs or annulus-diagram graphs defined in as follows: A *circle graph* is the intersection graph of chords of a circle, that is lines (curves) drawn in the disk whose endpoints are the points intersecting the boundary. If instead of in a disc, we draw the chords on a different surface Σ with boundary and looking at their intersection graphs, we get other classes of graphs, which we call Σ-*diagram graphs*. Looking at chord diagrams in the crosscap / Möbius strip gives us *crosscap-diagram graphs*. This class of graphs is closed under pivot-minors and each crosscap-diagram graph can be obtained from some circle graph by performing a rank 1 pertubation. Looking at the chord diagrams in the annulus gives us *annulus-diagram graphs*. We can switch between the classes of crosscap-diagram graphs and annulus diagram graphs by performing a single local complementation (which again is a rank 1 pertubation). So the union of crosscap-diagram graphs and annulus-diagram graphs is closed under vertex-minors. Moreover, each annulus-diagram graphs can be obtained from circle graphs by performing a rank 2 pertubation.

37. Low Distortion Embeddings (Marc Distel)

For a map f from a metric space X to a metric space Y, let α, β be minimal such that for $u, v \in X$, $\frac{1}{\alpha}\mathrm{dist}_X(u,v) \leqslant \mathrm{dist}_Y(f(u), f(v)) \leqslant \beta\mathrm{dist}_X(u,v)$; the *distortion* of f is defined to be the product $\alpha\beta$. We then say that X can be embedded in Y with distortion at most $d := \alpha\beta$. For a given class of graphs $\mathcal{G}$, is there a constant d such that every graph in the class embeds into ℓ_1^∞, the infinite dimensional reals with the 1-norm, with distortion at most d? In particular, I ask if we can take $\mathcal{G}$ to be the class of all planar graphs of treewidth at most 3. The statement is true if $\mathcal{G}$ is the class of all graphs with treewidth at most 2 (`https://ieeexplore.ieee.org/document/4691008`) or any class with bounded pathwidth (https://arxiv.org/abs/1708.04073). The case where $\mathcal{G}$ is the class of all planar graphs is a long standing open problem from https://link.springer.com/article/10.1007/s00493-004-0015-x which my problem aims to work towards.