On recognition algorithms and structure of graphs with restricted induced cycles

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Abstract

We call an induced cycle of length at least four a hole. The parity of a hole is the parity of its length. Forbidding holes of certain types in a graph has deep structural implications. In 2006, Chudnovksy, Seymour, Robertson, and Thomas famously proved that a graph is perfect if and only if it does not contain an odd hole or a complement of an odd hole. In 2002, Conforti, Cornuéjols, Kapoor and Vuškovíc provided a structural description of the class of even-hole-free graphs. In Chapter 3, we provide a structural description of all graphs that contain only holes of length ℓ for every $\ell \geq 7$.

Analysis of how holes interact with graph structure has yielded detection algorithms for holes of various lengths and parities. In 1991, Bienstock showed it is NP-Hard to test whether a graph G has an even (or odd) hole containing a specified vertex $v \in V(G)$. In 2002, Conforti, Cornuéjols, Kapoor and Vuškovíc gave a polynomial-time algorithm to recognize even-hole-free graphs using their structure theorem. In 2003, Chudnovsky, Kawarabayashi and Seymour provided a simpler and slightly faster algorithm to test whether a graph contains an even hole. In 2019, Chudnovsky, Scott, Seymour and Spirkl provided a polynomial-time algorithm to test whether a graph contains an odd hole. Later that year, Chudnovsky, Scott and Seymour strengthened this result by providing a polynomial-time algorithm to test whether a graph contains an odd hole of length at least ℓ for any fixed integer $\ell \geq 5$. In Chapter 2, we provide a polynomial-time algorithm to test whether a graph contains an even hole of length at least ℓ for any fixed integer $\ell \geq 4$.

Contents

Chapter 1

Introduction

1.1 Prior Publication and Joint Work

All results presented in this thesis are joint work with my advisor Paul Seymour. Chapter 2 contains results that have been published on the arXiv as [30] and have been submitted to a journal. I have also presented the results of Chapter 2 at the Waterloo Graph Coloring Conference at University of Waterloo in Ontario, Canada on September 27, 2019, The New York State Regional Graduate Mathematics Conference on March 28, 2020 online and the Princeton Applied and Computational Math Graduate Student Seminar on September 17, 2019.

Chapter 3 is unpublished joint work with Paul Seymour and presents a structural description of graphs which are chordal or only have holes of some fixed length ℓ for some $\ell \geq 7$. When $\ell \geq 7$ and odd we provide a complete structural characterization. Another group consisting of Jake Horsfield, Myriam Preissmann, Ni Luh Dewi Sintiari, Cléophée Robin, Nicolas Trotignon and Kristina Vušković has been working on the same problem independently. They have proved a complete structural description for the case where $\ell \geq 7$ and odd in an as yet unpublished manuscript [40].

Section 1.2 gives generally known graph theory definitions. Sections 1.3 and 1.4 describe results related to the contents of this thesis.

1.2 Definitions

A *finite simple graph* is a pair (V, E) where V is a finite set of *vertices* and E consists of subsets of V of cardinality two called *edges*. We will assume all graphs we discuss are finite and simple.

Let G be a graph. We denote the set of edges of G as $E(G)$ and the set of vertices of G as $V(G)$.

We call $|V(G)|$ the *order* of G and denote it by $|G|$. We denote an edge $\{x, y\}$ by xy. If xy is an edge we say x is *adjacent* to y and we say x and y are *neighbors*. We say x and the edge xy are *incident* to each other. For $v \in V(G)$ the set of neighbors of v is denoted by $N_G(v)$ and is called the *neighborhood* of v. In cases where G is not ambiguous we simply denote the neighborhood of v by $N(v)$. For a set of vertices $S \subseteq V(G)$ we define $N(S)$ to be the set $\cup_{v\in S}\ N(v) \setminus S$. If G is a graph and S is subset of $V(G)$ such that every vertex in $V(G) \setminus S$ has a neighbor in S we call S a *dominating set* of G. We reserve the symbol G^c for the *complement* of G which is the graph defined as follows: $V(G) = V(G^c)$ and $E(G^c)$ satisfies the property that for every two distinct $x, y \in V(G)$, $xy \in E(G)$ if and only if $xy \notin E(G^c)$.

If G and H are graphs we call H a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G and if for every two distinct $x, y \in V(H)$, $xy \in E(G)$ if and only if $xy \in E(H)$ we say H is an *induced subgraph* of G. In other words H is an induced subgraph of G if it can be obtained from G by deleting vertices and any edges incident to deleted vertices. If H is an induced subgraph of G we say G *contains* H. In this case we say H is the subgraph of G induced by $V(H)$ and denote H by $G[V(H)]$. For $X \subseteq V(G)$ we denote the subgraph of G induced by $V(G) \setminus X$ by $G \setminus X$. If X consists of a single vertex x we will abbreviate this notation to $G \setminus x$.

We say a graph G and a graph H are isomorphic if there is a bijective function $f : V(G) \to V(H)$ satisfying $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. In this case we call f an isomorphism. We can now describe containment more formally. If H, G are graphs and G contains a subgraph isomorphic to H we say G contains H, otherwise we say G is H-free. If $\mathcal{H}$ is a set of graphs we say G is $\mathcal{H}$-free if G does not contain any graph in $\mathcal{H}$.

Let $k \geq 1$ be an integer. Let P be the graph consisting of a sequence of vertices $v_1, v_2, \ldots, v_k$ and edges between consecutive vertices. Then we say P is a *path* of length $k-1$ and denote it by P_k. We call v_1 and v_k the *ends* of P. We say $V(P) \setminus \{v_1, v_k\}$ is the *interior* of P and we denote it by P^*. Thus if P has length at most one, $P^* = \emptyset$. We denote P by v_1-v_2-$\ldots$-v_k or v_k-v_{k-1}-$\ldots$-v_1. We call the graph consisting of P and the edge v_1v_k a cycle of length k and denote it by v_1-v_2-$\ldots v_k$-v_1. We say the parity of a path or cycle is the parity of its length.

Let x and y be vertices. We call any path with ends x, y an xy-path. We say a graph G is connected if for every $x, y \in V(G)$, G contains some xy-path. We call C a connected component of a graph G if it is a maximal connected subgraph of G. For any x, y in the same connected component of a graph G we call the xy-path of minimum length a *shortest xy-path* and denote its length by $d_G(x, y)$. In cases where G is not ambiguous we will denote $d_G(x, y)$ by $d(x, y)$.

We call a graph *complete* if it contains all possible edges. We denote the complete graph on n

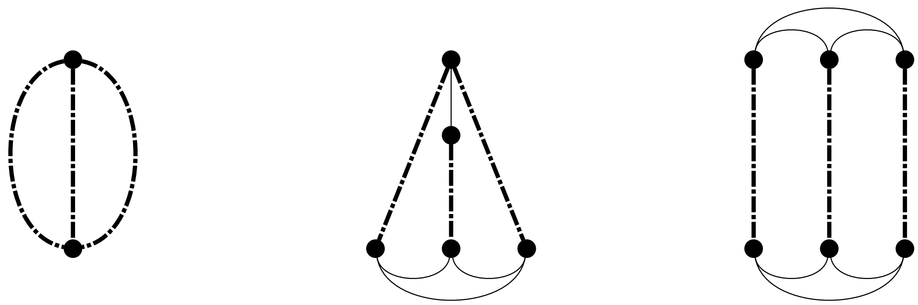

Figure 1.1: An illustration of a theta (left), pyramid (center) and prism (right). The thick dashed lines represent paths of length at least one.

vertices by K_n. We call a complete subgraph a *clique*. If G is a graph we say the *clique number* of G is the order of its largest clique and we denote it by $\omega(G)$. For we say $S \subseteq V(G)$ is a *stable set* if there are no edges between any two elements of S. We call the cardinality of the largest stable set in G the *stability number* of G and denote it by $\alpha(G)$. If $X, Y \subseteq V(G)$, we say X is *anticomplete* to Y if no vertex in X is equal or adjacent to a vertex in Y. We say X, Y are *complete* if X and Y are disjoint and x is adjacent to y for every $x \in X$ and $y \in Y$. Let G be a graph. If G is a connected graph and X is subset of $V(G)$ such that $G \setminus X$ is not connected we call X a *(vertex) cut-set*. If X is a cut-set of G and $G[X]$ is a clique we call X a *clique cut-set*. If the set consisting of a single vertex v is a cut-set we call v a *cut-vertex*. For every integer $k \geq 0$, we say a graph G is k-connected if $|V(G)| \geq k+1$ and G has no cut-set of cardinality k.

For any integer $n \geq 1$, we call an assignment $\phi : V(G) \rightarrow \{1, 2, 3 \ldots, n\}$ a *proper coloring using* n *colors* if for every $xy \in E(G)$, $\phi(x) \neq \phi(y)$. We call the smallest n for which G has a proper coloring using n colors the *chromatic number* of G and denote it by $\chi(G)$. A graph with chromatic number two is called a *bipartite* graph.

We will define few useful classes of graphs. A connected graph containing no cycles is called a *tree*. We call an induced cycle of length at least four a *hole* and we call the complement of a hole an *antihole*. A graph that does not contain any hole is called *chordal*.

A *theta* is a graph H consisting of two non-adjacent vertices u, v and three paths P_1, P_2, P_3 joining u, v with pairwise disjoint interiors. We call $\{u\}, \{v\}$ the terminating sets of H. A *prism* is a graph H consisting of two vertex disjoint triangles with vertex sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and three pairwise vertex-disjoint paths P_1, P_2, P_3, such that P_i has ends a_i and b_i for $i \in \{1, 2, 3\}$. We call $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ the *terminating sets of* H. A *pyramid* is a graph H consisting of a triangle with vertex set $\{a_1, a_2, a_3\}$ and a vertex r and three paths P_1, P_2, P_3 satisfying all of the following following:

- For each $i \in \{1, 2, 3\}$, P_i is an ra_i-path,
- At most one of P_1, P_2, P_3 has length one, and
- $P_1 \setminus a$, $P_2 \setminus a$, $P_3 \setminus a$ are pairwise vertex-disjoint paths.

We call r the *apex* of H and we call $\{a_1, a_2, a_3\}, \{r\}$ the *terminating sets of* H. If H is a theta, prism, or pyramid and P_1, P_2, P_3 are as in the definition of H we call P_1, P_2, P_3 the *constituent* paths of P. See Figure 1.1 for an illustration. For further background on graph theory see [32], [10] or [5].

1.2.1 Notation

For a positive integer k let $[k]$ denote the set $\{1, 2, \ldots, k\}$. For an integer $n \geq k$ let $[k, n]$ denote the set $\{k, k+1, k+2, \ldots, n\}$.

1.3 Overview

In this thesis we present two results related to what holes a graph contains. The first is a polynomial-time algorithm for every fixed integer $\ell \geq 4$ to determine whether an input graph contains a hole of length at least ℓ and is described in Chapter 2. The second is the subject of Chapter 3 and is a structural characterization of the class of graphs that do not have a hole of any length other than ℓ for any integer $\ell \geq 7$. On first glance these may seem to be only tangentially related results. However, the algorithm to detect even holes of length at least ℓ relies on a structural analysis of graphs that do not contain "obvious" even holes, such as even holes of length at most $2\ell + 2$. In this sense, both results are about describing the structure of graphs that do not contain holes of certain types ("graphs with restricted holes").

Forbidding holes of certain types in a graph has deep structural implications. A simple example of this is the fact that graphs are bipartite if and only if they do not contain an odd cycle. The most famous example of this is the class of perfect graphs. The spirit of perfect graphs is due to Tibor Gallai and his study of linear programming duality theory's applications to combinatorics [44]. A graph G is perfect if for every induced subgraph H of G, $\omega(H) = \chi(H)$. G is called *Berge* if G and G^c both do not contain any odd holes. In 1961, Claude Berge conjectured that a graph G is perfect if an only if G is Berge [4]. He also conjectured that a graph G is perfect if and only if its complement is perfect. After that, the study of perfect graphs became a field of its own, both due to their mathematical elegance and because of the two famous conjectures. The second conjecture was proven by Lovász in 1972 [49]. The first conjecture stayed open for over forty years and was

	Hole going through a predetermined vertex	Any length	A shortest hole	Length $\geq \ell$
Even	NP-Hard [7, 6]	$\mathcal{O}(\|G\|^9)$ [48]	$\mathcal{O}(\|G\|^{31})$ [14]	W[1]-hard [38] $\mathcal{O}(\|G\|^{9\ell+3})$ [30]
Odd	NP-Hard [7, 6]	$\mathcal{O}(\|G\|^8)$ [48]	$\mathcal{O}(\|G\|^{14})$ [19]	W[1]-hard [38] $\mathcal{O}(\|G\|^{20\ell+40})$ [18]
Any Parity	$\mathcal{O}(\|G\|^3)$	$\mathcal{O}(\|G\| + \|E(G)\|)$ [65, 64] [58, 59]	$\mathcal{O}(\|G\| * \|E(G)\|)$ [42]	NP-Hard $\mathcal{O}(\|G\|^{\ell})$

Table 1.1: A summary of algorithmic results for even/odd/general hole detection. The polynomial-time algorithm to determine whether a graph contains an even hole of length at least ℓ is joint work by Paul Seymour and me. It is described in Chapter 2 and has been submitted for journal publication as [30]. Section 1.4 contains a more detailed overview of the algorithmic results for detecting odd and even holes.

an active area of study. It was finally proved by Chudnovksy, Seymour, Robertson, and Thomas in 2006 and became known as the "strong perfect graph theorem". Their proof used a revolutionary technique of breaking graphs apart into more tractable pieces called graph decomposition [17].

The quest to prove the perfect graph theorem motivated the first major work on the structure of even-hole-free graphs. In 2002, Conforti, Cornuéjols, Kapoor and Vuškovíc provided a structural description of even-hole-free graphs using graph decomposition [27]. They then used this description to develop a polynomial-time algorithm to test whether a graph contains an even hole [28]. According Vuškovíc's survey paper on even-hole-free graphs [67], the work by Conforti et al. on even-hole-free graphs was motivated by the perfect graph conjecture. The group hoped that by studying even-hole-free graphs they would develop techniques that would be useful in the study of Berge graphs. The complement of any hole of length at least six contains a hole of length four. Thus, if G is even-hole-free, then C_5 is the only possible odd hole in G^c. This property makes even-hole-free graphs a reasonable proxy for Berge graphs.

Since the proof of the strong perfect graph theorem and the work on even-hole-free graphs by Conforti et al., many polynomial-time algorithms have been found detecting an even (or odd) hole in a graph [28, 21] and related problems such as finding a shortest even hole in an input graph [19] or finding an odd hole of length greater than some constant ℓ [18]. (See Table 1.1.) Chapter 2 provides a polynomial-time algorithm to determine whether an input graph contains an even hole of length greater than ℓ for some constant ℓ. Section 1.4 includes a survey of the main results and algorithms concerning even-hole-free graphs and odd-hole-free graphs. We have not yet examined the algorithmic implications of our structural description of monoholed graphs. However, I believe that the structural description of monoholed graphs in Chapter 3 could be exploited to answer algorithmic questions about monoholed graph as is done in the algorithm of Chapter 2. One particularly intriguing question is whether the class of monoholed graphs can be colored in polynomial-time. While coloring graphs is NP-Hard in general, Maffray, Penev, and Vušković gave a polynomial-time coloring algorithm for a subclass of monoholed graphs called "rings" [51] which we define in Subsection 1.4.1. Since rings are important in our structural description of ℓ-monoholed graphs it is possible some of the techniques of Maffray et al could be applied to the class of ℓ-monoholed graphs in general for integers $\ell \geq 7$.

1.4 Related Work

Both the results of Chapter 2 and Chapter 3 concern themselves with the structure of graphs with restricted holes. In this section we summarize some of the structural results and recognition algorithms for graphs with restricted holes.

1.4.1 Structural results

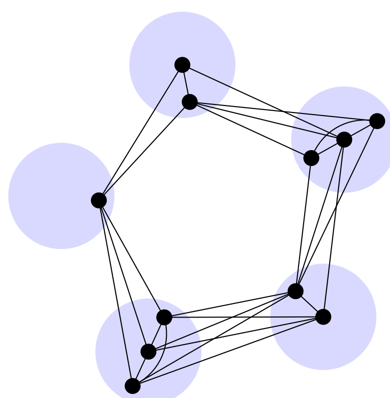

Figure 1.2: An example of a ring on five sets.

One of the most relevant areas to this thesis is the study of rings. Let $k \geq 3$ be an integer. We say a graph G is a *ring on k sets* if $V(G)$ can be partitioned into k sets $X_0, X_2, \ldots, X_{k-1}$ satisfying the following:

- $G[X_0], G[X_1], \ldots, G[X_{k-1}]$ are all cliques,

- For each $i \in \{0, 1, 2, \ldots, k-1\}$ and $x, x' \in X_i$, $N(x) \subseteq N(x')$ or $N(x') \subseteq N(x)$,

- For each $i \in \{0, 1, 2, \ldots, k-1\}$ and $x \in X_i$, $N(x) \subseteq X_{i-1} \cup X_i \cup X_{i+1}$ (where subscripts are taken modulo k) and
- For each $i \in \{0, 1, 2, \ldots, k-1\}$ there is some $x \in X_i$ such that x is complete to both X_{i-1} and X_{i+1}.

(See Figure 1.2 for an illustration.) Note that in Maffray, Penev, and Vušković's paper on coloring rings, k is assumed to be at least four. By definition, if G is a ring on k sets, then G does not contain a clique cut-set and every hole in G has length k. In Chapter 3, we show that if G is a ℓ-monoholed graph for some $\ell \geq 7$ then one of the following outcomes holds: G contains a clique cut-set, G contains a vertex v that is adjacent to every other vertex in G, G is chordal, G is a ring or G is a type of graph we call a "crowned k-corpus". Thus rings are one of the basic structural outcomes of our description of ℓ-monoholed graphs for $\ell \geq 7$. Interestingly, the study of rings originated because rings were also one of the basic structural outcomes in Boncompagni, Penev and Vušković's characterization graphs without certain types of thetas, prisms, pyramids and wheels as induced subgraphs [9].[1]

Hoàng and Trotignon construct rings on k sets with "unbounded rank-width" for any fixed integer $k \geq 3$ in forthcoming work [39]. Very informally, the rank-width of a graph is a way of measuring how complicated it is. Rank-width was introduced by Sang-il Oum and Paul Seymour in [56]. Many problems that are NP-complete in general are polynomial-time for the class of graphs with bounded rank-width such as deciding whether a graph has chromatic number at most some constant [47] or determining whether a graph has a Hamiltonian cycle [69]. Subjectively, graph classes with unbounded rank-width, like the class of monoholed graphs, can be argued to be more "interesting" than those with bounded rank-width.

While the perfect graph theorem is the most famous result about the structure of graphs with restricted holes, there are several other notable results that relate to major open questions in graph theory. A key question in structural graph theory concerns the induced subgraphs of graphs with large chromatic number. We need the following definitions. We call a subset $\mathcal{F}$ of the set of all graphs an *ideal* if $\mathcal{F}$ is closed under taking induced subgraphs. We say an ideal $\mathcal{F}$ is *χ-bounded* if there exists some function f such that for every $G \in \mathcal{F}$, $\chi(G) \leq f(\omega(G))$ and we call f the *χ-bounding*

[1] In the context of [9] a wheel is a graph H consisting of a cycle C and another vertex v adjacent to at least three elements of $V(C)$. H is called a universal wheel if v is complete to $V(C)$ and H is called a twin wheel v if $N(v) \cap V(C)$ consists of three consecutive vertices in $V(C)$. A wheel that is neither a twin wheel nor a universal wheel is called a proper wheel. Boncompagni et al. provide a characterization of graphs that contain no theta, pyramid, prism or proper wheels. Prisms, pyramids, thetas, and wheels are called Truemper configurations because they are important in a theorem by Truemper [66] that characterizes the graphs for which the edges can be labeled with integers in such a way that the sum of the labels on every induced cycle C has a prescribed parity $f(C)$ for any assignment f of parities to cycles. See [67] for a survey.

function of $\mathcal{F}$. Trivially, $\chi(G) \geq \omega(G)$ for every graph G. The set of all graphs is not χ-bounded. In fact, there exist triangle-free graphs with arbitrarily large chromatic number [71] [53].

Many of the results on χ-boundedness have to do with graphs with restricted holes. Erdős famously proved that for any cycle C, the ideal of C-free graphs is not χ-bounded [34]. But what about $\mathcal{C}$-free graphs where $\mathcal{C}$ is an infinite family of cycles? Berge graphs are χ-bounded by the strong perfect graph theorem. In 2016, Scott and Seymour showed that odd-hole-free graphs are χ-bounded with χ-bounding function $f(\kappa) = 2^{2^{\kappa}+2}$ [61], proving conjecture of Gyarfàs and Sumner. Addario-Berry, Chudnovsky, Havet, Reed and Seymour showed that every even-hole-free graph G contains a vertex whose neighborhood consists of the vertex set of the union of two cliques and thus $\chi(G) \leq 2\omega(G)$ [2, 24]. Bonamy, Charbit and Thomassé proved that every graph with sufficiently large chromatic number contains an induced cycle of length $0 \mod 3$ [8], answering a question of Kalai and Meshulam. Scott and Seymour later proved that for any $p \geq 0$ and $q \geq 1$ the ideal of graphs with no induced cycle of length $p \mod q$ is χ-bounded [62]. Thus, the class of ℓ-monoholed graphs for any fixed ℓ is χ-bounded. Maffray, Penev and Vušković give the optimal χ-bounding function for the class of rings with at least four sets in [51]. Gyarfàs and Sumner conjectured that for any $\ell \geq 0$ the ideal of graphs not containing any hole of length greater than ℓ is χ-bounded and they conjectured that the ideal of graphs not containing any odd hole of length greater than ℓ is χ-bounded in [23]. The first conjecture was proven by Chudnovsky, Scott, and Seymour in [20] and the second stronger conjecture was proven by Chudnovsky, Scott, Seymour and Spirkl in [23]. See the survey by Scott and Seymour for more background on χ-boundedness [60].

The Erdős-Hajnal conjecture states that for every graph H there exists an $\epsilon > 0$ such that every H-free graph G has a stable set or clique of cardinality at least $|G|^{\epsilon}$. This one of the most active open questions in structural graph theory. Recently, Chudnovsky, Scott, Seymour and Spirkl proved that the Erdős-Hajnal conjecture holds when H is a hole of length five [22].

1.4.2 Detecting (odd, even) Holes

The main result of Chapter 2 is an algorithm to determine whether an input graph G contains a hole of length at least ℓ and even for some fixed $\ell \geq 4$. In this section we will give an overview of prior work on problems related to detecting holes of specific parities. A summary of the results is given in Table 1.1.

In 1991, Bienstock proved that it is NP-hard to determine whether G contains a even (or odd) hole going through a specified vertex [7, 6], answering a question raised by Bruce Shepherd. Maffray and Trotignon extended this result to show that the problem remains NP-hard when we only consider triangle-free graphs as inputs. [50]. Note that it is trivial to test whether an input G contains a hole through a specified vertex v in time $\mathcal{O}(|G|^3)$: We enumerate all pairs of non-adjacent vertices $x, y \in N(v)$ and test if $G \setminus (N(v) \setminus \{x, y\})$ contains an xy-path (e.g. by using breath first search).

In 2002, Conforti, Cornuéjols, Kapoor and Vuškovíc [28] gave an approximately $\mathcal{O}(|G|^{40})$ algorithm to test whether a graph contains an even hole by their using their structural decomposition theorem from [27]. In 2003, Chudnovsky, Kawarabayashi, and Seymour [16] provided a simpler algorithm that searches for even holes without the use of a structural decomposition theorem for even-hole-free graphs in time $\mathcal{O}(|G|^{31})$. In forthcoming work [14], Cheong and Lu show that techniques of [16] can be used to find the shortest even hole in an input graph G or determine that G is even-hole-free in time $\mathcal{O}(|G|^{31})$.

Significantly faster algorithms have been found using decomposition theorems for even-hole-free graphs based on [27]. In 2008, da Silva and Vuškovíc published a strengthening of the decomposition theorem of [27] along with an algorithm using the new decomposition theorem to test whether a graph is even-hole-free in time $\mathcal{O}(|G|^{19})$ [31]. In 2015, Chang and Lu [12] gave an $\mathcal{O}(|G|^{11})$ algorithm to determine whether a graph contains an even hole using the decomposition theorem of [31]. Lai, Lu and Thorup improved this running time to $\mathcal{O}(|G|^{9})$ in 2020 [48] by modifying the algorithm of [12] to improve the running-time of its subroutines.

Detecting an odd hole remained open until 2020 when Chudnovksy, Scott, Seymour and Spirkl provided an algorithm to detect an odd hole in G in time $\mathcal{O}(|G|^{9})$ [21]. In 2020, Lai, Lu and Thorup improved this running time to $\mathcal{O}(|G|^{8})$ [48]. In the same year, Chudnovsky, Scott, and Seymour [19] gave an algorithm that determines whether a graph G has an odd hole and returns the minimum length of an odd hole in G if one exists in time $\mathcal{O}(|G|^{14})$.

Chudnovsky, Scott and Seymour give a $\mathcal{O}(|G|^{20\ell+40})$ algorithm to test whether G contains an odd hole of length at least ℓ, where $\ell \geq 5$ is given as a constant, in 2019 [18]. Paul Seymour and I give an algorithm to test whether G contains an even hole of length at least ℓ in time $\mathcal{O}(|G|^{9\ell+3})$ in forthcoming work [30]. Chapter 2 describes a variant of this algorithm.

It would be nice for long even hole (and long odd hole) detection to remain polynomial-time when ℓ is considered to be part of the input rather than a constant. Unfortunately, this seems highly unlikely. Sepehr Hajebi provided a proof in private communication [38] that detecting long holes with specific residues is W[1]-hard and thus not fixed parameter tractable unless the central conjecture of parameterized complexity theory is false. More precisely, for all integers r, m with $m \geq 2$ and $0 < r \leq m$ if there was an algorithm that on input G, ℓ determined whether G contains a hole C of length at length at least ℓ and $|E(C)| \cong r \mod m$ in time $\mathcal{O}(f(\ell) * p(|G|))$ where f is some computable function and p is a polynomial, then the central conjecture of parameterized complexity theory would be false.

1.4.3 Other related algorithmic results

Prior to the discovery of the first odd-hole detection algorithm by Chudnovsky et al. in 2020, polynomial-time algorithms had been found for detecting odd holes in certain restricted graph classes. In 1987, Hsu presented an algorithm for detecting odd holes in planar graphs in time $\mathcal{O}(|G|^3)$ [41]. In 2009, Schrem, Stern and Golumbic provided an algorithm for detecting odd holes in claw-free[2] graphs in time $\mathcal{O}(|G| * |E(G)|^2)$ using an approach based on breadth-first-search [63]. This result was improved four years later when Kennedy and King provided an algorithm to detect odd holes in claw-free graphs in time $\mathcal{O}(|E(G)|^2 + |G|^2 \log(|G|))$ [46] using structural results of Fouquet [36] and Chudnovsky and Seymour [25]. In 2006, Conforti, Cornuéjols, Liu, Xinming, Vušković, and Zambelli provided a polynomial-time algorithm to test for odd holes in graphs with bounded clique number [29].

Porto provided an algorithm to test whether a planar graph G is even-hole-free in time $\mathcal{O}(|G|^3)$ [57]. Itah and Rodeh provided an $\mathcal{O}(|G||E(G)|)$ alogrithm to find the girth of a planar graph [43] in 1978. This result was subsequently improved by Djidev [33], by the min cut algorithm of Chalermsook, Fakcharoenphol and Nanongkai [11] and by Weimann and Yuster [70]. Chang and Lu provided a linear time algorithm to determine the girth of a planar graph in 2011 [13]. The complexity of determining whether G contains a hole of length at least five is $\mathcal{O}(|E(G)|^2 + |G|)$ by an algorithm of Nikolopoulos, and Palios [54, 55].

In his 1992 paper Bienstock also showed that it is NP-hard to determine whether an input graph contains a hole through two prespecified vertices [7]. However, when the problem is restricted to planar graphs it becomes solvable in polynomial-time: In fact, for any fixed integer $k \geq 0$, Kawarabayashi and Kobayashi give a linear time algorithm to test whether a planar graph contains a hole going through k fixed vertices. Moreover, for every $\epsilon \geq 0$ they provide an algorithm that on input a planar graph G and $L \subseteq V(G)$ of cardinality $o((\frac{\log |G|}{\log \log |G|})^{2/3})$ tests whether G contains a hole going through every vertex in L in time $\mathcal{O}(|G|^{2+\epsilon})$ [45].

The previous results were all concerned with algorithms to decide whether G contained various types of cycles as an induced subgraph. There has also been work on algorithms to decide whether

[2]The claw is the graph consisting of four vertices v_1, v_2, v_3, v_4 such that v_2, v_3, v_4 are pairwise non-adjacent and v_1 is adjacent to each of v_2, v_3, v_4.

G has a cycle of length k as subgraph for some input k, to find a cycle of length k if one exists and to count the number of cycles of length k in G. For instance, Alon, Yuster and Zwick present several results of this type in [3].

Chapter 2

Detecting a Long Even Hole

2.1 Technical Overview

The main result of this chapter is the following:

Theorem 2.1.1. *For each integer $\ell \geq 4$, there is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *Decides whether G has an even hole of length at least ℓ.*

Running time: $\mathcal{O}(|G|^{108\ell-22})$

Our algorithm combines approaches described in [16] and [18].The new algorithm uses a technique called "cleaning", as do the algorithms of [16],[18] and many other algorithms to detect induced subgraphs. We test for the existence of long even holes directly as is done in the algorithm of [16].

Here is an outline of the method. Throughout this paper $\ell \geq 4$ is a fixed number and a *long* hole or path is a hole or path of length at least ℓ. If C is a hole in G, a vertex v of $V(G) \setminus V(C)$ is *C-major* if there is no subpath of C of length three containing all neighbors of v in $V(C)$. A hole C is *clean* if it has no C-major vertex.

- First, we test for the presence in the input graph G of certain kinds of induced subgraphs ("short" long even holes, "long jewels of bounded order", "long thetas", "long prisms") that are detectable in polynomial time and whose presence would imply that G contains a long even hole. We call these kinds of subgraphs "easily-detected configurations." We may assume these tests are unsuccessful. Then, we run an algorithm that outputs either that G contains no shortest long even hole such that $E(G) \setminus E(C)$ includes a specific type of edge we call "major" or that G contains a long even hole. We may assume it outputs that G contains no shortest long even hole with a major edge.

- Second, we generate a *cleaning list*, a list of polynomially many subsets of $V(G)$ such that if C is a long even hole of minimum length in G (*a shortest long even hole*) then for some set X in the list, X contains every C-major vertex and no vertex of C. This process depends on the absence of easily-detected configurations and major edges.

- Third, for every X in our cleaning list we check whether $G \setminus X$ contains a clean shortest long even hole. This depends on the absence of easily-detected configurations, major edges, and major vertices.

We remark that we are calling long prisms easily detectable configurations, because they are detectable in polynomial time in graphs without long thetas as an induced subgraph. However for a general graph G, deciding whether G contains a long prism is NP-complete; Maffray and Trotignon's proof [50] that deciding whether G contains a prism is NP-complete can easily be adjusted to prove that deciding whether G contains a long prism is NP-complete. We are able to detect long thetas by invoking the "three-in-a-tree" algorithm given in [26]. The detection of long prisms makes up the bulk of what is novel in this paper and is the computationally most expensive step of our algorithm.

The approach of determining whether G contains an even hole by first testing whether G contains a prism or a theta was outlined in [16]. Moreover, Chudnovsky and Kapadia gave an algorithm to decide whether G contains a theta or a prism in [15]. Their algorithm does not directly translate to long theta and long prism detection, but we were able to use a similar algorithm structure to detect long prisms when G contains no long theta. Finally, when G has no easily detectable configurations, we detect a clean shortest long even hole C by guessing three evenly spaced vertices along C and taking shortest paths between them as in [18].

2.2 The easily-detected configurations

We begin with a test for what we call "short" long even holes:

Theorem 2.2.1. *For each integer $k \geq \ell$, there is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *Decides whether G has a long even hole of length at most k.*

Running Time: $\mathcal{O}(|G|^k)$.

Proof. We enumerate all vertex sets of size $\ell, \ell+1, \dots, k$ and for each one, check whether it induces a long even hole. □

We need the following easily-detected configuration of [18]: Let $u, v \in V(G)$ and let Q_1, Q_2 be induced paths between u, v of different parity. Let P be an induced path between u, v of length at least ℓ, such that P^* is disjoint from and anticomplete to Q_1^*, Q_2^*. We say the subgraph induced on $V(P \cup Q_1 \cup Q_2)$ is a *long jewel of order* $\max(|V(Q_1)|, |V(Q_2)|)$ *formed by* Q_1, Q_2, P. Any graph containing a long jewel has a long even hole, since the holes $P \cup Q_1$ and $P \cup Q_2$ are both long holes and one of them is even.

We need the following easy result given as Theorem 2.2 of [18].

Theorem 2.2.2. *There is an algorithm with the following specifications.*

Input: *A graph G and an integer $k \geq 0$.*

Output: *Decides whether G has a long jewel of order at most k.*

Running Time: $\mathcal{O}(|G|^{2k+\ell})$.

A *theta* is a graph consisting of two non-adjacent vertices u, v and three paths P_1, P_2, P_3 joining u, v with pairwise disjoint interiors and we say P_1, P_2, P_3 *form* a theta. A *long theta* is a theta where at most one of the three paths that form it has length less than ℓ. If G contains a long theta it contains a long hole because for every distinct $i, j \in \{1, 2, 3\}$, $V(P_i) \cup V(P_j)$ induces a long hole and at least two of P_1, P_2, P_3 must have the same parity. We use the "three-in-a-tree" algorithm given as the main result of [26] to detect long thetas:

Theorem 2.2.3. *There is an algorithm with the following specifications:*

Input: *A graph G and three vertices v_1, v_2, v_3 of G.*

Output: *Decides whether there is an induced subgraph T of G with $v_1, v_2, v_3 \in V(T)$ such that T is a tree.*

Running Time: $\mathcal{O}(|G|^4)$.

Chudnovsky and Seymour's algorithm in [26] to detect a theta in a graph G can easily be adjusted to detect a long theta:

Theorem 2.2.4. *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *Decides whether G contains a long theta.*

Running Time: $\mathcal{O}(|G|^{2\ell+7})$.

Proof. Our algorithm is as follows. For every $v \in V(G)$, we enumerate all sets of three paths P_1, P_2, P_3 starting at v of lengths $2, \ell$ and ℓ, respectively, such that P_i^* is disjoint from and anticomplete to P_j^* and $V(P_i) \cup V(P_j)$ induces a path or a cycle for any distinct $i, j \in \{1, 2, 3\}$. For each $i \in \{1, 2, 3\}$ denote the end of P_i that is not equal to v by w_i. We construct G', the graph $G \setminus (N(P_1^* \cup P_2^* \cup P_3^* \cup \{v\}) \setminus \{w_1, w_2, w_3\})$.

- If $w_1 = w_2 = w_3$, we output that G contains a long theta.
- If $w_i = w_j$ and $w_k \neq w_i$ for some distinct $i, j, k \in \{1, 2, 3\}$, we perform the following: we test whether G' contains a $w_i w_k$-path. If it does, we output that G contains a long theta.
- If w_1, w_2, w_3 are three distinct vertices we perform the following: we test whether G' contains a tree T with $w_1, w_2, w_3 \in V(T)$. If it does, we output that G contains a long theta.

If there are no more choices of v, P_1, P_2, P_3 remaining we output that G has no long theta.

Correctness follows from the fact that G has a long theta if and only if for some choice of v, P_1, P_2, P_3, the graph G' contains some tree T with $w_1, w_2, w_3 \in V(T)$. There are $\mathcal{O}(|G|^{2\ell+3})$ choices for v, P_1, P_2, P_3, so the total running time is $\mathcal{O}(|G|^{2\ell+7})$. □

Lai, Lu and Thorup provide a faster algorithm for the three-in-a-tree problem in [48]. Using their $\mathcal{O}(|E(G)|(\log|G|)^2)$ algorithm we can reduce the running time for detecting a long theta to $\mathcal{O}(|G|^{2\ell+5}(\log|G|)^2)$. This improvement does not affect the asymptotic running time of our long even holes detection algorithm.

For brevity, it is convenient to describe enumerating all subgraphs of a certain type as "guessing" subgraphs of that type. In this language the three-in-a-tree algorithm can be written as follows: We guess the paths P_1, P_2 and P_3 and test whether w_1, w_2, w_3 are contained in some induced tree of G'.

A *prism* is a graph consisting of two vertex-disjoint triangles on $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and three pairwise vertex-disjoint paths P_1, P_2, P_3 such that P_i has ends a_i and b_i for any $i \in \{1, 2, 3\}$. A *long prism* is a graph K consisting of two triangles on $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ called *bases* and three pairwise vertex-disjoint paths P_1, P_2, P_3 such that all of the following conditions hold:

- The bases of K are vertex disjoint or $a_i = b_i$ for exactly one $i \in \{1, 2, 3\}$.

- For every $i \in \{1,2,3\}$, P_i is has ends a_i and b_i.
- At most one of P_1, P_2, P_3 has length less than ℓ.

We call P_1, P_2, P_3 the *constituent paths of* K. The next section will describe our algorithm to test whether G contains a long prism when G contains no long thetas.

2.3 Long prisms

We call a path P with ends x, y an *xy-path.* For $a, b \in V(P)$, we denote the subpath of P with ends a, b by aPb. Let P, Q be vertex disjoint paths. Let p be an end of P and let q be an end of Q. If p is adjacent to q we denote the path whose edge set consists of $E(P) \cup E(Q) \cup \{p, q\}$ by P-Q. For a graph G and $x, y \in V(G)$, we call the length of a shortest xy-path in G the *G-distance* between x and y and denote it by $d_G(x, y)$. For a vertex v we denote the set of its neighbors as $N(v)$. For a set of vertices $S \subseteq V(G)$ we define $N(S)$ to be the set $\cup_{v \in S} N(v) \setminus S$.

Theorem 2.3.1. *For each integer $\ell \geq 4$, there is an algorithm with the following specifications:*

Input: *A graph G containing no long theta.*

Output: *Decides whether G contains a long prism.*

Running Time: $\mathcal{O}(|G|^{108\ell-22})$.

We will use the same notation as in the definition of long prisms. The outline of the algorithm is similar to that of [15]. For a constituent path P_i of K we define P_i', P_i'' to be the subpaths of P_i whose vertex sets consist of all vertices with P_i-distance at most $\ell - 1$ from a_i, b_i, respectively. Thus P_i' and P_i'' each have one end equal to a_i, b_i, respectively. We denote the other ends of P_i' and P_i'' by s_i and t_i respectively. We define a *frame* F of a long prism K to be the graph obtained by taking the union of the following graphs:

- The triangle induced by $\{a_1, a_2, a_3\}$.
- The triangle induced by $\{b_1, b_2, b_3\}$.
- The paths P_i', P_i'' for each $i \in \{1,2,3\}$.

Note that for every $i \in \{1,2,3\}$ if P_i has length less than $2\ell - 3$, then $P_i \subseteq F$. We call the frame of K *tidy* if no vertex in the frame has a neighbor in $V(G) \setminus V(K)$ except for vertices equal to s_i or t_i for some $i \in \{1,2,3\}$ such that $P_i \not\subseteq F$. We call a long prism K' *shorter* than a long prism

K, if $|V(K')| < |V(K)|$. Let K be a shortest long prism in G with paths P_1, P_2, P_3. Let u, v be distinct, non-adjacent vertices in $V(P_i)$ for some $i \in \{1, 2, 3\}$. We call a shortest uv-path Q *good* if no vertex of Q^* has neighbors in $V(K) \setminus (V(uP_iv) \setminus \{u, v\})$. If Q is not good it is *bad*. Hence, for any $i \in \{1, 2, 3\}$, if $P_i \subseteq F$ and F is tidy, then $s_iP_it_i$ is the unique s_it_i-path and thus all shortest s_it_i-paths are good.

Suppose all shortest s_it_i-paths are good for each $i \in \{1, 2, 3\}$. Then the problem becomes easy because of the following: we first guess the frame F of K. If $V(P_1) \subseteq F$, we set Q_1 to be the empty graph. Otherwise, we set Q_1 to be a shortest s_1t_1-path. We delete all vertices not in $V(F) \cup V(Q_1)$ with neighbors in Q_1^*. Since Q_1 is good we have not deleted any vertex of $V(K) \setminus V(P_1)$. We repeat this process on P_2, P_3 to obtain Q_2, Q_3. Then, $V(F) \cup_{i=1}^{3} V(Q_i)$ induces a shortest long prism.

We call a vertex $q \in V(G) \setminus V(K)$ K-*major* if there is no three-vertex subpath of K containing all neighbors of q in $V(K)$. In order to arrange that all shortest s_it_i-paths are good we generate a "path-cleaning" list of polynomially many sets of vertices such that for some X in the list X is disjoint from K and X contains a vertex from every bad shortest s_it_i-path. This process is described in Subsection 2.3.1.

In order to generate this cleaning list we require that G contains no K-major vertices. Thus we need another phase of cleaning where we generate a cleaning list of polynomially many sets of vertices such that for some X in the list X is disjoint from K and X contains all K-major vertices. This phase is described in Subsection 2.3.2.

2.3.1 Path-cleaning

We use the notation from the definition of prism and frame throughout this section. Let $i \in \{1, 2, 3\}$, let u, v be non-adjacent vertices in $V(P_i)$, and suppose a_i, u, v, b_i occur in order along P_i. For a bad shortest uv-path Q we define ζ_Q to be the vertex in Q^* with minimum Q-distance to v with a neighbor in $V(K) \setminus V(uP_iv)$.

In this section we will provide a cleaning algorithm that generates a list of polynomially many sets of vertices such that for any shortest long prism K with a tidy frame and vertices u, v in the same constituent path of K for some X in the list, X contains a ζ_Q for every bad shortest uv-path Q for which ζ_Q is not K-major and $X \cap V(K) = \emptyset$. This algorithm depends on G containing no long thetas. We will apply this algorithm twice, once to help clean major vertices and once to clean a vertex from all bad shortest s_it_i-paths for each $i \in \{1, 2, 3\}$.

For a vertex q with a neighbor in $V(P_i)$ for some $i \in \{1, 2, 3\}$, let $\alpha_i(q)$ denote the element of

$N(q) \cap V(P_i)$ with minimum P_i-distance to a_i. Similarly, let $\beta_i(q)$ denote the element of $N(q) \cap V(P_i)$ with minimum P_i-distance to b_i. If F is tidy, it follows that $q \notin V(F)$ and the paths $a_i P_i \alpha_i(q)$ and $b_i P_i \beta_i(q)$ both have length at least ℓ.

We need the following lemma:

Lemma 2.3.2. *Let G be a graph without long thetas and K be a shortest long prism in G. Let P_1, P_2, P_3 be the constituent paths of K. Suppose K has a tidy frame. Let u, v be distinct non-adjacent vertices in $V(P_1)$. Suppose Q, Q' are bad shortest uv-paths such that ζ_Q, $\zeta_{Q'}$ each have neighbors in $V(P_2)$ and are not K-major. Then there is a $\zeta_Q\zeta_{Q'}$-path of length at most $\ell + 1$ with interior contained in $V(P_2)$.*

Proof. Suppose not. Then the P_2-distance between any neighbor of ζ_Q in $V(P_2)$ and any neighbor of $\zeta_{Q'}$ in $V(P_2)$ is at least ℓ. Without loss of generality suppose a_1, u, v, b_2 occur in order along P_1.

(1) *The set of neighbors of ζ_Q in $V(K) \setminus V(uP_1v)$ does not consist of exactly two adjacent vertices. The same statement holds for $\zeta_{Q'}$.*

Suppose ζ_Q has exactly two neighbors y_1, y_2 in $V(P_2)$ and they are adjacent. Then we obtain a shorter long prism induced by $(V(K) \setminus V(a_1P_1v)) \cup V(\zeta_Q Q v)$, a contradiction. This proves (1).

(2) *There is an induced $\zeta_Q\zeta_{Q'}$-path W disjoint from $V(K) \setminus V(uP_1v)$ of length greater than one.*

There is a $\zeta_Q\zeta_{Q'}$-path $W \subseteq \zeta_Q Q v \cup v Q' \zeta_{Q'}$, so we only need to show that W is not a single edge. Suppose that ζ_Q and ζ_Q' are adjacent. Without loss of generality suppose that $a_2, \alpha_2(\zeta_Q), \alpha_2(\zeta_{Q'}), b_2$ occur in order on P_2. Let K' denote the graph from K obtained by replacing the path $\alpha_2(\zeta_Q)P_2\beta_2(\zeta_{Q'})$ with $\alpha_2(\zeta_Q)$-ζ_Q-$\zeta_{Q'}$-$\beta_2(\zeta_{Q'})$. Since K has a tidy frame, K' is a long prism and it is shorter than K. Since $\zeta_Q, \zeta_{Q'}$ are not K-major, G contains K' as an induced subgraph, a contradiction. This proves (2).

Since ζ_Q is not K-major, ζ_Q has at most one neighbor in $V(K)$ not equal to $\alpha_2(\zeta_Q)$, $\beta_2(\zeta_Q)$. Similarly, $\zeta_{Q'}$ has at most one neighbor in $V(K)$ not equal to $\alpha_2(\zeta_{Q'}), \beta_2(\zeta_{Q'})$. Let $H = ((N(\zeta_Q) \cup N(\zeta_{Q'})) \cap V(K)) \setminus \{\alpha_2(\zeta_Q), \alpha_2(\zeta_{Q'}, \beta_2(\zeta_Q), \beta_2(\zeta_{Q'}\}$. Then $V(W) \cup V(P_2) \cup V(P_3) \setminus H$ induces a long theta, a contradiction. □

Theorem 2.3.3. *There is an algorithm with the following specifications:*

Input: *A graph G containing no long theta.*

Output: *A list of $\mathcal{O}(|G|^{4\ell+2})$ subsets of $V(G)$ with the following property: for every shortest long prism K, if K has a tidy frame and u, v are distinct non-adjacent vertices in the same constituent path of K then there is a set X in the list such that:*

- *X is disjoint from $V(K)$ and*
- *$X \cap V(Q) \neq \emptyset$ for every bad shortest uv-path Q such that ζ_Q is not K-major.*

Running Time: $\mathcal{O}(|G|^{4\ell+3})$.

Proof. The algorithm is as follows:

We enumerate all pairs (R_2, R_3) of disjoint paths of length at most 2ℓ. For each $i \in \{2, 3\}$, we set X_i to be the set of all vertices in $V(G) \setminus V(R_i)$ with neighbors in R_i^*. We output $X_2 \cup X_3$ and move on to the next pair of paths.

Let K be a shortest long prism with constituent paths P_1, P_2, P_3. Suppose K has a tidy frame and that $u, v \in V(P_1)$. We claim there is a choice of (R_2, R_3) such that $X_1 \cup X_2$ is disjoint from $V(K)$ and $X_1 \cup X_2$ contains ζ_Q for every bad shortest uv-path Q such that ζ_Q is not K-major. Suppose there is a bad uv-path Q such that ζ_Q is not K-major and ζ_Q has a neighbor in $V(P_2)$. Then for some choice of R_2, we have that $R_2 \subseteq P_2$ and R_2^* contains all vertices with P_2-distance at most $\ell - 1$ from some neighbor of ζ_Q in $V(P_2)$. Hence, $\zeta_Q \in X_2$. By Lemma 2.3.2, $\zeta_S \in X_2$ for every bad shortest v_1v_2-path S such that ζ_S is not K-major and ζ_S has a neighbor in $V(P_2)$. By construction, X_2 is disjoint from $V(K)$. Since the case where ζ_Q has a neighbor on P_3 is symmetric this completes the proof of correctness.

There are $\mathcal{O}(|G|^{4\ell+2})$ possibilities for (R_1, R_2) and constructing X_1 and X_2 takes $\mathcal{O}(|G|)$ so the running time and list length are as claimed. □

Corollary 2.3.4. *There is an algorithm with the following specifications:*

Input: *A graph G containing no long theta.*

Output: *A list of $\mathcal{O}(|G|^{12\ell+6})$ subsets of $V(G)$ with the following property: If K is a shortest long prism, K has a tidy frame and there are no K-major vertices, then there is a set X in the list such that:*

- *X is disjoint from $V(K)$ and*
- *X contains a vertex of every bad shortest s_it_i-path for each $i \in \{1, 2, 3\}$.*

Running Time $\mathcal{O}(|G|^{12\ell+6})$.

Proof. We run the algorithm of Theorem 2.3.3 on input G to generate a list $\mathcal{L}$. For each choice of $X, Y, Z \in \mathcal{L}$, we output $X \cup Y \cup Z$. □

2.3.2 Major Vertices on Prisms

For a vertex $x \notin V(K)$ with a neighbor in $V(P_i)$ for some $i \in \{1, 2, 3\}$ we denote by $A_i(x)$, $B_i(x)$, the paths $\alpha_i(x)P_ia_i$ and $\beta_i(x)P_ib_i$, respectively.

Lemma 2.3.5. *Let K be a shortest long prism in G with constituent paths P_1, P_2, P_3. Suppose K has a tidy frame. If q is a K-major vertex, then q has neighbors in at least two different constituent paths of K.*

Proof. Suppose all neighbors of q in $V(K)$ are contained in $V(P_1)$. Then $\alpha_1(q)P_1\beta_1(q)$ has length strictly greater than two. We obtain a shorter prism by replacing $\alpha_1(q)P_1\beta_1(q)$ in P_1 with the path $\alpha_1(q)q\beta_1(q)$. Since K' has the same frame as K, it follows that K' is a long prism, a contradiction. □

Lemma 2.3.6. *Let K be a shortest long prism. Suppose K has a tidy frame. If q is a K-major vertex, then q has three pairwise non-adjacent neighbors in $V(K)$.*

Proof. Suppose not. By Lemma 2.3.5, we may assume q has a neighbor in $V(P_1)$ and in $V(P_2)$. Since K has a tidy frame, we may assume q has no neighbors in $V(P_3)$. If q has exactly one neighbor in $V(P_1)$ and exactly one neighbor in $V(P_2)$, then $V(P_1) \cup V(P_2) \cup \{q\}$ induces a long theta. So we may assume q has exactly two neighbors in $V(P_1)$ and that they are adjacent. Then we obtain a shorter long prism with bases $\{a_1, a_2, a_3\}$ and $\{\alpha_1(q), \beta_1(q), q\}$ and constituent paths $A_1(q)$, $A_2(q)$ and $B_1(q)$-P_3, a contradiction. □

We will show that for any shortest long prism K, if K has a tidy frame, there is a bounded size set of K-major vertices with certain structural properties that can be exploited to clean K-major vertices. We first need to state a few definitions. For a K-major vertex q we say a vertex $v \in V(K)$ is *q-internal* if $v \in V(\alpha_i(q)P_i\beta_i(q))$ for some $i \in \{1, 2, 3\}$ such that q has a neighbor in $V(P_i)$. Otherwise, $v \in V(K)$ is called *q-external*.

Lemma 2.3.7. *Let G be a graph containing no long thetas and let K be a shortest long prism in G. Let P_1, P_2, P_3 be the constituent paths of K. Suppose K has a tidy frame. Let x and y be distinct non-adjacent K-major vertices. Suppose y has two non-adjacent x-external vertices. Then,* *there exists an $i \in \{1,2,3\}$ such that $N(x) \cap V(P_i) \neq \emptyset$ and y has a neighbor $w \in V(P_i)$ satisfying* $\min\{d_{P_i}(w, \alpha_i(x)),\ d_{P_i}(w, \beta_i(x))\} \leq \ell - 3$.

Proof. Suppose that for each $i \in \{1,2,3\}$, either x has no neighbor in $V(P_i)$ or y has no neighbors in $V(P_i)$ with P_i-distance at most $\ell - 3$ from $\alpha_i(x)$ or $\beta_i(x)$.

(1) *For all distinct $i, j \in \{1,2,3\}$, if x has neighbors in both $V(P_i)$ and $V(P_j)$, y does not have x-external neighbors in both $V(P_i)$ and $V(P_j)$.*

Suppose x has neighbors in both $V(P_i)$ and $V(P_j)$ and y has x-external neighbors in both $V(P_i)$ and $V(P_j)$. Then for all $k \in \{i, j\}$, there is a subpath Q_k of P_k of length at least $\ell - 2$ with one end an x-external neighbor of y and the other end $\alpha_k(x)$ or $\beta_k(x)$ such that x-Q_k-y is an induced xy-path. Let M_k denote the subpath of P_k with interior equal to $V(Q_k)$ for each $k \in \{i, j\}$. By definition of $\alpha_i(x)$, $\alpha_j(x)$, $\beta_i(x)$, $\beta_j(x)$ and Lemma 2.3.6, x has a neighbor in $V(K) \setminus (V(M_i) \cup V(M_j))$. Since y has no neighbors adjacent to $\alpha_i(x)$, $\alpha_j(x)$, $\beta_i(x)$ or $\beta_j(x)$, it follows from Lemma 2.3.6 that y has a neighbor in $V(K) \setminus (V(M_i) \cup V(M_j))$.

Since K has a tidy frame, $K \setminus (V(M_i) \cup V(M_j))$ is connected. Hence there is an xy-path B with interior in $V(K) \setminus (V(M_i) \cup V(M_j))$. But then, x-Q_i-y, x-Q_j-y and B form a long theta, a contradiction. This proves (1).

(2) *x does not have neighbors in all of three of $V(P_1), V(P_2), V(P_3)$.*

Suppose x has neighbors in all three of $V(P_1), V(P_2), V(P_3)$. By (1) we may assume that the x-external neighbors of y are contained in $V(P_1)$. Then, we may assume y has an x-internal neighbor in $V(P_2)$. Hence, $\alpha_2(x)$ and $\beta_2(x)$ are not equal or adjacent. Let M_1 denote the induced xy-path with interior contained in $V(\beta_2(y)P_2\beta_2(x))$. We may assume that either y has a neighbor in $V(A_1(x))$ and a neighbor in $V(B_1(x))$ or that y has two non-adjacent neighbors in $V(A_1(x))$. In the first case, M_1, y-$A_1(y)$-$A_2(x)$-x, y-$B_1(y)$-$B_3(x)$-x form a long theta, a contradiction. Hence, the second case holds. There are two long induced xy-paths M_2 and M_3 with interiors contained in $V(A_1(x) \cup A_2(x))$ and such that M_2^* is disjoint from and anticomplete to M_3^*. Then M_1, M_2, M_3 form a long theta, a contradiction. This proves (2).

By (2) we may assume that x has no neighbors in $V(P_3)$. By Lemma 2.3.5, it follows that x

has neighbors in both $V(P_1)$ and $V(P_2)$.

(3) *If* $\{i,j\} = \{1,2\}$*, y does not have two non-adjacent x-external neighbors in $V(P_i)$ and an x-internal neighbor in $V(P_j)$.*

Suppose y has two non-adjacent x-external neighbors in $V(P_i)$ and an x-internal neighbor in $V(P_j)$. By (1), y has no x-external neighbors in $V(P_j)$. By definition of x-internal, $\alpha_j(x)$ is not equal or adjacent to $\beta_j(x)$. Let M_1 and M_2 denote the induced xy-paths with interiors in $V(\beta_j(y)P_1\beta_j(x))$ and $V(\beta_i(x)P_i\beta_i(y))$, respectively. Then M_1, M_2, and x-$A_j(x)$-$A_i(y)$-y form a long theta, a contradiction.

By (1), (3), Lemma 2.3.5 and Lemma 2.3.6, y has a neighbor in $V(P_3)$. By Lemma 2.3.5 and Lemma 2.3.6, we may assume that x has two non-adjacent neighbors in $V(P_1)$ and at least one neighbor in $V(P_2)$.

(4) *Either y has exactly one neighbor in $V(P_3)$ or y has exactly two neighbors in $V(P_3)$ and they are adjacent.*

Suppose y has two non-adjacent neighbors in $V(P_3)$. There are two long induced xy-paths M_1, M_2 with interiors in $V(P_1 \cup P_3)$ such that M_1^* is disjoint from and anticomplete to M_2^*. If y has a neighbor in $V(P_2)$, there is an xy-path M_3 with interior in $V(P_2)$, such that M_1, M_2 and M_3 form a long theta, a contradiction. So by Lemma 2.3.5 and 2.3.6, we may assume y has a neighbor in $V(P \backslash B_1(x))$. Let Q_1 denote the long induced xy-path whose interior is a subset of $V(B_1(x) \cup B_3(y))$. Let Q_2 denote the induced xy-path with interior contained in $V(\alpha_1(x)P_1\alpha_1(y))$. Then Q_1, Q_2 and x-$A_2(x)$-$A_3(y)$-y form a long theta, a contradiction. This proves (4).

(5) *Either y has exactly one x-external neighbor in $V(P_1 \cup P_2)$ or exactly two x-external neighbors in $V(P_1 \cup P_2)$ and they are adjacent.*

By (4), y has at least one x-external neighbor in $V(P_1 \cup P_2)$. Suppose y has two non-adjacent x-external neighbors in $V(P_1 \cup P_2)$. We may assume that either y has two non-adjacent neighbors in $V(A_1(x) \cup A_2(x))$ or y has a neighbor in $V(A_1(x) \cup A_2(x))$ and a neighbor in $V(B_1(x) \cup B_2(x))$. In the first case, there are two long induced xy-paths M_1, M_2 with interiors in $V(A_1(x) \cup A_2(x))$ such that no vertex of M_1^* is disjoint from and anticomplete to M_2^*. Then M_1, M_2 and x-$B_1(x)$-$B_3(y)$-y

form a long theta, a contradiction.

Hence, the second case holds. By (1), we have that one of $V(P_1)$, $V(P_2)$ contains no x-external neighbors of y. If $V(P_2)$ contains no x-external neighbors of y, there is a long theta formed by the two induced xy-paths with interiors contained in $V(A_1(x))$ and $V(B_1(x))$, respectively, and the path x-$A_2(x)$-$A_3(y)$-y, a contradiction. Hence, $V(P_1)$ contains no x-external neighbors of y. Let Q_1 denote the xy-path with interior contained in $V(\beta_2(x)P_2\beta_2(y))$. Then Q_1, x-$A_1(x)$-$A_2(y)$-y and x-$B_1(x)$-$B_3(y)$-y form a long theta, a contradiction. This proves (5).

By (4) and (5), there is an edge $e \in E(P_1 \cup P_2)$ and $e' \in E(P_3)$ such that every x-external neighbor of y in $V(K)$ is incident with e or e' and e, e' each contain at least one x-external neighbor of y. By Lemma 2.3.6, y has an x-internal neighbor in $V(P_i)$ for some $i \in \{1, 2\}$. Since x has two non-adjacent neighbors in $V(P_1)$ we may assume without loss of generality that y has an x-internal neighbor in $V(P_1)$. Let M_1 denote the induced xy-path with interior in $V(\alpha_1(x)P_1\alpha_1(y))$ and let M_2 denote the induced xy-path with interior in $V(\beta_1(x)P_1\beta_1(y))$. If $e \in E(A_1(x))$, there is a long theta formed by M_1, M_2 and y-$B_3(y)$-$B_2(x)$-x. Hence, we may assume that $e \in E(A_2(x))$. Let M_3 denote the long induced xy-path with interior in $V(\alpha_2(y)P_2\alpha_2(x))$. Then M_1, M_3 and x-$B_1(x)$-$B_3(y)$-y form a long theta, a contradiction. $\square$

We use the results from this section to prove that there exists pair consisting of a set of K-major vertices and a set of paths each of which is contained in K with certain helpful properties for cleaning. We begin with some definitions. For a set of paths $\mathcal{P}$ we denote $\cup_{P \in \mathcal{P}} P^*$ by $\mathcal{P}^*$. Let G be a graph containing no long thetas and let K be a shortest long prism in G. Let P_1, P_2 be distinct constituent paths of K and let F be the frame of K. Let S be the set of K-major vertices that have a neighbor in $V(P_1)$ and a neighbor in $V(P_2)$. Let $H \subseteq S$ and let $\mathcal{Q}$ be a set of paths such that each $Q \in \mathcal{Q}$ is a subpath of P_1, P_2 or P_3. We call the ordered pair $(H, \mathcal{Q})$ a (K, P_1, P_2)-*contrivance* if it satisfies the following two properties:

- If S is non-empty, then H contains a vertex $v \in S$ maximizing $|E(A_1(v))|$ over all $v \in S$, $\alpha_1(v) \in \mathcal{Q}^*$, and $N(H) \cap V(A_1(v)) \subseteq \mathcal{Q}^*$ and
- Every vertex $w \in S \setminus H$ has a neighbor in $H \cup \mathcal{Q}^*$.

We will show that if G has no long thetas, K is a shortest long prism in G and K has a tidy frame, then for every choice of two distinct constituent paths P_1, P_2 of K there is a (K, P_1, P_2)-contrivance $(H, \mathcal{Q})$ with $|H|$ and $\sum_{Q \in \mathcal{Q}} |V(Q)|$ bounded. Thus we will be able to guess a (K, P_1, P_2)-contrivance in our cleaning algorithm. We need the following lemma:

Lemma 2.3.8. *Let G be a graph containing no long thetas and K be a shortest long prism in G. Suppose K has a tidy frame. Let P_1, P_2, P_3 be the constituent paths of K. Suppose x and y are K-major vertices satisfying all of the following:*

- *x and y are non-adjacent,*
- *x has a neighbor in $V(P_1)$ and y has a neighbor in $V(A_1(x))$ and*
- *If $v \in V(K)$ is adjacent to y, then $d_K(v, \alpha_i(x)), d_K(v, \beta_i(x)) \geq \ell - 1$ for any $i \in \{1, 2, 3\}$.*

Then y has exactly two x-external neighbors in $V(K)$ and they are adjacent elements of $V(A_1(x))$.

Proof. By Lemma 2.3.7, we need only show that y does not have a unique neighbor in $V(A_1(x))$. Suppose that y has a unique neighbor $w \in V(A_1(x))$. Then by Lemma 2.3.7, w is the unique x-external neighbor of y in $V(K)$. By Lemma 2.3.6, we may assume x has neighbor in $V(P_2)$. Let M_1, M_2 denote the two paths of the cycle x-$A_1(x)$-$A_2(x)$-x with ends x and w. Since K has a tidy frame and $d_{P_1}(w, \alpha_1(x)) \geq \ell - 1$, the paths M_1 and M_2 each have length at least ℓ. Let J be the set of vertices in $V(K) \setminus V(M_1 \cup M_2)$ that are anticomplete to $V(M_1 \cup M_2)$. By Lemma 2.3.6, x and y each have a neighbor in J. $G[J]$ is connected so there is an induced xy-path M_3 with interior in J. But then M_1, M_2 and M_3-w form a long theta, a contradiction. □

Lemma 2.3.9. *Let G be a graph containing no long thetas and K be a shortest long prism in G. Suppose K has a tidy frame. Let P_1, P_2 be two constituent paths of K. Then, there is a (K, P_1, P_2)-contrivance $(H, \mathcal{Q})$ in G satisfying $|H| \leq 3$, $|\mathcal{Q}| \leq 14$ and $\sum_{Q \in \mathcal{Q}} |V(Q)| \leq 28\ell - 12$.*

Proof. Let S be the set of K-major vertices with both a neighbor in $V(P_1)$ and a neighbor in $V(P_2)$. We may assume without loss of generality that $S \neq \emptyset$. Let $x \in S$ maximize $|E(A_1(x))|$. For any $i \in \{1, 2, 3\}$, vertex $v \in V(P_i)$ and non-negative integer k, we denote the set consisting of all vertices b satisfying $d_{P_i}(a, b) \leq k$ as $N^k_{P_i}(a)$.

Let $\mathcal{Q}_x$ denote the set of paths contained in K whose interiors are equal to $N^{\ell-1}_{P_i}(\alpha_i(x))$ or $N^{\ell-1}_{P_i}(\beta_i(x))$ for some $i \in \{1, 2, 3\}$. Hence, $|\mathcal{Q}_x| \leq 6$. Let S_x denote the set of vertices $s \in S$ that have no neighbor in $\mathcal{Q}^*_x$. We may assume without loss of generality that $(\{x\}, \mathcal{Q}_x)$ is not a (K, P_1, P_2)-contrivance. Thus, $S_x \neq \emptyset$.

Let $y \in S_x$ maximize $|E(A_2(y))|$ over all $y \in S_x$. Let $\mathcal{Q}_y$ denote the set of paths contained in K whose interiors are equal to $N^{\ell}_{P_i}(\alpha_i(y))$ or $N^{\ell-1}_{P_i}(\beta_i(y))$ for some $i \in \{1, 2, 3\}$. Hence, $|\mathcal{Q}_y| \leq 6$. By Lemma 2.3.8, y has exactly two neighbors in $V(A_1(x))$ and they are adjacent. Hence

$N(y) \cap V(A_1(x)) \subseteq \mathcal{Q}_y^*$. Let S_{xy} be the set of vertices $s \in S_x$ that have no neighbor in $\mathcal{Q}_x^* \cup \mathcal{Q}_y^* \cup \{x, y\}$. We may assume without loss of generality that $(\{x, y\}, \mathcal{Q}_x \cup \mathcal{Q}_y)$ is not a (K, P_1, P_2)-contrivance. Hence, $S_{xy} \neq \emptyset$.

(1) *Let $v \in S_{xy}$. Then v has exactly two x-external neighbors in $V(K)$ and they are adjacent elements of $V(A_1(x))$ and v has exactly two y-external neighbors and they are adjacent elements of $V(A_2(y))$.*

Apply Lemma 2.3.8 to x, v and to y, v. This proves (1).

Let z be a vertex in S_{xy}. Let p_1, q_1 denote the x-external neighbors of z in $V(P_1)$. Let p_2, q_2 denote the y-external neighbors of z in $V(P_2)$. Let $\mathcal{Q}_z$ denote the set consisting of the two paths contained in K whose interior is equal to $N_{P_1}^{\ell-2}(p_i) \cup N_{P_1}^{\ell-2}(q_i)$ for some $i \in \{1, 2\}$. Hence, $N(z) \cap V(A_1(x)) \subseteq \mathcal{Q}_z^*$.

Let $H = \{x, y, z\}$. Let $\mathcal{Q} = \mathcal{Q}_x \cup \mathcal{Q}_y \cup \mathcal{Q}_z$. We claim that $(H, \mathcal{Q})$ is a $(K,\ P_1,\ P_2)$-contrivance. By choice of H, $\mathcal{Q}$ it is enough to show that every $t \in S \setminus H$ has a neighbor in $H \cup \mathcal{Q}^*$. Suppose some $t \in S \setminus H$ has no neighbors in $H \cup \mathcal{Q}^*$. Then, by (1) there is a long induced tz-path M_1 with interior in $V(A_1(x))$ and a long induced tz-path M_2 with interior in $V(A_2(y))$. Let h_1, h_2 denote the two vertices of $V(P_1 \cup P_2) \setminus V(A_1(x) \cup A_2(y))$ with a neighbor in $V(A_1(x) \cup V(A_2(y))$. Let J denote the graph $K \setminus (V(A_1(x) \cup A_2(y)) \cup \{h_1, h_2, a_3\})$. Since $t, z \in S_{xy}$, both t and z are non-adjacent to each of h_1, h_2, a_3. Hence, by Lemma 2.3.6, t, z each have a neighbor in $V(J)$. Then since J is connected there is an induced tz-path M_3 with interior in $V(J)$. Hence, M_1, M_2, M_3 form a long theta, a contradiction. It follows that $(H, \mathcal{Q})$ is a $(K,\ P_1,\ P_2)$-contrivance.

By construction, we have that $|H| = 3$, $|\mathcal{Q}| \leq 14$. The number of vertices in the paths in $\mathcal{Q}$ are as follows:

$$|V(Q_i)| \leq \begin{cases} 2\ell - 1 & \text{if } Q \in \mathcal{Q}_x \cup \mathcal{P}_y. \\ 2\ell - 2 & \text{if } Q \in \mathcal{Q}_z. \end{cases}$$

Since $|\mathcal{Q}_x \cup \mathcal{Q}_y| \leq 12$ and $|\mathcal{Q}_z| = 2$, it follows that $\sum_{Q \in \mathcal{Q}} V(Q) \leq 28\ell - 16$ as claimed. □

We apply the results from this section to obtain the following cleaning algorithm for major vertices of shortest long prisms.

Theorem 2.3.10. *There is an algorithm with the following specifications:*

Input: *A graph G containing no long theta.*

Output: *A list of $\mathcal{O}(|G|^{32\ell-10})$ subsets of $V(G)$, with the following property: For every shortest long prism K and choice of two distinct constituent paths P_1, P_2 of K, if K has a tidy frame, then there is some X in the list such that:*

- *X is disjoint from $V(K)$ and*
- *X contains all K-major vertices with neighbors in both $V(P_1)$ and $V(P_2)$.*

Running Time: $\mathcal{O}(|G|^{32\ell-8})$.

Proof. We enumerate all triples $(H, \mathcal{Q})$ satisfying all of the following:

- H is a set of at most three vertices in G,
- $\mathcal{Q}$ is a set of at most 14 paths of G, and
- $\sum_{Q \in \mathcal{Q}} |V(Q)| \leq 28\ell - 16$.

For each such pair we perform the following. If H is empty, we output the empty set. Otherwise, we guess a vertex $a \in V(G)$ and a vertex α in $\mathcal{Q}^*$. For each $Q \in \mathcal{Q}$, let X_Q be the set of vertices in $V(G) \setminus V(Q)$ with neighbors in Q^*. Let $X = \cup_{Q \in \mathcal{Q}} X_Q$. We run the algorithm of Theorem 2.3.3 on $G \setminus X$ to generate a list $Y_1, Y_2, \ldots, Y_k$ of subsets of $V(G)$.

For each $i \in \{1, 2, \ldots, k\}$, let G_i be the graph $G \setminus ((X \cup Y_i \cup H \cup N(H)) \setminus \mathcal{Q}^*)$. We compute the union of the interiors of all shortest $a\alpha$-paths in G_i and denote it by R_i. Let Z_i be the set of vertices of $V(G) \setminus \mathcal{Q}^*$ with a neighbor in $V(R_i)$ and a neighbor in H. We output $H \cup X \cup Z_i$.

We prove the algorithm is correct. Let K be a shortest long prism in G and let P_1, P_2 be distinct constituent paths of K. Suppose K has a tidy frame. Then it follows from Lemma 2.3.9 that for some choice of $(H, \mathcal{Q})$ the pair $(H, \mathcal{Q})$ is a (K, P_1, P_2)-contrivance. Let S denote the set of K-major vertices with a neighbor in $V(P_1)$ and a neighbor in $V(P_2)$. By definition of (K, P_1, P_2)-contrivance, H is empty if and only if S is empty. We may assume S and H are both non-empty. Thus, every vertex in $S \setminus (H \cup N(H))$ has a neighbor in X. Let A denote the path $aP\alpha$. Let x be a vertex in S maximizing $|E(A_1(x))|$ over all $x \in S$. For some choice of a, α, the paths $A_1(x)$ and A are equal so we may assume $A = A_1(x)$. It follows that every vertex in S has a neighbor in $V(A)$. By construction, X is disjoint from $V(K)$. Hence, by Theorem 2.3.3, there exists an $i \in \{1, 2, \ldots k\}$ such that Y_i is disjoint from $V(K)$ and Y_i contains a vertex of every bad shortest $a\alpha$-path Q such that ζ_Q is not K-major. By definition of (K, P_1, P_2)-contrivance, $H \cup N(H) \cup \mathcal{Q}^*$ contains all vertices in S. Hence all shortest $a\alpha$-paths in G_i are good. Since $N(H) \cap V(A) \subseteq \mathcal{Q}^*$, it follows that $A^* \subseteq R_i$.

Thus, Z_i contains all vertices in $S \setminus (H \cup Y_i)$. Since K has a tidy frame, a_1 has no neighbors in H and by definition of (K, P_1, P_2)-contrivance, $N(H) \cap V(A) \subseteq \mathcal{Q}^*$. Since all shortest $a\alpha$-paths in G_i are good, it follows that Z_i is disjoint from $V(K)$. Hence, the list satisfies the properties from the claim.

There are $\mathcal{O}(|G|^{28\ell-12})$ choices for $(H, \mathcal{Q})$ and a. For each choice we find X in time $\mathcal{O}(|G|)$ and run the algorithm of Theorem 2.3.3 to generate a list of $\mathcal{O}(|G|^{4\ell+2})$ subsets $Y_1, Y_2, \dots, Y_k$ of $V(G)$ in time $\mathcal{O}(|G|^{4\ell+3})$. For each set in the list we compute Z_i in $\mathcal{O}(|G|^2)$. Hence the total running is $\mathcal{O}(|G|^{32\ell-8})$ and the length of the output is $\mathcal{O}(|G|^{32\ell-10})$.

□

Corollary 2.3.11. *There is an algorithm with the following specifications:*

Input: *A graph G containing no long theta.*

Output: *A list of $\mathcal{O}(|G|^{96\ell-30})$ subsets of $V(G)$, with the following property: For every shortest long prism K and choice of two distinct constituent paths P_1, P_2 of K, if K has a tidy frame, then there is some X in the list such that:*

- *X is disjoint from $V(K)$ and*
- *X contains all K-major vertices with neighbors in both $V(P_1)$ and $V(P_2)$.*

Running Time: $\mathcal{O}(|G|^{96\ell-30})$.

Proof. We apply the algorithm of Theorem 2.3.10 to obtain a cleaning list $X_1, X_2, \dots, X_k$ of length $\mathcal{O}(|G|^{32\ell-6})$ in time $\mathcal{O}(|G|^{32\ell-4})$. We output $X_a \cup X_b \cup X_c$ for each choice $a, b, c \in \{1, 2 \dots, k\}$. Correctness follows from Lemma 2.3.6. □

2.3.3 The long prism detection algorithm

We can now prove the main result of this section, which we restate.

Theorem 2.3.1 *For each integer $\ell \geq 4$ there is an algorithm with the following specifications:*

Input: *A graph G containing no long theta.*

Output: *Decides whether G contains a long prism.*

Running Time: $\mathcal{O}(|G|^{108\ell-22})$.

Proof. The algorithm is as follows: We guess a set J of at most $6\ell - 6$ vertices and a set D of at most 6 vertices. We construct the set X of all vertices in $V(G) \setminus (J \cup D)$ with neighbors in J.

We apply the algorithm described in Corollary 2.3.11 to $G \setminus X$ and obtain a cleaning list $Y_1, Y_2 \dots Y_p$. For each $i \in \{1, 2, \dots, p\}$, we apply the algorithm described in Corollary 2.3.4 to generate another cleaning list $Z_1^i, Z_2^i, \dots Z_{k_i}^i$. We guess two vertices x_1, y_1 in $J \cup D$. We search for a shortest x_1y_1-path Q_1 in $G \setminus (X \cup Y_i \cup Z_j^i)$. We construct the set A of all vertices in $V(G) \setminus (J \cup X \cup Y_i \cup Z_j^i)$ with neighbors in Q_1^*. Then we guess two vertices x_2, y_2 in J and we search for a shortest x_2y_2-path Q_2 in $G \setminus (X \cup Y_i \cup Z_j^i \cup A)$ and construct the set B of all vertices in $V(G) \setminus (J \cup X \cup Y_i \cup Z_j^i \cup A)$ with neighbors in Q_2^*. Finally, we guess two vertices x_3, y_3 in J we search for a shortest x_3y_3-path Q_3 in $G \setminus (X \cup Y_i \cup Z_j^i \cup A \cup B)$. We test whether $S \cup V(Q_1) \cup V(Q_2) \cup V(Q_3)$ induces a long prism.

Now, we prove the output is correct. Let K be a shortest long prism in G and F be the frame of K. Then for some guess of J and D, the set $J \cup D$ is equal to $V(F)$ and D is the set of vertices in $V(F)$ with a neighbor in $V(K) \setminus V(F)$. Hence, $G \setminus X$ contains K and K has a tidy frame in $G \setminus X$. By Corollary 2.3.11 it follows that for some choice of $i \in \{1, 2, \dots, p\}$, there are no K-major vertices in $G \setminus (X \cup Y_i)$. Therefore by Corollary 2.3.4, for some choice of $j \in \{1, 2, \dots, k_i\}$, all shortest s_it_i-paths are good in $G \setminus (X \cup Y_i \cup Z_j^i)$ for every $i \in \{1, 2, 3\}$ such that $P_i \not\subseteq F$. For each $i \in \{1, 2, 3\}$, we may assume x_i, y_i equal s_i, t_i since $s_i, t_i \in V(F)$. Since Q_1 is good, there is a long prism induced by $V(F \cup Q_1 \cup P_2 \cup P_3)$. Thus Q_2 exists and is a good shortest s_2t_2-path. Similarly, Q_3 exists and is a good shortest s_3t_3-path for K. By choice of A, B it follows that, Q_1, Q_2, Q_3 are pairwise vertex disjoint and their interiors are pairwise anticomplete. Thus since F is tidy, $J \cup V(Q_1 \cup Q_2 \cup Q_3)$ induces a long prism K'. Since Q_p is good for each $p \in \{1, 2, 3\}$, it follows that K' is a shortest long prism in G.

There are $\mathcal{O}(|G|^{6\ell})$ guesses to check for $J \cup D$. For each of these we obtain a cleaning list $Y_1, Y_2 \dots Y_p$ of length $\mathcal{O}(|G|^{96\ell-30})$ in time $\mathcal{O}(|G|^{96\ell-30})$. For each Y_i in the list we generate another cleaning list $Z_1, Z_2, \dots, Z_t$ of length $\mathcal{O}(|G|^{12\ell+6})$ in time $\mathcal{O}(|G|^{12\ell+6})$. Finding Q_i for each $i \in \{1, 2, 3\}$ and testing whether $J \cup V(Q_1 \cup Q_2 \cup Q_3)$ is a long prism takes $\mathcal{O}(|G|^2)$. Hence, the running time is $\mathcal{O}(|G|^{108\ell-22})$. □

2.4 Shortest Even Holes with Major Edges

Let C be a shortest long even hole. For u, v distinct and non-adjacent vertices in $V(C)$ we call an induced uv-path Q a *shortcut* if $V(Q)$ contains no C-major vertices and Q has length less than $d_C(u, v)$. In this section we will focus on a type of shortcut of length three called a "major edge".

Let q_1, q_2 be vertices in $V(G)$ such that u-q_1-q_2-v is a shortcut. Let P_1 be a minimal path of C containing all neighbors of q_1 in $V(C)$ such that P_1 has length at most two. Let a_1, b_1 denote the ends of P_1. Define P_2, a_2, b_2 similarly for q_2. If $|E(P_1)| = |E(P_2)| = 2$ we call q_1q_2 a *C-major edge*. Since u-q_1-q_2-v is a shortcut, P_1 and P_2 share at most one vertex. We may assume a_1, b_1, b_2, a_2 occur in order along C. Let A denote the path of C with ends a_1, a_2 that does not contain b_1 or b_2 and let B denote the path of C with ends b_1, b_2 that does not contain a_1 or a_2. For $i \in \{1, 2\}$, let A_i be the subpath of A whose vertex set consists of all $v \in V(A)$ satisfying $d_A(v, a_i) \leq \ell$ and let B_i be the subpath of B whose vertex set consists of all $v \in V(B)$ satisfying $d_A(v, b_i) \leq \ell$. We call the graph induced by $V(A_1) \cup V(P_1) \cup V(A_2) \cup V(B_1) \cup V(P_2) \cup V(B_2) \cup \{q_1, q_2\}$ the *C-frame* of q_1q_2. We say the C-frame of a C-major edge q_1q_2 is *tidy* if $\cup_{i=1}^{2}(A_i^* \cup B_i^* \cup V(P_i) \cup \{q_i\})$ is anticomplete to $V(G) \setminus V(C)$. For $i \in \{1, 2\}$, let A_i have ends a_i, r_i and let B_i have ends b_i, s_i. Let A_0 denote the path r_1Ar_2 and let B_0 denote the path r_1Br_2. Then, C is equal to A_1-P_1-B_1-B_0-B_2-P_2-A_2-A_0. As in the long prism detection algorithm, we call an induced r_1r_2-path Q *good* if Q^* is disjoint from and anticomplete to $V(C) \setminus V(A_0)$. If an induced r_1r_2-path is not good it is *bad*. We define good and bad induced s_1s_2-paths similarly. For a vertex $v \in V(G) \setminus V(C)$ with neighbors in $V(A)$ we define $\alpha_A(v)$ and $\beta_A(v)$ to be the neighbors of v in $V(A)$ with minimum A-distance from a_1 and a_2 respectively. We define $\alpha_B(v), \beta_B(v)$ similarly.

We will prove that if G is a graph containing no long prisms or long thetas, C is a shortest long even hole in G, and q_1q_2 is a C-major edge with a tidy C-frame then all shortest r_1r_2-paths and all shortest s_1s_2-paths are good. Consequently, we will obtain an algorithm to detect whether a graph G that contains no long thetas or long prisms contains a long even hole C such that there is a C-major edge in $E(G)$.

In the rest of the section we will use the notation established in the definitions of major edge and its frame with respect to a long even hole.

Lemma 2.4.1. *Let G be graph with no long prism or long theta and let C be shortest long even hole in G. Then every C-major vertex has three pairwise non-adjacent neighbors in $V(C)$.*

Proof. Let x be a C-major vertex and suppose that there exist vertices $a_1, a_2, b_1, b_2 \in V(C)$ such that $N(x) \cap C \subseteq \{a_1, a_2, b_1, b_2\}$, a_1, b_1 are adjacent and a_2, b_2 are adjacent. Without loss of generality a_1, b_1, b_2, a_2 occur in order along $V(C)$. Let A denote the path of C with ends a_1, a_2 that does not contain b_1 or b_2. Let B denote the path of C with ends b_1, b_2 that does not contain a_1 or a_2. By 2.5.5, A, B each have length at least ℓ. Since x is C-major, x must have at least one neighbor in $\{a_1, b_1\}$ and at least one neighbor in $\{a_2, b_2\}$. If x is adjacent to all of a_1, a_2, b_1, b_2, it follows that $V(C) \cup \{x\}$ induces a long prism, a contradiction. If x is adjacent to exactly one of $\{a_1, b_1\}$ and x is adjacent to exactly one of $\{a_2, b_2\}$ then $V(C) \cup \{x\}$ induces a long theta, a contradiction. Hence we may assume x is adjacent to a_1, a_2, b_2 and x is non-adjacent to b_1. Then $V(A) \cup \{x\}$ and $V(B) \cup \{a_1, x\}$ both induce long holes that are shorter than C, so both holes must both be odd. But then $|E(A)|$ and $|E(B)| + 1$ are both odd, contradicting that C is even. □

Lemma 2.4.2. *Let G be a graph containing no long thetas or long prisms and let C be a shortest long even hole in G. Suppose G contains a C-major edge q_1q_2 and that the C-frame of q_1q_2 is tidy. Then for every $x \in V(G) \setminus V(C)$, x does not not have neighbors in both $V(A)$ and $V(B)$ where A, B are as in the definition of C-major edge.*

Proof. Suppose x has neighbors in both $V(A)$ and $V(B)$. Then since the C-frame of q_1q_2 is tidy, x is C-major. Then, by Lemma 2.4.1 we may assume that x has two non-adjacent neighbors in $V(A)$. Then since the C-frame of q_1q_2 is tidy, x-$\alpha_A(x)Aa_1$-q_1, x-$\beta_A(x)Aa_2$-q_2-q_1, and x-$\alpha_B(x)Bb_1$-q_1 form a long theta, a contradiction. □

Lemma 2.4.3. *Let G be a graph with no long theta or long prism and let C be a shortest long even hole in G. Suppose some edge q_1q_2 of G is a C-major edge and the C-frame of q_1q_2 is tidy. Then, all r_1r_2-paths are good and all s_1s_2-paths are good.*

Proof. Suppose there is some bad r_1r_2-path R. Let ζ denote the vertex of R^* with minimum R-distance to r_1 among all vertices in R^* with neighbors in $V(C) \setminus V(A_0)$. Let R_ζ denote the subpath of R with ends r_1, ζ. By Lemma 2.4.2, the neighbors of ζ in $V(C)$ are contained in $V(B)$.

(1) *ζ does not have two non-adjacent neighbors in $V(B)$.*

Suppose that it does. Then the paths q_1-a_1Ar_1-$r_1R_\zeta\zeta$, q_1-$b_1B\alpha_B(\zeta)$-ζ and q_1-q_2-$b_2B\beta_B(\zeta)$-ζ form a long theta. This proves (1).

(2) *ζ has exactly two neighbors in $V(C)$ and they are adjacent elements of $V(B)$.*

Suppose that it doesn't. By (1) we may assume ζ has a unique neighbor in $w \in V(B)$. Then since the C-frame of q_1q_2 is tidy, the paths q_1-a_1Ar_1-$r_1R_\zeta\zeta$-w, q_1-q_2-b_2Bw, q_1-b_1Bw form a long theta, a contradiction. This proves (2).

Since R is a shortest r_1r_2-path and ζ has no neighbors in $V(A)$, it follows that r_2 has no neighbors in R^*_ζ.

(3) R^*_ζ is not anticomplete to A^*_0.

Suppose that it it is. Then R^*_ζ is anticomplete to $V(C) \setminus \{r_1\}$. Since R is a shortest r_1r_2-path and the C-frame of q_1q_2 is tidy, it follows that the paths r_1-R_ζ-ζ-$\alpha_B(\zeta)Bb_1$-q_1, r_1Aa_2-q_2-q_1, and r_1Aa_1-q_1 form a long theta, a contradiction. This proves (3).

Let z minimize $d_A(z, r_1)$ among all vertices in A^*_0 with neighbors in R^*_ζ. Let γ minimize $d_{R_\zeta}(\gamma, \zeta)$ amongst all neighbors of z in R^*_ζ.

(4) γ *is not the only neighbor of* z *in* $V(R_\zeta)$.

Suppose that it is. By definition of γ, γ is not equal to r_1 and so z is not adjacent to r_1. Then the paths $\gamma R_\zeta\zeta$-$\alpha_B(\zeta)Bb_1$-q_1, $\gamma R_\zeta r_1$-r_1Aa_1-q_1, γ-zAa_2-q_2-q_1 form a long theta, a contradiction. This proves (4).

Let η denote the neighbor of z in $V(R_\zeta)$ at minimum R_ζ-distance from r_1. By (4), η is not equal to γ.

(5) η *is adjacent to* γ

Suppose that η and γ are not adjacent. Then the paths z-$\eta R_\zeta r_1$-r_1Aa_1-q_1, z-$\gamma R_\zeta\zeta$-$\alpha_B(\zeta)Bb_1$-q_1, and zAa_2-q_2-q_1 form a long theta, a contradiction. This proves (5).

It follows from (2) and (5), that there is a long prism with bases induced by $\{z, \gamma, \eta\}$ and $\{\zeta, \alpha_B(\zeta), \beta_B(\zeta)\}$ and constituent paths equal to zAa_2-$b_2B\beta_B(\zeta)$, $\eta R_\zeta r_1$-r_1Aa_1-$b_1B\alpha_B(\zeta)$ and $\gamma R_\zeta\zeta$, a contradiction. $\square$

Theorem 2.4.4. *For each integer $\ell \geq 5$ there is an algorithm with the following specifications:*

Input: *A graph G without long thetas or long prisms.*

Output: *Outputs that G contains a long even hole or that G contains no shortest long even hole C with a C-major edge.*

Running Time: $\mathcal{O}(|G|^{4\ell+10})$.

Proof. The algorithm is as follows. We guess a set J of at most $4\ell+4$ vertices and two sets D_1, D_2 each of at most 2 vertices. We construct the set X of all vertices in $V(G) \setminus (J \cup D_1 \cup D_2)$ with neighbors in J.

If D_1 has cardinality two, set W_1 to be a shortest path between the two vertices in D_1 in $G \setminus X$ if it exists. Otherwise, set W_1 to be the empty path. Let Y be the set of vertices in $V(G) \setminus (X \cup V(W_1))$ with neighbors in W_1^*.

If D_2 has cardinality two, set W_2 to be a shortest path between the two vertices in D_2 in $G \setminus (X \cup Y)$ if it exists. Otherwise, set W_2 to be the empty path.

We test whether $J \cup D_1 \cup D_2 \cup V(W_1 \cup W_2)$ induces a long even hole with a major edge. If so, output that G contains a long even hole. If there are no more guesses to check, output that G contains no shortest long even hole C with a C-major edge.

Now we prove the output is correct. Suppose there is a shortest long even hole C in G such that $E(G) \setminus E(C)$ includes a C-major edge q_1q_2. Let F denote the C-frame of q_1q_2. Then for some choice of J, D_1, D_2, the set of vertices $J \cup D_1 \cup D_2$ is equal $V(F)$ and $D_1 \cup D_2$ is the set of vertices in $V(F)$ with neighbors in $V(C) \setminus V(F)$. Hence C and q_1q_2 are both contained in $G \setminus X$ and q_1q_2 has a tidy C-frame in $G \setminus X$. Let $r_1, r_2, s_1, s_2, A_0, B_0$ be as in the definition of C-frame. We may assume that $D_1 = \{r_1, r_2\}$ and $D_2 = \{s_1, s_2\}$. Hence, W_1 exists and it is a good r_1r_2-path by Lemma 2.4.3. Let C' be the graph induced by $V(C \cup W_1) \setminus A_0^*$. It follows that C' is a shortest long even hole of $G \setminus X$, the edge q_1q_2 is C'-major and the C'-frame of q_1q_2 is tidy in $G \setminus X$. It follows from the choice of Y that C' is contained in $G \setminus (X \cup Y)$. Thus, W_2 exists and it is a good s_1s_2-path by Lemma 2.4.3. Let C'' be the graph induced by $V(C' \cup W_2) \setminus B_0^*$. By choice of Y, it follows that C' is a shortest long even hole in $G \setminus (X \cup Y)$. Moreover, q_1q_2 is a C''-major edge. Since $V(C'') \cup \{q_1, q_2\}$ equals $J \cup D_1 \cup D_2 \cup V(W_1 \cup W_2)$ the algorithm outputs that G contains a shortest long even hole with a major edge.

For the running time, there are $\mathcal{O}(|G|^{4\ell+8})$ guesses of J, D_1 and D_2 to check. For each such guess we compute W_1 and W_2 in time $\mathcal{O}(|G|^2)$ so the running time $\mathcal{O}(|G|^{4\ell+10})$ as claimed. □

2.5 Detecting a clean shortest long even hole

In this section we provide an algorithm to detect a clean shortest long even hole in a "candidate", a graph that contains no easily detectable configurations or shortest long even holes with major edges. More rigorously, G is a *candidate* if it contains no long even hole of length at most $2\ell+2$, no long jewel of order at most $\ell+3$, no long theta, no long prism and no shortest long even hole C such that $E(G)$ contains a C-major edge.

We will need the following properties of shortest paths between vertices in a shortest long even hole of a candidate.

Theorem 2.5.1. *Let C be a clean shortest long even hole in a candidate G. Let u, v be distinct, non-adjacent vertices in $V(C)$. Let L_1, L_2 be the two paths of C with ends u and v where $|E(L_1)| \leq |E(L_2)|$. Then:*

(i) L_1 is a shortest uv-path in G and

(ii) for every shortest uv-path P in G, either $P \cup L_2$ is a clean shortest long even hole in G or $|E(L_1)| = |E(L_2)|$ and $P \cup L_1$ is a clean shortest long even hole in G.

We begin by proving the first statement of Theorem 2.5.1 in a more general form that can be used for cleaning major vertices of shortest long even holes. We need the following lemmas:

Lemma 2.5.2. *Let G be a candidate and let C be a shortest long even hole in G. Let Q be a shortcut of C. Denote the vertices of Q that are adjacent to an end of Q by q_1, q_k. Suppose $Q^* \setminus \{q_1, q_k\}$ is disjoint from and anticomplete to $V(C)$. Then, one of q_1, q_k has two non-adjacent neighbors in $V(C)$.*

Proof. Since q_1 is not C-major, there is a path P_1 of C of length at most two such that all neighbors of q_1 in $V(C)$ are elements of $V(P_1)$. Choose P_1 to be minimal. Define P_2 similarly for q_k. Suppose for a contradiction that both P_1 and P_2 have length at most one. If P_1 and P_2 both have length 0 then $G[V(C \cup Q)]$ forms either a long theta or a long jewel of order less than ℓ, contradicting that G is a candidate. Hence, we may assume P_1 has length one.

Suppose P_2 has length one. If $|E(L_1)| \geq \ell+2$, then $G[V(C \cup Q)]$ is a long prism, a contradiction. It follows that $|E(Q)|, |E(L_1)| \leq \ell+1$. L_2 has length greater than ℓ since $|E(C)| > 2\ell$. Thus, $L_2 \cup Q$ is a long hole and it is shorter than C, so it is odd. Therefore Q and L_1 have different parities. But then L_1, Q and L_2 form a long jewel of order at most $\ell+1$, a contradiction.

Hence, we may assume that P_2 has length 0. Denote the two vertices of P_1 as a_1, b_1 where $a_1 \in V(L_1)$ and $b_1 \in V(L_2)$. Denote the end of Q adjacent to q_k by v. Since C has length at least $2\ell+3$, the path $b_1 L_2 v$ has length greater than ℓ. Then $b_1 L_2 v$-$q_k Q q_1$-b_1 is a long hole and it is shorter than C, so it is odd. Hence, $b_1 L_2 v$ has a different parity than $q_k Q q_1$-b_1 and thus $b_1 L_2 v$ has a different parity than $q_k Q q_1$-a_1. Since C is even, $b_1 L_2 v$ has a different parity than $a_1 L_1 v$, so $a_1 L_1 v$ and $q_k Q q_1$-a_1 have the same parity. Then, $a_1 L_1 v$-$q_k Q q_1$-a_1 is an even hole and it is shorter than C, so it is not long. Thus, $|E(a_1L_1v)|, |E(q_1Qv)| < \ell$. Hence, b_1-a_1L_1v, b_1-q_1Qv, and b_1L_2v form a long jewel of order less than $\ell + 1$, a contradiction. □

Lemma 2.5.3. *Let C be a shortest long even hole in a graph G. Let u, v be distinct and non-adjacent vertices in $V(C)$. Let Q be an induced uv-path of length at most $d_C(u, v)$ such that $V(Q)$ contains no C-major vertices. Suppose no proper subpath of Q is a shortcut for C. Suppose there is some $q \in Q^*$ such that q is not adjacent to an end of Q and q has a neighbor $w \in V(C)$. Let R denote the path of C with ends u, w whose interior does not contain v. Let S denote the path of C with ends w, v whose interior does not contain u. Then $d_C(u, w) = |E(R)|$ and $d_C(w, v) = |E(S)|$.*

Proof. Suppose the lemma does not hold. By symmetry we may assume, $|E(R)| > d_C(u, w)$. Since Q is induced, $w \neq v$ and so $|E(S)| \geq 1$. Let L denote the path of C with ends u, v that does not go through w. Then since uQq-w is not a shortcut for C, it follows that $|E(L)|+|E(S)| \leq d_Q(u, q)+1 \leq |E(Q)|$. However, $|E(L)| + |E(S)| \geq |E(Q)| + 1$, a contradiction. □

Theorem 2.5.4. *Let G be a candidate and let C be a shortest long even hole in G. Then C has no shortcut.*

Proof. Suppose G is a minimal counterexample. Then G has a clean shortest long even hole C with a shortcut Q. We may assume C and Q are chosen to minimize $|E(Q)|$. Let u, v be the ends of Q. Let L_1 and L_2 be the two paths of C joining u, v with $|E(L_1)| \leq |E(L_2)|$. Denote the vertices of Q by u-q_1-q_2-$\ldots$-q_k-v in order. It follows that $|E(L_1)|, |E(L_2)| > k + 1$. Since Q contains no major vertices, $k > 1$. Since q_1 is not C-major, there is a path P_1 of C of length at most two such that all neighbors of q_1 in $V(C)$ lie in P_1. Choose $V(P_1)$ to be minimal. Define P_2 similarly for q_k.

(1) *P_1 and P_2 are vertex-disjoint.*

Suppose not. Then $|E(L_1)| \leq 4$. Since $|E(L_1)| > k + 1 \geq 3$, it follows that $k = 2$ and $|E(L_1)| = 4$. Thus P_1 and P_2 both have length two. Hence, L_1, u-q_1-q_2-v and L_2 form a long jewel of order four, a contradiction. This proves (1).

(2) *One of $q_2, \ldots, q_{k-1}$ has a neighbor in $V(C)$.*

Suppose not. By Lemma 2.5.2 we may assume P_1 has length two. Let C' be the hole obtained by replacing the central vertex of P_1 with q_1. If $\{q_2, \ldots q_k\}$ contains any C'-major vertices, $k = 2$

and by Lemma 2.4.1 q_1q_2 is a C-major edge, a contradiction. Thus, we may assume $\{q_2, q_3, \ldots, q_k\}$ contains no C'-major vertices. If u is the middle vertex of P_1, the path q_1Qq_k-v is a shortcut of C' and it is shorter than Q, a contradiction. Hence, we may assume u is an end of P_1. Denote the other end of P_1 by z.

Suppose $z \in L_2^*$. Then vL_2z-q_1 has length greater than k. The path L_1-q_1 has length at least $k+3$ so q_1Qq_k-v is a shortcut of C', a contradiction. Therefore, we may assume $z \in L_1^*$. Since $|E(zL_1v)| > k-1$ and $|E(L_2)| > k$, it follows that q_1Qq_k-v is a shortcut of C', a contradiction. This proves (2).

We will show that none of $q_2, \ldots, q_{k-1}$ has neighbors in $V(C)$ for a contradiction.

(3) *Suppose $q_i \in \{q_2, q_3 \ldots, q_{k-1}\}$ has a neighbor w in $V(C)$. Let R denote the path of C with ends u, w that does not go through v. Let S denote the path of C with ends w, v that does not go through u. Let $x = i+1-|E(R)|$, let $y = k-i+2-|E(S)|$. Then $x, y \in \{0, 1\}$ and at at most one of x, y is equal to zero.*

By Lemma 2.5.3 and the fact that Q is a shortest shortcut it follows that $|E(R)| \leq i+1$ and $|E(S)| \leq k-i+2$. Since Q is a shortcut, $|E(R)| + |E(S)| > k+1$, and the claim follows. This proves (3).

(4) *None of $q_2, \ldots, q_{k-1}$ have a neighbor in $V(L_1)$.*

Suppose that for some $i \in \{2, 3, \ldots, k-1\}$, q_i has a neighbor $w \in V(L_1)$. Since Q is a shortest uv-path, $w \neq u, v$. Let R_1, S_1 be the subpaths of L_1 with ends u, w and w, v, respectively.

Suppose $q_j \in \{q_2, q_3, \ldots, q_{k-1}\}$ has a neighbor $z \in V(L_2)$. Let R_2, S_2 denote the subpaths of L_2 with with ends u, z and z, v, respectively. Then, $d_C(w, z) = \min\{|E(R_1)| + |E(R_2)|,\ |E(S_1) + |E(S_2)|\}$. Let Q' denote the path w-q_iQq_j-z. By (3) it follows that $|E(R_1)| \geq i$ and $|E(R_2)| \geq j$. Thus, $|E(R_1)|+|E(R_2)| > |j-i|+3 > |E(Q')|$ since $i, j \geq 2$. Similarly, $|E(S_1)|+|E(S_2)| > |E(Q')|$. But then Q' is a shortcut, contradicting that Q is a shortest shortcut. Hence, none of $q_2, q_3, \ldots, q_{k-1}$ has a neighbor in $V(L_2)$.

We denote the vertices of L_2 by u-b_1-b_2-$\ldots$-b_m-v in order. Let $x = i+1-|E(R_1)|$ and let $y = k-i+2-|E(S_1)|$. Suppose $x = y = 0$. Then $|E(L_1)| = k+3$, and L_1, L_2 and Q all have the same parity. Since $G[V(Q \cup L_2)]$ does not contain a long even hole, we may assume b_1 is adjacent to q_1. But then b_1-q_1Qq_i-w is shortcut since:

$$d_C(b_1, w) = \min\{|E(L_2)| - 1 + |E(S_1)|,\ |E(R_1)| + 1\} \geq \min\{2k - i + 4,\ i + 2\} > i + 1.$$

We reach a contradiction as b_1-q_1Qq_i-w is shorter than Q.

Thus by (3), we may assume $x_1 = 0$ and $y_1 = 1$. Hence, $|E(L_1)| = k + 2$. Denote the vertices of L_1 by u-a_1-a_2-$\dots$-a_{k+1}-v in order. Then, $w = a_{i+1}$. For all $j \in \{1, 2\}$, q_1 is not adjacent to b_j, because otherwise b_j-q_1Qq_i-a_{i+1} would be a shortcut with length less than $|E(Q)|$. It follows from (3) that if $q_h \in Q^*$ is adjacent to $a_j \in L_1^*$, then $j \in \{h, h + 1\}$. Let C' denote the hole induced by $V(L_2) \cup \{q_1, q_2, \dots q_i, a_{i+1}, a_{i+2}, \dots a_{k+1}\}$. It follows that C' is a clean shortest long even hole in G. If $i \neq k - 1$, then C' is a clean shortest long even hole in $G[V(C \cup Q)]$. But in that case, q_iQq_k-v is a shortcut for C' in $G[V(C \cup Q)]$ and q_iQq_k-v is shorter than Q, a contradiction.

Hence, $i = k - 1$. Since $i \geq 2$, it follows that $k \geq 3$. The vertices q_k and b_{m-1} are non-adjacent because otherwise a_k-q_{k-1}-q_k-b_{m-1} is a shortcut and, since $k > 2$, it is a shorter than Q, a contradiction. Suppose q_k is not adjacent to b_m. Then $L_2 \cup Q$ is a hole. Since G contains no long even holes of length less than 2ℓ, it follows that $|E(L_2)| \geq \ell$. But then q_{k-1}-q_k-v, q_{k-1}-a_k-a_{k+1}-v and $q_{k-1}Qu$-L_2 form a long jewel of order 3, a contradiction. Hence q_k is adjacent to b_m. We may assume Q was chosen to maximize the distance along C between its ends. It follows that $m \leq k+2$. Hence $m = k + 1$, because L_1 and L_2 have the same parity.

If q_{k-1} is adjacent to a_{k-1}, since $k \geq 3$ the paths b_{k+1}-v-a_{k+1}-a_k-a_{k-1}, b_{k+1}-q_k-q_{k-1}-a_{k-1} and L_2-uL_1a_{k-1} form a long jewel of order four, a contradiction. Thus, q_{k-1} is non-adjacent to a_{k-1}. Consider the cycle C'' obtained from C by replacing the path b_{k+1}-v-a_{k+1}-a_k with b_{k+1}-q_k-q_{k-1}-a_k. It follows that C'' is a shortest long even hole in G. Since G has no long even hole of length at most $2\ell + 2$ and $|E(C)| = 2k + 2$, it follows that $k > 3$. Consequently, C'' is a clean shortest long even hole in $G[V(C \cup Q)]$. But uQq_{k-1} is a shortcut for C'' and it is shorter than Q, a contradiction. This proves (4).

By (2) and (4), some $q_i \in \{q_2, q_3, \dots, q_k\}$ has a neighbor $w \in V(L_2)$ and we may assume that $|E(L_1)| < |E(L_2)|$. Since C is even, it follows that $|E(L_1)|+2 \leq |E(L_2)|$. Moreover, $|E(L_1)| > k+2$, so $|E(L_2)| \geq k + 4$. Let R_2, S_2 be the subpaths of L_2 between u, w and w, v, respectively. Since Q is a shortest shortcut it follows from Lemma 2.5.3 that $|E(R_2)| \leq i + 1$ and $|E(S_2)| \leq k - i + 2$ so $|E(L_2)| \leq k + 3$, a contradiction. □

We will need the following analogue of 3.4 in [18] in order to prove the second statement of Theorem 2.5.1.

Lemma 2.5.5. *Let G be a graph containing no long jewel of order at most k and no long even hole of length less than $k + \ell$. Let C be a shortest long even hole in G and let $v \in V(G)$ be a C-major vertex. Then every path of C that contains all neighbors of v in $V(C)$ has length greater than k.*

Proof. Suppose that P is a path of C of length at most k and that P contains all of neighbors of v in $V(C)$. Choose P to be minimal. Denote the ends of P as a, b. Let Q be the other path of C with ends a and b. We have $|E(Q)| \geq \ell$ and $|E(P)| \geq 3$. So $V(Q) \cup \{v\}$ induces a long hole shorter than C. So Q is odd and thus P is odd. But then P, a-v-b and Q form a long jewel of order at most k, a contradiction. $\square$

We will now prove the second statement of Theorem 2.5.1:

Theorem 2.5.6. *Let G be a candidate. Let C be a clean shortest long even hole in G. Let u, v be distinct and non-adjacent vertices in $V(C)$. Let L_1 and L_2 be the two paths of C with ends u and v. Let Q be a shortest uv-path. Then $L_2 \cup Q$ or $L_1 \cup Q$ is a clean shortest long even hole.*

Proof. We may assume $|E(L_1)| \leq |E(L_2)|$. By Theorem 2.5.4, L_1 and Q have the same length. Denote the vertices of L_1 by u-a_1-a_2-$\ldots$-a_k-v in order, denote the vertices of Q by u-q_1-q_2-$\ldots$-q_k-v in order and denote the vertices of L_2 by u-b_1-b_2-$\ldots$-b_m-v in order. We proceed by induction on k.

We show that the theorem holds if $k = 1$. Consider the cycle C' obtained by replacing the middle vertex of L_1 with q_1. Since C is clean, C' is a shortest long even hole. Suppose x is a C'-major vertex. Let P be a minimum length path of C containing all neighbors of x in $V(C')$. Then by Lemma 2.5.5, P has length at least $\ell + 3 \geq 7$. Since x is not C-major, it follows that x is adjacent to q_1 and x is adjacent to some $w \in V(L_2)$ with $d_{C'}(q_1, w) \geq \ell + 3 \geq 7$. Hence $d_C(u, w) > 6$. But then u-q_1-x-w is a shortcut, contradicting Theorem 2.5.4. Hence, we may assume $k \geq 2$.

We begin by proving $L_1 \cup Q$ or $L_2 \cup Q$ is a shortest long even hole.

(1) If $k = 2$, then $L_1 \cup Q$ or $L_2 \cup Q$ is a shortest long even hole.

Suppose $k = 2$, and thus $|E(L_1)| = 3$. Since $|E(C)| \geq 9$ and C has no shortcuts, q_1 and q_2 have no neighbors in L_2^*. Thus $L_2 \cup Q$ is a shortest long even hole. This proves (1).

(2) If there exists some $q_i \in \{q_1, q_2, \ldots, q_k\}$ such that $q_i \in V(C)$, then $L_1 \cup Q$ or $L_2 \cup Q$ is a

shortest long even hole.

Suppose for some $i \in \{1, 2, \ldots, k\}$, $q_i \in V(C)$. Let R denote the path of C with ends u, q_i that does not contain v and let S denote the path of C with ends q_i, v that does not contain u. By Theorem 2.5.4 and Lemma 2.5.3, it follows that R, S have lengths $d_C(u, q_i)$ and $d_C(q_i, v)$ respectively. Thus R has length at most i and S has length at most $k - i + 1$ by Theorem 2.5.4. So $|E(R \cup S)| = k + 1$ and we may assume $R \cup S = L_1$. By induction, the cycle C' obtained from C by replacing R with the path u-q_1-q_2-$\ldots$-q_i and the cycle obtained from C' by replacing S with the path q_i-q_{i+1}-$\ldots$-q_k-v are both clean shortest long even holes. But then $Q \cup L_2$ is a shortest long even hole. This proves (2).

(3) The vertex set $\{q_2, q_3, \ldots, q_{k-1}\}$ is anticomplete to $V(L_1)$ or it is anticomplete to $V(L_2)$.

Suppose that it isn't. Hence, for some $i, j \in \{2, 3 \ldots, k - 1\}$, q_i has a neighbor $x \in L_1^*$ and q_j has a neighbor $y \in L_2^*$ and we may assume $i \leq j$. By (2), we may assume $Q^* \cap V(C) = \emptyset$. Let R_1, S_1 be the subpaths of L_1 between u and x and between x and v, respectively. Let R_2, S_2 be the subpaths of L_2 between u and y and between y and v, respectively. Then by Theorem 2.5.4 and Lemma 2.5.3, $|E(R_1)| \leq i + 1$, $|E(S_1)| \leq k - i + 2$, $|E(R_2)| \leq j + 1$ and $|E(S_2)| \leq k - j + 2$. Since $|E(L_1)|$, $|E(L_2)| \geq k + 1$, it follows that $|E(R_1)| \geq i$, $|E(S_1)| \geq k - i + 1$, $|E(R_2)| \geq j$ and $|E(S_2)| \geq k - j + 1$. By Theorem 2.5.4, the distance between x and y in G is equal to the length of $S_1 \cup S_2$ or $R_1 \cup R_2$. So $d_G(x, y) \geq \min\{i + j + 2, 2k - i - j + 4\} > j - i + 2$. But the path x-q_i-q_{i+1}-$\ldots$-q_{j-1}-q_j-y has length $j - i + 2$, so it is a shortcut, a contradicting Theorem 2.5.4. This proves (3).

(4) If none of $q_2, q_3, \ldots, q_{k-1}$ have neighbors in $V(C)$, then $L_1 \cup Q$ or $L_2 \cup Q$ is a shortest long even hole.

Suppose none of $q_2, q_3, \ldots, q_{k-1}$ have neighbors in $V(C)$. By (1) we may assume $k \geq 2$ and by (2) we may assume $V(C) \cap Q^* = \emptyset$. Since C is clean there is a path P_1 of C with length at most two containing all neighbors of q_1 in $V(C)$. Choose P_1 to be minimal. Define P_2 similarly for q_k. Suppose P_1 has length two. Denote the ends of P_1 by w, z. Since the theorem holds if $k = 1$, the cycle C' obtained by replacing the middle vertex of P_1 with q_1 is a clean shortest long even hole. By Theorem 2.5.4, C' has no shortcut, so we may assume $d_C(v, w) \leq k - 1$. Then $z = u$ and we may assume $w = a_2$.

If q_k is adjacent to b_m, it follows that q_1-q_2-…-q_k-b_m is a shortcut of C', a contradiction. Suppose q_k is adjacent to b_{m-1}. Denote the hole u-b_1-b_2-…-b_{m-1}-q_k-q_{k-1}-…-q_1-u by C''. Since $|E(C)| \geq 2\ell + 3$, it follows that C'' is long. Since L_1, L_2, Q all have the same parity, C'' is even. But C'' is shorter than C, a contradiction. Hence, q_k has no neighbor in L_2^*. But then $L_2 \cup Q$ is a shortest long even hole. Thus, we may assume P_1 does not have length two. Similarly, P_2 does not have length two.

We may assume q_1 is adjacent to b_1, because otherwise $L_2 \cup Q$ is a shortest long even hole. It follows that $d_G(b_1, v) \leq k + 1$, so $d_C(b_1, v) \leq k + 1$ by Theorem 2.5.4. Therefore, $|E(L_2)| = k + 1$. We may assume q_k is adjacent to a_k, because otherwise $L_1 \cup Q$ is a shortest long even hole. But $|E(C)| \geq 2\ell + 3$, so $G[V(C \cup Q)]$ is a long prism, a contradiction. This proves (4).

(5) Suppose that $|E(L_2)| \geq |E(L_1)| + 2$ and for some $q_i \in \{q_2, q_3, \ldots, q_{k-1}\}$, q_i has a neighbor $w \in L_2^*$. Then, $L_1 \cup Q$ or $L_2 \cup Q$ is a shortest long even hole

By (2) we may assume $V(C) \cap Q^* = \emptyset$. By (3) none of $q_2, q_3, \ldots, q_{k-1}$ have neighbors in $V(L_1)$. Let R_2 and S_2 denote the subpaths of L_2 between u and w and between w and v respectively. By Theorem 2.5.4 and Lemma 2.5.3, $|E(R_2)| \leq i+1$ and $|E(S_2)| \leq k-i+2$. Since $|E(L_2)| \geq k+3$, it follows that $|E(R_2)| = i + 1$ and $|E(S_2)| = k - i + 2$. Hence, $|E(C)| = 2k + 4$ and $|E(L_2)| = |E(L_1)| + 2$. Since $|E(C)| \geq 2\ell + 3$, $L_1 \cup Q$ is a long even cycle, so we may assume $L_1 \cup Q$ is not an induced subgraph of G. Hence, we may assume q_1 is adjacent to a_j for some $j \in \{1, 2\}$. Then $d_C(w, a_j) \geq \min\{i + 2, 2k - i + 1\} > i + 1$. But b_j-q_1-q_2-…-q_i-w has length $i + 1$, a contradiction. This proves (5).

(6) $L_1 \cup Q$ or $L_2 \cup Q$ is a shortest long even hole.

Suppose neither $L_1 \cup Q$ nor $L_2 \cup Q$ is a shortest long even hole. By (2), $Q^* \cap V(C) = \emptyset$. By (3) and (5), we may assume $V(L_2)$ is anticomplete to $\{q_2, q_3, \ldots q_{k-1}\}$. Hence, by (4), we may assume that some $q_i \in \{q_2, q_3, \ldots, q_{k-1}\}$ has a neighbor $a_j \in V(L_1^*)$. Since $G[V(Q \cup L_2)]$ contains no long even hole we may assume q_1 is adjacent to b_1 and non-adjacent to b_2. Hence $d_G(b_1, v) \leq k + 1$, so by Theorem 2.5.4 $d_C(b_1, v) \leq k + 1$. Thus $|E(L_2)| \leq k + 2$. Since L_1 and L_2 have the same parity, $|E(L_2)| = k + 1$. Since b_1-q_1-q_2-…-q_i-a_j is not a shortcut, it follows from Lemma 2.5.3 that $j \leq i + 1$. Since a_j-q_i-q_{i+1}-…q_k-v is not a shortcut, it follows from Lemma 2.5.3 that $j \geq i$. Suppose $i = j$. Let C' denote the cycle obtained from C by replacing the path a_i-a_{i-1}-…a_1-u-b_1

with the path a_i-q_i-q_{i-1}-…-q_1-b. Then C' is a clean shortest long even hole by induction. But then q_i-q_{i+1}-…-q_k-v is a shortcut for C', contradicting Theorem 2.5.4. Hence, we may assume $i = j-1$ and that for all $c, d \in \{2, 3, \ldots, k-1\}$, if a_c is adjacent to q_d then $c = d-1$. Without loss of generality we may assume i is chosen to be the smallest element of $\{2, 3, \ldots, k-1\}$ such that q_i is adjacent to a_{i-1}. The paths b_1-u-a_1-a_2-…-a_{i-1}, b_1-q_1-q_2-…-q_i-a_{i-1} and b_1-b_2-…-b_m-v-a_k-a_{k-1}-…-a_{i-1} form a long jewel of order i. Hence, $i > \ell + 3$. Then, if a_1 is not adjacent to q_1, the cycle u-a_1-a_2-…-a_{i-1}-q_i-q_{i-1}-…-q_1-u is long even hole shorter than C, a contradiction. Thus, q_1 is adjacent to a_1. But then, q_1-L_2-vQq_i, q_1Qq_i and q_1-$a_1L_1a_{i-1}$-q_i form a long theta, a contradiction. This proves (6).

Let C' denote $L_2 \cup Q$. By (6) we may assume C' is a shortest long even hole. It remains to show C' is clean. Suppose there is a C'-major vertex x. Since x is not C-major, x has a neighbor in Q^*. Since Q^* is a shortest path there is a subpath M_1 of Q of length at most two containing all neighbors of x in $V(Q)$. Choose M_1 to be minimal. Thus x has a neighbor in L_2^*. Since x is not C-major, there is a subpath M_2 of L_2 of length at most two containing all neighbors of x in $V(L_2)$. Choose M_2 to be minimal. Let M_1 have ends y_1, z_1 where u, y_1, z_1, v are in order on Q. Let M_2 have ends y_2, z_2 where u, y_2, z_2, v are in order in L_2. Let Y denote the path of C' with ends y_1 and y_2 that contains u and let Z denote the path of C' with ends z_1 and z_2 that contains v. By Lemma 2.5.5, $M_1 \cup Y \cup M_2$ and $M_1 \cup Z \cup M_2$ both have length at least $\ell + 3$. Thus M_1 and M_2 are vertex disjoint and do not contain u or v. Since $|E(C)| \geq 2\ell + 3$ one of Y, Z has length at least ℓ, say Y. Hence the hole obtained by adding y_1-x-y_2 to Y is long and shorter than C, so Y is odd. Thus the path $M_1 \cup Z \cup M_2$ is odd.

Suppose M_1 has length two and denote its vertices by q_{i-1}-q_i-q_{i+1} in order. Let Q' be the path obtained from Q by replacing q_i with x. Since $L_2 \cup Q'$ is not an induced subgraph of G, by (6) we have that $L_1 \cup Q$ is an induced of G and $|E(L_1)| = |E(L_2)| = k+1$. Then $k+1 \geq \ell$. Since there are no long thetas, $L_1 \cup Q$ is not a hole. Thus q_i has a neighbor w in L_1^*. Then $d_G(w, y_2), d_G(w, z_2) \leq 4$. So by Theorem 2.5.4, it follows that $d_C(w, y_2), d_C(w, z_2) \leq 4$. Since $|E(C)| > 9$, we may assume y_2-z_2-v-w is a path of C. Hence, $w = a_k$. By Lemma 2.5.5, the path y_2-z_2-vQq_i has length at least $\ell + 3$. Thus, $i \leq k - \ell$ and it follows that uQq_i-a_k has length less than k. But by Lemma 2.5.3, $d_C(u, a_k) = k$ so uQq_i-a_k is a shortcut for C, contradicting Theorem 2.5.4.

Thus M_1 has length at most one, and so Z has length at least ℓ. Hence Z has odd length. Since C' is an even hole and Y, Z both have odd length, it follows that M_1 and M_2 have the same parity. If M_1 and M_2 both have length equal to zero, $G[V(C' \cup Q)]$ is a long theta, a contradiction. If

M_1 and M_2 both have length equal to one, $G[V(C' \cup Q)]$ is a long prism, a contradiction. So, M_1 has length equal to zero and M_2 has length equal to two. Let C'' be the cycle obtained from C by replacing the middle vertex of M_2 by x. Since $k \geq 2$, C'' is a clean shortest long even hole. Then by (6), $L_1 \cup Q$ is a long even hole. But then $G[V(C \cup Q)]$ is a long theta, a contradiction. □

We can now give the main result of the section.

Theorem 2.5.7. *There is an algorithm with the following specifications:*

Input: *A candidate G.*

Output: *Decides either that G has a long even hole or that there is no clean long even hole in G.*

Running Time: $\mathcal{O}(|G|^5)$.

Proof. If C is a clean shortest long even hole let $u, v, w \in V(C)$ be chosen so that each of $d_C(u,v)$, $d_C(w,v)$, $d_C(w,z)$ is equal to $\lceil |C|/3 \rceil$ or $\lfloor |C|/3 \rfloor$. Here is the algorithm: Guess u, v, w, find a shortest path between each pair of them and test whether these three paths form a long even hole. If so, output that G has a long even hole, otherwise output G has no long even hole. Correctness follows from Theorem 2.5.1. □

2.6 Cleaning a shortest long even hole

Let C be a shortest long even hole in G. For a C-major vertex x, we call a subpath P of C of length at least two an *x-gap* if both ends of P are neighbors of x and no interior vertex of P is adjacent to x. Thus, adding x to P yields a hole. For a pair of non-adjacent C-major vertices x, y we call a path P of C an *xy-gap* if $V(P)$ is the interior of an xy-path. For a path P with ends a, b we call $v \in V(P)$ a *midpoint* of P if it maximizes the value $\min\{d_P(v,a), d_P(v,b)\}$ among all vertices in $V(P)$. We begin with the following observations about gaps of major vertices on shortest long even holes.

Lemma 2.6.1. *Let G be a graph and let C be a shortest long even hole in G. Let x, y be a pair of non-adjacent C-major vertices. Suppose the neighbors of x in $V(C)$ are contained in a y-gap P. Then there is an xy-gap of length at most $\lceil \frac{\ell}{2} \rceil - 3$ contained in P.*

Proof. Suppose not. Let v_1, v_2 denote the ends of P. For $i \in \{1,2\}$, let P_i denote the xy-gap contained in P with one end equal to v_i. Then P_1, P_2 each have length at least $\lceil \frac{\ell}{2} \rceil - 2$. Let P_3 be the subpath of C with interior equal to $V(P) \setminus (V(P_1 \cup P_2)$. Thus $P = P_1 \cup P_2 \cup P_3$. Since

x is C-major, P_3 has length at least three and since y is C-major the path $C \setminus P^*$ has length at least three. It follows that $V(P) \cup \{y\}$ induces a long hole and it is shorter than C. Hence, $|E(P_1)| + |E(P_2)| + |E(P_3)|$ is odd. Since $V(P_1) \cup V(P_2) \cup \{x, y\}$ induces a long hole shorter than C, it follows that $|E(P_1)| + |E(P_2)|$ is odd. Hence, $|E(P_3)|$ is even. So $(V(C) \setminus P_3^*) \cup \{x\}$ induces a long even hole and it is shorter than C, a contradiction. □

Lemma 2.6.2. *Let G be a candidate and let C be a shortest long even hole in G. Let x, y be non-adjacent C-major vertices. Let P be a path of C such that P is a y-gap, every neighbor of x in $V(P)$ has P-distance at least $\ell - 1$ from an end of P and x has no neighbor in $V(C)$ that is adjacent to an end of P. Let x have a neighbor in $V(P)$. Then x has exactly two neighbors in $V(P)$ and they are adjacent.*

Proof. Denote the ends of P by a, b. By Lemma 2.6.1, it follows that x has a neighbor in $V(C) \setminus V(P)$. Since x is not adjacent to any vertex at C-distance at most 1 from an end of P and y has three pairwise non-adjacent neighbors in $V(C)$ by Lemma 2.4.1, there is an xy-path M_1 with $V(M_1)$ disjoint from and anticomplete to $V(P)$. Suppose $N(x) \cap V(P)$ consists of a single vertex w. Then there is a long theta formed by wPa-y, wPb-y, w-xM_1y, a contradiction. So we may assume that x has two non-adjacent neighbors in $V(P)$. Then there are two long xy-paths M_2, M_3 with interior in $V(P)$ such that M_2 and M_3 have disjoint and anticomplete interiors. But then, M_1, M_2 and M_3 form a long theta, a contradiction. □

We will prove the existence of a bounded-sized collection of paths and vertices of G with useful properties for cleaning C-major vertices of a shortest long even hole C called a "C-contrivance". A C-contrivance is analogous to a (K, P_1, P_2)-contrivance where K is a shortest long prism and P_1, P_2 are distinct constituent paths of K. The techniques used to prove that G contains a C-contrivance are similar to those used in the proof of Lemma 2.3.9. More formally, for a graph G and a shortest long even hole C in G, we call a triple $(A, \mathcal{B}, m)$ a *C-contrivance* if the following conditions all hold:

1. A is a set of C-major vertices.
2. $\mathcal{B}$ is set of paths of C.
3. m is a vertex in $V(C)$.
4. Every C-major vertex has a neighbor in $A \cup (\cup_{B \in \mathcal{B}} B^*)$.
5. There is a path P of C with both ends in $\cup_{B \in \mathcal{B}} B^*$ such that m is a midpoint of P, all C-major vertices have a neighbor in $V(P)$ and $N(A) \cap V(P) \subseteq \cup_{B \in \mathcal{B}} B^*$.

6. Every vertex in A has a neighbor in $\cup_{B\in\mathcal{B}}B^*$.

If $(A,\mathcal{B},m)$ is a C-contrivance, we say that A, $\mathcal{B}$ and m *form* a C-contrivance. We guess a C-contrivance as a part of our cleaning algorithm, so it is critical to prove the existence of a C-contrivance with some bound on $|A|,|\mathcal{B}|$ and the lengths of the paths in $\mathcal{B}$. We don't need to have a bound on the length of P.

We call a C-contrivance $(A,\mathcal{B},m)$ *useful* if $|A|\leq 3$, $|\mathcal{B}|\leq 6$, all paths in $\mathcal{B}$ have length at most $2\ell-5$ and at most two paths in $\mathcal{B}$ have length greater that $\ell+2$. By definition of C-contrivance, $\cup_{B\in\mathcal{B}}B^*\neq\emptyset$ so some path in $\mathcal{B}$ must have length at least two.

Lemma 2.6.3. *Let G be a candidate and let C be a shortest long even hole in G. Then there is a there is a useful C-contrivance.*

Proof. We may assume that C is not clean. Let a_1 be a C-major vertex with an a_1-gap P such that P has maximum length among all paths of C that form an x-gap for some C-major vertex x. Denote the ends of P by v_1,v_2. It follows that every C-major vertex has a neighbor in $V(P)$. For $i\in\{1,2\}$ let B_i be the path of C whose vertex set consists of all vertices of $V(P)$ with P-distance at most $\ell-1$ from v_i and the two vertices of $V(C)\setminus V(P)$ with C-distance at most two from v_i.

Let S be the set of C-major vertices with no neighbor in $B_1^*\cup B_2^*\cup\{a_1\}$. Let m be a midpoint of P. We may assume that $S\neq\emptyset$, because otherwise the triple consisting of $(\{a_1\},\{B_1,B_2\},m)$ is a useful C-contrivance. For each $w\in S$, we define P_w to be the w-gap with $v_1\in P_w^*$. Let a_2 be an element of S maximizing $|E(P_{a_2})\setminus E(P)|$. Let v_3 denote the end of P_{a_2} that is not contained in $V(P)$. Since a_2 has no neighbor in $B_1^*\cup B_2^*\cup\{a_1\}$ and a_2 has at least one neighbor in $V(P)$, it follows from Lemma 2.6.2 that a_2 has exactly two neighbors in $V(P)$ and they are adjacent. Denote them by v_4,v_5, where v_4 is an end of P_{a_2}. Let R denote the path $v_1P_{a_2}v_3$. It follows that every C-major vertex has a neighbor in $B_1^*\cup B_2^*\cup\{a_1\}$ or a neighbor in $V(R)$. Let B_3 be the path of C whose vertex set consists of all vertices of $V(P_{a_2})$ with P_{a_2}-distance at most $\ell-1$ from v_3 and the two vertices of $V(C)\setminus V(P_{a_2})$ with C-distance at most two from v_3. Let B_4 be the path of C whose vertex set consists of all vertices of $V(P_{a_2})$ with P_{a_2}-distance at most $\ell-1$ from v_4 and the two vertices of $V(C)\setminus V(P_{a_2})$ with C-distance at most two from v_4. Then $v_5\in B_4^*$.

Let T denote the set of C-major vertices with no neighbor in $\cup_{i=1}^4 B_i^*\cup\{a_1,a_2\}$. Let $t\in T$. Then t is not equal to a_1 or a_2. Since t has a neighbor in $V(R)$ it follows from Lemma 2.6.2 applied to a_2,t and P_{a_2} that a_3 has exactly two neighbors in $V(P_{a_2})$ and they are adjacent. Denote them by x_1,x_2. Since t has a neighbor in $V(R)$ and t is not adjacent to v_1,v_3, it follows that $x_1,x_2\in V(R)$. Since t has a neighbor in $V(P)$, it follows from Lemma 2.6.2 applied to a_1,t and P that t has exactly two neighbors in P^*. Denote them by x_3, x_4. Since x_1, x_2 are the only neighbors of t in $V(P_{a_2})$ it follows that $x_3, x_4 \in P^* \setminus V(P_{a_2})$.

We may assume $T \neq \emptyset$ because otherwise $\{a_1, a_2\}$, $\{B_1, B_2, B_3, B_4\}$ and m form a useful C-contrivance. Let a_3 be an arbitrary element of T. Denote the neighbors of a_3 in $V(R)$ by v_6, v_7 and the denote the neighbors of a_3 in $P^* \setminus V(P_{a_2})$ by v_8, v_9. Let B_5 be the path of C containing all vertices of $V(C)$ with C-distance at most $\ell - 3$ from v_6 or v_7. Let B_6 be the path of C containing all vertices of $V(C)$ with C-distance at most $\ell - 3$ from v_8 or v_9.

Let $A = \{a_1, a_2, a_3\}$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_6\}$. We claim $(A, \mathcal{B}, m)$ is a useful C-contrivance. It follows from the choice of $B_1, B_2, \ldots, B_6$ that $v_i \in \cup_{j=1}^{6} B_j^*$ for every $i \in \{1, 2, \ldots, 9\}$. Hence, $(A, \mathcal{B}, m)$ satisfies condition 6. Moreover, $(A, \mathcal{B}, m)$ satisfies conditions 1, 2 and 3 and the conditions of usefulness by construction, so we need only prove it satisfies 4 and 5.

Every C-major vertex has a neighbor in $V(P)$ and $N(A) \cap V(P) = \{v_1, v_2, v_4, v_5, v_8, v_9\}$. Since P has ends v_1, v_2 and m is a midpoint of P, condition 5 is satisfied. Suppose there is some C-major vertex w that has no neighbor in $A \cup (\cup_{i=1}^{6} B_i^*)$. Then $w \in T$. Hence, w has exactly two neighbors in $V(R)$ and they are adjacent and w has exactly two neighbors in $V(P) \setminus V(P_{a_2})$ and they are adjacent. Since w has no neighbors in $B_5^* \cup B_6^*$ it follows that there are two long induced a_3w-paths M_1, M_2 with $M_1^* \subseteq R$ and $M_2^* \subseteq V(P) \setminus V(R)$. Moreover, since a_3, w have no neighbors in B_1^*, it follows that M_1^* is anticomplete to M_2^*. Since w and a_3 each have no neighbor in $B_2^* \cup B_3^*$, it follows that w, a_3 have no neighbors in $V(C)$ that are adjacent to v_2 or v_3. Hence, it follows from Lemma 2.4.1 that there is a wa_3-path M_3 with M_3^* disjoint from and anticomplete to $V(P) \cup V(R)$. But then M_1, M_2, M_3 form a long theta, a contradiction. □

We can now prove the main result of this section.

Theorem 2.6.4. *There is an algorithm with the following specifications:*

Input: *A candidate G.*

Output: *A list of $\mathcal{O}(|G|^{8\ell+2})$ sets with the following property: For every shortest long even hole C there is some X in the list such that X contains all C-major vertices and $X \cap V(C) = \emptyset$.*

Running Time: $\mathcal{O}(|G|^{8\ell+4})$

Proof. We guess a set A of at most three vertices in $V(G)$ and a vertex m in $V(G)$. We guess a list $\mathcal{B}$ of at most 6 paths of G, $B_1, B_2, \ldots B_k$ such that all paths in $\mathcal{B}$ have length at most $2\ell - 5$, at least one path in $\mathcal{B}$ has length greater than one, and at most two paths in $\mathcal{B}$ have length greater than $\ell + 2$.

Let $\mathcal{B}^*$ denote $\cup_{i=1}^k B_i^*$. By definition, $\mathcal{B}^* \neq \emptyset$. Let Y be the set of vertices in $V(G) \setminus (\cup_{i=1}^k V(B_i))$ with neighbors in $\mathcal{B}^*$. Guess two vertices d_1, d_2 in $\mathcal{B}^*$. Let R, S be union of the vertex sets of all shortest d_1m-paths and d_2m-paths in $G \setminus ((Y \cup N(A)) \setminus \mathcal{B}^*)$ respectively. Let Z be the set of vertices in $V(G) \setminus (Y \cup \mathcal{B}^*)$ with a neighbor in A and a neighbor in $R \cup S \setminus \{d_1, d_2\}$. Output $Y \cup Z$.

We will now prove the output is correct. Suppose C is a shortest long even hole in G. Then by Lemma 2.6.3, G contains a useful C-contrivance. Thus, for some guess of $(A, \mathcal{B}, m)$, the triple $(A, \mathcal{B}, m)$ is a useful C-contrivance. By construction, Y is disjoint from $V(C)$ and Y contains every C-major vertex in $V(G)$ with a neighbor in $\mathcal{B}^*$. By condition 6 in the definition of C-contrivance, A is contained in Y. Let P denote the path of C with ends d_1, d_2 that contains m. By condition 5 in the definition of C-contrivance, we may assume that d_1, d_2 are chosen such that every C-major vertex has a neighbor in $V(P)$, m is a midpoint of P and $N(A) \cap V(P) \subseteq \mathcal{B}^*$. Let P_1 denote the subpath of P with ends d_1, m and let P_2 denote the subpath of P with ends d_2, m. Since m is a midpoint of P, it follows that P_1 and P_2 have lengths $d_C(d_1, m)$ and $d_C(d_2, m)$, respectively. Since $N(A) \cap V(P) \subseteq \mathcal{B}^*$, the paths P_1, P_2 are both subgraphs of $G \setminus ((Y \cup N(A)) \setminus \mathcal{B}^*)$. By condition 4 in the definition of C-contrivance, $Y \cup N(A)) \setminus \mathcal{B}^*$ contains all C-major vertices. Hence, it follows from Theorem 2.5.4 that P_1 and P_2 are shortest paths between d_1, m and d_2, m, respectively, in $G \setminus ((Y \cup N(A)) \setminus \mathcal{B}^*)$. Hence, $P^* \subseteq (R \cup S) \setminus \{d_1, d_2\}$.

We prove Z contains all C-major vertices in $V(G) \setminus Y$. Suppose $w \in V(G) \setminus Y$ is a C-major vertex. By condition 4, it follows that w has a neighbor in A. Since $d_1, d_2 \in \mathcal{B}^*$ and all C-major vertices have a neighbor in $V(P)$, it follows that w has a neighbor in $(R \cup S) \setminus \{d_1, d_2\}$.

We prove Z is disjoint from $V(C)$. Suppose there exists some $z \in V(C) \cap Z$. Then since $N(A) \cap V(P) \subseteq \mathcal{B}^*$ it follows that $z \notin V(P)$. Since m is a midpoint of P, it follows that $d_C(z, m) \geq \frac{|E(P)|}{2} + 1$. Since $z \in Z$, we may assume there is some shortest d_1m-path in $G \setminus ((Y \cup N(A)) \setminus \mathcal{B}^*)$ and $q \in V(Q) \setminus \{d_1\}$ such that z is adjacent to q. Then, the path qQm has length strictly less than $\frac{|E(P)|}{2}$ so z-qQm has length strictly less than $d_C(z, m)$. Since $Y \cup N(A)) \setminus \mathcal{B}^*$ contains all C-major vertices, z-qQm contains no C-major vertices. Hence, z-qQm is a shortcut, contradicting Theorem 2.5.4. This completes the proof of correctness.

We will now prove the bounds on the running time and list length. There are $\mathcal{O}(|G|^{8\ell+2})$ guesses of $(A, \mathcal{B}, m)$ to check. For each such guess, we compute Y, Z in time $\mathcal{O}(|G|^2)$. Hence the list outputted has length $\mathcal{O}(|G|^{8\ell+2})$ and the total running time is $\mathcal{O}(|G|^{8\ell+4})$. □

2.7 The Algorithm

We can now prove our main result which we restate:

Theorem 2.1.1 *For each integer $\ell \geq 4$ there is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *Decides whether G has an even hole of length at least ℓ.*

Running Time: $\mathcal{O}(|G|^{108\ell-22})$.

Proof. Our algorithm is as follows. We begin by testing if G is a candidate by performing the following steps. We apply the algorithm of Theorem 2.2.1 to test whether G contains a long even hole of length at most $2\ell + 2$ in time $\mathcal{O}(|G|^{2\ell+2})$, we apply the algorithm of Theorem 2.2.2 to test whether G contains a long jewel of order at most $\ell+3$ in time $\mathcal{O}(|G|^{3\ell+6})$ and we apply the algorithm of Theorem 2.2.4 to test whether G contains a long theta in time $\mathcal{O}(|G|^{2\ell+7})$. We may assume these tests fail, because otherwise G contains a long even hole. Then, we apply the algorithm of Theorem 2.3.1 to test whether G contains a long prism in time $\mathcal{O}(|G|^{108\ell-22})$. We may assume this test fails, because otherwise G contains a long even hole. Next, we apply the algorithm of Theorem 2.4.4 which outputs either that G contains a long even hole or that G contains no shortest long even hole C such that $E(G) \setminus E(C)$ contains a C-major edge in time $\mathcal{O}(|G|^{4\ell+10})$. We may assume the algorithm outputs that G contains no shortest long even hole with a major edge. Consequently, G is a candidate.

Thus we are able to apply the algorithm given in Theorem 2.6.4 to obtain a cleaning list for major vertices of length $\mathcal{O}(|G|^{8\ell+2})$ in time $\mathcal{O}(|G|^{8\ell+4})$. For every X in the list we use the algorithm of Theorem 2.5.7 to test whether $G \setminus X$ has a clean shortest long even hole in time $\mathcal{O}(|G|^5)$. If we have not found a clean shortest long even hole in $G \setminus X$ for any X in the list, we output that G has no long even hole.

Correctness follows from Theorems 2.5.7 and 2.6.4. For the running time, testing whether G is a candidate takes time $\mathcal{O}(|G|^{108\ell-22})$ and determining whether a candidate contains a shortest long even hole takes time $\mathcal{O}(|G|^{8\ell+7})$. Hence, the total running time is $\mathcal{O}(|G|^{108\ell-22})$. □

Chapter 3

Monoholed Graphs

In the next several chapters we will describe the structure of graphs where every hole has length ℓ for some integer $\ell \geq 7$. We call G *ℓ-monoholed* if G only contains holes of length ℓ for some integer ℓ. When the value of ℓ is not ambiguous we will refer to G as *monoholed*. We need the following easy fact about monoholed graphs:

Fact 3.0.1. *Let G be an ℓ-monoholed graph for some $\ell \geq 5$. Suppose C is a hole in G. Then for every $v \in V(G) \setminus V(C)$, either v is complete to $V(C)$, v is anticomplete to $V(C)$ or there is a path P of C of length at most such that $N(v) \cap V(C) = V(P)$.*

Proof. Suppose some $v \in V(G) \setminus V(C)$ has both a neighbor and a non-neighbor in $V(C)$. Let P be a path of C containing all neighbors of v in $V(C)$ and choose P to be minimal. We may assume P has length at least two. Then $V(C) \cup \{v\} \setminus P^*$ induces a hole of length $|E(C)| - |E(P)| + 2$. Since G is ℓ-monoholed it follows that P has length two. Since $\ell \geq 5$, $V(P) \cup \{v\}$ does not induce a hole so v is adjacent to the interior vertex of P. □

3.1 Introducing the structure of ℓ-monoholed graphs

A well-known class of bipartite graphs called half-graphs comes up frequently in our analysis. This class was first named by Erdős and Hajnal in [35]. For an integer $n \geq 1$ we say H_n is the bipartite graph on with vertices $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ and edge set $\{x_i y_j \mid i, j \in [k], i \geq j\}$. (See Figure 3.1). We call a graph G a *half-graph* if G is contained in H_n for some n. (Note that in [35], half-graphs only referred to the set of graphs H_n for every integer n.) It follows from the definition of half-graph, that a graph G is a half-graph if and only if it contains no induced two edge matching.

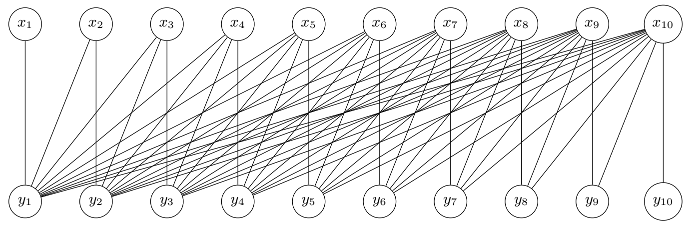

Figure 3.1: An illustration of H_{10}

Let X, Y, Z be disjoint sets of vertices and suppose there is a half-graph H_1 between X and Y and a half-graph H_2 between Y and Z. Let H denote the graph with vertex set $X \cup Y \cup Z$ and edge set $E(H_1 \cup H_2)$. We say H_1 and H_2 are *compatible with respect to* X if for every $x, x' \in X$, $N_H(x) \subseteq N_H(x')$ or $N_H(x') \subseteq N_H(x)$.

The following class of graphs comes up repeatedly in our analysis. We call H a *threshold graph* if the vertex set of H can be partitioned into a stable set S and the vertex set of a clique K and the edges between S and K form a half-graph. Threshold graphs were first introduced by Chvátal and Hammer in [68]. For further background on threshold graphs see Chapter 10 of Golumbic's *Algorithmic Graph Theory and Perfect Graphs* [37] or Mahadev and Peled's book on the subject [52]. We will need a theorem of [68]:

Theorem 3.1.1 (Chvátal and Hammer, 1973)**.** *A graph H is a threshold graph if and only if it contains no P_4, C_4, or two-edge matching.*

Proof. Let K be a maximal clique in H. Let S denote $V(H) \setminus V(K)$. Suppose some $s_1, s_2 \in S$ are adjacent. Since K is a maximal clique there are distinct vertices $k_1, k_2 \in V(K)$ such that k_1 is not adjacent to s_1 and k_2 is not adjacent to s_2. Since s_1s_2 and k_1k_2 are not an induced two edge matching, we may assume s_1 is adjacent to k_2. If k_1 and k_2 are adjacent s_1-s_2-k_1-k_2-s_1 is an induced C_4, a contradiction. So k_1 is not adjacent to k_2. But then, k_1-k_2-s_1-s_2 is an induced P_4, a contradiction.

Thus, S is a stable set. Let $s_1, s_2 \in S$ and suppose there is some $k_1 \in N(s_1) \setminus N(s_2)$. Then $N(s_2) \subseteq N(s_1)$ for if there is a $k_2 \in N(s_2) \setminus N(s_1)$, then s_1-k_1-k_2-s_2 is an induced P_4, a contradiction. Hence there is a half graph between $V(K)$ and S. □

A type of graph called a transitive closure of a tree comes up repeatedly in our analysis. Given a tree T with root $r \in V(T)$ we call the graph obtained from T by replacing every induced path from the root to a leaf the *transitive closure of* T and denote it as $\mathcal{T}(T, r)$. See Figure 3.2 for an illustration.

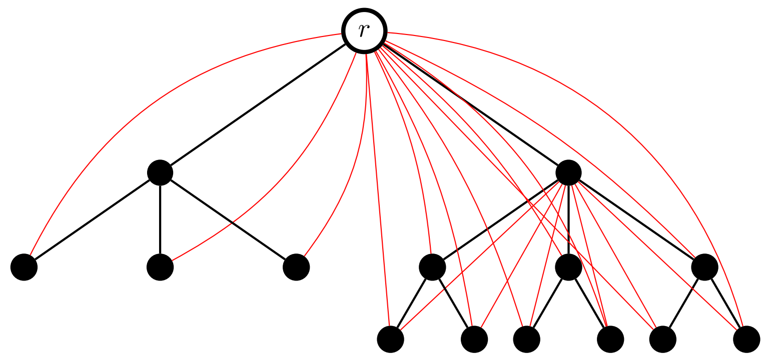

Figure 3.2: An example of a transitive closure of a tree. Let T be the tree drawn in black and r the root of T. Then the transitive closure is the union of T and the red edges.

3.1.1 Inflated graphs

We use an object called an "inflated graph" throughout our analysis. We call $\mathcal{H}$ an inflated-graph if $\mathcal{H}$ is obtained from a graph G by replacing every $v \in V(G)$ by a non-empty clique K_v and for every edge $xy \in E(G)$ we add a connected half-graph between K_x and K_y so that for every $v \in V(G)$ the half graphs between K_x and K_y for $x, y \in N(v)$ are pairwise compatible with respect to K_v. We say $V(K_v)$ for $v \in V(G)$ are the *bags* of $\mathcal{H}$. Informally, inflated graphs can be thought of as a generalization of rings.

If $\mathcal{X}$ is a set of bags of an inflated graph we denote the union of all bags in $\mathcal{X}$ as $V(\mathcal{X})$. We call an inflated-graph $\mathcal{M}$ a *sub-inflated-graph* if there is some injective function f from the bags of $\mathcal{M}$ to the bags of $\mathcal{H}$ such that for any bag B of M, $B \subseteq f(B)$. We say $\mathcal{M}$ is an *underlying inflated-graph* of $\mathcal{H}$ if $\mathcal{M}$ is a sub-inflated-graph of H and f is a bijection. We say M is an *underlying graph* of $\mathcal{H}$ if M is an underlying inflated-graph of $\mathcal{H}$ and every bag of M has size exactly one. For any $v \in V(M)$ we say the bag $f(B)$ and v *correspond* to each other.

It follows from the definition that G is an underlying graph of $\mathcal{H}$ with the function f mapping v to K_v for every $v \in V(G)$. Moreover, all underlying graphs of an inflated-graph are isomorphic. If B_1, B_2 are bags of $\mathcal{H}$ and B_1, B_2 correspond to adjacent vertices in M we call them *adjacent bags* or *neighboring bags*. We denote the set of neighboring bags of a bag B as $\mathcal{N}(B)$.

For $u \in K_x$ and $v \in K_y$ for some two bags K_x, K_y of $\mathcal{H}$ we say the $\mathcal{H}$-underlying distance between u and v is the distance between x and y in G.

We call $\mathcal{H}$ a *inflated path* or *inflated cycle* if the graph underlying $\mathcal{H}$ is a path or a cycle, respectively. In this case, we say the *length* of H is the length of its underlying graph. For an inflated path $\mathcal{P}$ we call the union of all bags corresponding to interior vertices of its underlying graph *interior bags* of $\mathcal{P}$ and denote it as $\mathcal{P}^*$. We call the bags corresponding to ends of the underlying graph of $\mathcal{P}$ *end bags* of $\mathcal{P}$. Note, a ring on n sets is an inflated C_n for any integer $n \geq 3$.

Lemma 3.1.2. *Let $\mathcal{H}$ be an inflated-graph. Then the following statements all hold:*

(a) For every bag B of $\mathcal{H}$ there is a $v \in V(B)$ that is complete to $V(\mathcal{N}(B))$

(b) For any two vertices $u, v \in V(H)$ if u and v are contained in different bags of $\mathcal{H}$ then there is an underlying graph of $\mathcal{H}$ containing both u and v.

(c) Let $\mathcal{S}$ be an sub-inflated-graph of $\mathcal{H}$ and suppose that every bag of S has size one. Then there is an underlying graph G of $\mathcal{H}$ such that $\mathcal{S}$ is contained in G as an induced subgraph.

Proof. Let B be a bag of $\mathcal{H}$ and let D be a neighboring bag of B. Then by definition of connected half graph, some $v \in V(B)$ is complete to $V(D)$. Let D' an arbitrary neighboring bag of B. Then v is complete to $V(D')$ since the half graphs between B, D and B, D' have equivalent linear orders with respect to B. This proves (a)).

Suppose $\mathcal{S}$ is a sub-inflated-graph of $\mathcal{H}$. By (a) for each bag B of $\mathcal{H}$ there is a vertex v_B that is complete to all neighboring bags of $\mathcal{H}$. Let $\mathcal{B}$ be the set of bags of $\mathcal{H}$ that do not contain any vertices of $\mathcal{S}$. By (a) every $B \in \mathcal{B}$ contains a vertex v_B complete to all neighboring bags of B. Then by definition of sub-inflated-graph $V(S) \cup \{V_B \mid B \in \mathcal{B}\}$ induces an underlying graph of $\mathcal{H}$. This proves (c).

This completes the proof because (c) implies (b) since the graph induced by any two vertices in different bags of $\mathcal{H}$ is a sub-inflated-graph of $\mathcal{H}$ with bags of size one. ☐

We will now state the main result of this chapter.

Theorem 3.1.3. *Let G be an ℓ-monoholed graph for some $\ell \geq 7$ one of the following conditions holds:*

(a) G contains a vertex that is adjacent to every other vertex in $V(G)$.

(b) G contains a clique cutset,

(c) G is chordal,

(d) G is an inflated ℓ-hole,

(e) G is a type of inflated graph we call a "crowned k-corpus". If ℓ is odd, then G is type of crowned k-corpus we call a "k-pyramidoid" for some $k \geq 3$.

Note, the inflated graphs in case (e) are not yet defined. As their definitions are both somewhat technical we will describe them later in this chapter. For now it is fully characterize the structure of k-pyramidoids, and thus the structure of ℓ-monoholed graphs when $\ell \geq 7$ and odd. When $\ell \geq 8$ and even we do not have a complete structure theorem for crowned k-corpuses but we can give a strong description of their structure. For the rest of this chapter we will assume that ℓ is some integer greater than 7.

3.2 Spines and Spiders

In this section we discuss a special kind of graph called a "mated k-spider" with a "nice" structure. We will prove that for any ℓ-monoholed graph G, G contains a k-mated spider for some $k \geq 3$ or G satisfies one of conditions (a), (b), (c) or (d) of Theorem 3.1.3. We will fully characterize the structure of mated spiders in Section 3.3. In future sections we go on to fully characterize ℓ-monoholed graphs by choosing a mated k-spider S in G with k maximum and seeing how $G \setminus S$ attaches to S when G does not satisfy conditions (a), (b), (c) or (d) of Theorem 3.1.3.

For $k \geq 3$, we call a graph S a *k-spider* if it vertices of degree one are $t_1, t_2, \ldots, t_k$ called its *toes* and is minimally connected under deleting vertices with these toes. For $i, j \in [k]$, let $d_S(i, j)$ denote $d_S(t_i, t_j)$. When the choice of spider S is not ambiguous we will simply write $d(i, j)$ for $d_S(i, j)$. We call a path P of S an *leg* if one end of P is a toe and P is a maximal path satisfying that all internal vertices of P have degree two. It follows that for each $i \in [k]$ there is a unique leg L_i with one end equal to t_i and L_i has length at least one. For each $i \in [k]$, we refer to L_i as the *t_i-leg* of S. For each $i \in [k]$, let a_i denote the end of L_i that is not equal to t_i. We refer to $a_1, a_2, \ldots, a_k$ at the *joints* and for each $i \in [k]$ we call a_i the *t_i-joint* of S. Let A be the graph obtained from S by deleting $L_i \setminus a_i$ for every $i \in [k]$. By definition, for all $i, j \in [k]$, if $i \neq j$ then $V(L_i \setminus a_i)$ is anticomplete to $V(L_j \setminus a_j)$. It follows that A is connected. We call A the *body* of S. See Figure 3.3. We call two k-spiders S, S' *mates* if they have the same set of toes, their vertex sets are anticomplete except for their toes and for every $i, j \in [k]$ with $i \neq j$, $d_S(i, j) + d_{S'}(i, j) = \ell$. We call a spider *mated* if it has a mate.

We will make repeated use of an easy fact about graphs with no clique cutset.

Fact 3.2.1. *Let G be a graph with no clique cutset. Let H be a proper induced subgraph of G. Then there is a connected induced subgraph S of $G \setminus V(H)$, such that $G[V(H \cup S)]$ contains no induced subgraph.*

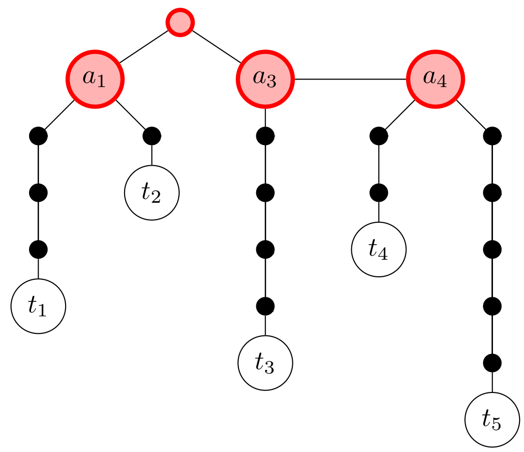

Figure 3.3: An example of a 5-spider. The vertices of the body are drawn in red. Note $a_1 = a_2$ and $a_4 = a_5$.

Proof. Suppose not. Let C be a component of $G \setminus V(H)$. Then the vertices of H that have neighbors in $V(C)$ form a clique. So $N(H) \cap V(C)$ is a clique cutset in G. □

We need the following lemma:

Lemma 3.2.2. *Let G be a ℓ-monoholed graph. Suppose G contains an inflated C_ℓ $\mathcal{H}$. Let $w \in V(G) \setminus V(\mathcal{H})$ and suppose w has a neighbor in $V(\mathcal{H})$. Then either, w is complete to $V(C)$ or the following conditions all hold:*

(a) There are some three consecutive bags of B_1, B_2, B_3 of $\mathcal{H}$ such that the neighbors of v in $V(\mathcal{H})$ are contained in $V(B_1 \cup B_2 \cup B_3)$,

(b) If w has neighbors in both $V(B_1)$ and $V(B_3)$, w is complete to $V(B_2)$,

(c) The graphs between $B_1, B_2 \cup \{w\}$ and $B_3, B_2 \cup \{v\}$ are half graphs and they are compatible with respect to $B_2 \cup \{w\}$, and

(d) $G[V(\mathcal{H}) \cup \{w\}]$ has a clique cutset or $G[V(\mathcal{H}) \cup \{w\}]$ induces an inflated C_ℓ whose bags can be obtained from the bags of $\mathcal{H}$ by adding w to B_2.

Proof. Suppose $w \in V(G) \setminus V(H)$ has both a neighbor and a non-neighbor in $V(\mathcal{H})$.

If w has a non-neighbor in a bag X of $\mathcal{H}$, then w cannot have neighbors in both adjacent bags of X in $\mathcal{H}$. (3.1)

Let Y, Z denote the neighboring bags of X in $\mathcal{H}$ and Suppose w has a neighbor $y \in Y$ and neighbor $z \in Z$. By Lemma 3.1.2, there is some $y' \in Y$ and $z \in Z$ are complete to $V(X)$. Hence,

$G[\{y, y', x, z, z', w\}]$ includes a hole of length at most 6, a contradiction. This proves (3.1).

It follows that w does not have a neighbor in every bag of $\mathcal{H}$. Let $\mathcal{Q}$ be a sub-inflated-graph of $\mathcal{H}$ containing all neighbors of w in $V(\mathcal{H})$. Choose $\mathcal{Q}$ to contain a minimum number of bags. Then $\mathcal{Q}$ is an inflated-path. If $\mathcal{Q}$ has length less than three the Lemma holds because any inflated-path of length three obtained by adding neighboring bags of the end bags of $\mathcal{Q}$ to $\mathcal{Q}$ satisfies (a). Hence we assume Q has length at least two. Suppose $\mathcal{Q}$ has length greater than three. Since $\mathcal{Q}$ is minimal, w has a neighbor u, v in the end bags of $\mathcal{H}$. Then by Lemma 3.1.2, there is an underlying graph C of $\mathcal{H}$ with $u, v \in V(C)$. It follows from the definition of $\mathcal{Q}$ that $C \setminus \mathcal{Q}^* \cup \{w\}$ is a hole of length $\ell - m + 2$ where m denotes the length of $\mathcal{Q}$. Hence, (a) holds and (b) hold by applying (3.1).

Let B_1, B_2, B_3 be the bags of $\mathcal{Q}$ in order. Suppose there is some $b_1 \in V(B_1)$, $b_2 \in V(B_2)$ and $b_3 \in V(B_3)$ such that b_1 is adjacent to b_2, w is adjacent to b_3, w is not adjacent to b_2 and b_1 is not adjacent to b_3. By Lemma 3.1.2 there is an underlying hole C of $\mathcal{H}$ with $b_1, b_3 \in V(\mathcal{H})$. Let P be the path from C obtained by deleting the vertex of B_2 in C. Then $V(C) \cup \{b_2, w\}$ induces a hole of length $\ell + 1$. Hence, (c) holds.

Suppose $G[V(\mathcal{H}) \cup \{w\}$ has no clique cutset. By definition of inflated graph, we need only show that w has a neighbor in both $V(B_1)$ and B_3 to prove (d). Suppose the neighbors of w in $\mathcal{H}$ are contained in $V(B_1 \cup B_2)$. Then since the half-graphs between B_1, B_2 and B_3, B_2 are compatible with respect to B_2, if $b_1 \in V(B_1)$ is adjacent to w then b_1 is complete to $V(B_2)$. Hence, $N(w) \cap V(\mathcal{H})$ is the vertex set of a clique, a contradiction. Thus, (d) holds. □

Theorem 3.2.3. *Let G be a ℓ-monoholed graph for some $\ell \geq 6$. Then one of the following holds:*

(a) G contains a vertex that is adjacent to every other vertex in $V(G)$.

(b) G contains a clique cutset,

(c) G is chordal,

(d) G is an inflated ℓ-hole,

(e) G contains a pair of mated k-spiders for some $k \geq 3$.

Proof. We may assume that (a), (b), (c), and (d) do not hold. G contains an inflated ℓ-hole $\mathcal{C}$ because it is not chordal. Choose $\mathcal{C}$ to maximize $|V(\mathcal{C})|$. Let W be the set of vertices in $V(G) \setminus V(C)$ that are complete to $V(\mathcal{C})$. Since $\mathcal{C}$ is not a clique and G is C_4-free, $G[W]$ must be a clique. $V(G) \neq V(\mathcal{C}) \cup W$ because (a) and (d) do not hold. Since G does not contain a clique cutset there is some connected graph X contained in $G \setminus (V(\mathcal{C}) \cup W)$ such that $V(C)$ has two non-adjacent neighbors in $V(\mathcal{C}) \cup W$. Since W is complete to $V(\mathcal{C})$ it follows that $V(X)$ must have two non-adjacent neighbors in $V(\mathcal{C})$. Choose X to be minimal. Then, X is a path. $G[V(\mathcal{C} \cup X)]$ does not contain a clique cutset. So by Lemma 3.2.2 if X is a single vertex x, then x can be added to a bag of $\mathcal{C}$ to obtain a larger inflated hole, contradicting the maximality of $\mathcal{C}$. Thus $|X| \geq 2$. Let the vertices of X be x_1-x_2-$\ldots$-x_n in order.

Then by minimality of X, there are some two non-adjacent vertices $v, w \in V(\mathcal{C})$ such that x_1 is adjacent to v and x_n is adjacent to w. Then v and w cannon be in the same bag of $\mathcal{C}$. By Lemma 3.1.2 there is a hole C underlying $\mathcal{C}$ with $x, y \in V(C)$. By minimality of X, there are paths P_1, P_2 of A of length at most one containing all neighbors of x_1 and x_n in $V(A)$, respectively. Choose P_1, P_2 to be minimal. For $i \in \{1, 2\}$, let the ends of P_i be a_i, b_i. We may assume a_1, b_1, b_2, a_2 occur in order in C. Let A be the path of C with ends a_1, a_2 that does not contain b_1 or b_2 and let B be the path of C' with ends b_1, b_2 that does not contain a_1 or a_2.

If some vertex $v \in V(X) \setminus \{x_1, x_2\}$ has a neighbor in $z \in V(C) \setminus V(B)$ then $z \in V(A)$ and A is a path of length two. The same statement holds with A and B exchanged. (3.2)

Suppose x_i has a neighbor in $z \in V(C) \setminus V(B)$ for some $i \in [2, n-1]$. By minimality of X, z is adjacent to all of a_1, a_2, b_1, b_2. Hence, $a_1 = b_1$, $a_2 = b_2$. Then a_1 and a_2 are not adjacent, so A is the path a_1-z-a_2. (See Figure 3.4 for an illustration). This proves (3.2).

No vertex in $V(X) \setminus \{x_1, x_n\}$ has a neighbor in $V(C')$. (3.3)

Suppose x_i has a neighbor in $z \in V(C)$ for some $i \in [2, n-1]$. By (3.2) we may assume $z \in V(A)$. Then by (3.2), A is the path a_1-z-a_2. Since G has length at least 7, (3.2) implies that no vertex in $V(X) \setminus \{x_1, x_n\}$ has a neighbor in $V(C) \setminus V(A)$. It follows that $V(C \cup X) \setminus \{z\}$ induces a hole. Hence $|E(X)| + 2 = |E(A)| = 2|$. But X is a path of length at least one, a contradiction. This proves (3.3).

It follows from (3.3) that $V(X \cup A)$ and $V(X \cup B)$ induce cycles in G of length $|E(A)| + |E(X)| + 2$ and $|E(B)| + |E(X)| + 2$, respectively. Suppose $a_1 = a_2$. Then B has length at least $\ell - 2$, so $V(X \cup B)$ induces a cycle of length greater than ℓ. Thus, $a_1 \neq b_1$ and by symmetry $a_2 \neq b_2$.

Let M denote the graph $G[V(C \cup X)]$. It follows that M is a prism, pyramid, or theta depending on the lengths of P_1, P_2. Moreover since G is ℓ-monoholed the constituent paths of M will have lengths in $\{\frac{\ell}{2} - 1, \frac{\ell-1}{2}, \frac{\ell}{2}\}$. Since $\ell \geq 7$ every constituent path of M has length at least three. Let

T be a set consisting of one vertex from the interior of each constituent path of M. Then M is the union of a pair of mated spiders with toes T.

□

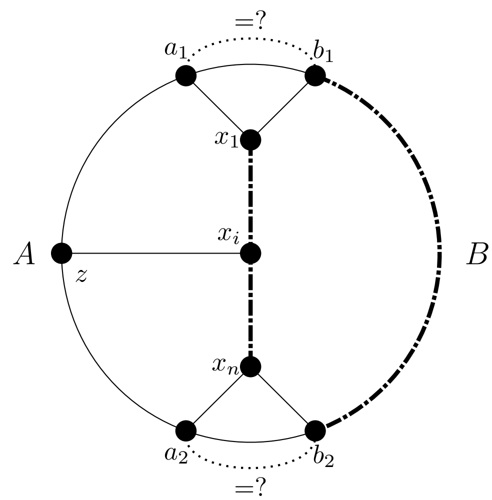

Figure 3.4: An illustration of the proof of Statement 3.2 of Theorem 3.2.3. Thick dashed lines indicate a path of length greater than zero. The vertices a_1 and b_1 are connected by dots and an =? to indicate that a_1 may be equal to b_1.

3.3 The structure of mated k-spiders

In this section we will fully characterize the structure of graphs consisting of the union of a pair of mated k-spiders in an ℓ-monoholed graph. In particular, we will prove that the union of mated k-spiders is a k-theta or a "k-spine" and if ℓ is odd it is a k-pyramid.

Let us begin with the definition of a k-spine: Let H be a graph such that $V(H)$ is the the union of the vertex sets of paths $P_1, P_2, \ldots, P_k$ for some $k \geq 3$. For each $i \in [k]$ let a_i and b_i denote the ends of P_i. Let $\ell = 2n + 2$. We call H an *generalized k-prism* if both of the following conditions hold:

- For every two distinct $i, j \in [k]$, $V(P_i)$ is anticomplete to $V(P_j)$ except possibly a_i is equal or adjacent to a_j or b_i is equal or adjacent to b_j.
- $[k]$ can be partitioned into (possibly empty) sets Q, R, S, T satisfying all of the following:
  - The graphs $G[\{a_i \mid i \in [k]\}]$ and $G[\{b_i \mid i \in [k]\}]$ both do not contain cut-vertices.
  - For any two distinct $i, j \in [k]$, $a_i = a_j$ if and only if i and j are both in T and $b_i = b_j$ if and only if i and j are both in S,

– For every $i \in Q$, P_i has length $n-1$,

– For every $i \in R \cup S \cup T$, P_i has length n,

– The sets $\{a_i \mid i \in Q \cup S\}$ and $\{b_i \mid i \in Q \cup T\}$ are stable,

– The sets $\{a_i \mid i \in R\}$ and $\{b_i \mid i \in R\}$ induce cliques,

– For any $i \in S$, $j \in R$ and $k \in T$, a_i is adjacent to a_j and and b_j is adjacent to b_k,

– If $t \in T$, a_t is complete to every other vertex in $\{a_1, a_2, \dots, a_k\}$ and if $s \in S$, b_s is complete to every other vertex in $\{b_1, b_2, \dots, b_k\}$,

– There is a half graph between $\{a_i \mid i \in Q\}$ and $\{a_j \mid j \in R\}$ and a half graph between $\{b_i \mid i \in Q\}$ and $\{b_j \mid j \in R\}$, and for any $i \in Q$ and $j \in R$, a_i is adjacent to a_j if and only if b_i is not adjacent to b_j.

See Figure 3.5 for an illustration. We call Q, R, S, T a *defining partition* of H. Note H may have multiple defining partitions. In general we work with defining partitions that maximize the cardinality of R and thus S is non-empty if and only if $|S| \geq 2$ and T is non-empty if and only if $|T| \geq 2$. We call the paths $P_1, P_2, \dots, P_k$ the *constituent paths* of H and we call the sets $\{a_1, a_2, \dots, a_k\}$ and $\{b_1, b_2, \dots, b_k\}$ the *terminating sets* of H. Note that by definition k-prisms with constituent paths of length $\frac{\ell}{2} - 1$ are generalized k-prism, but k-thetas and k-pyramids are not generalized k-pyramids. We call a graph H a *k-spine* if it is a k-prism, k-theta, or generalized k-prism. We say the terminating sets of a prism are the two vertex sets of its bases and the terminating sets of a theta are each consist of one of the vertices of degree greater than two. We are now ready to state the main result of this chapter.

Theorem 3.3.1. *Let G be an ℓ-monoholed graph for some $\ell \geq 7$. Then one of the following conditions holds:*

(a) G contains a vertex that is adjacent to every other vertex in $V(G)$.

(b) G contains a clique cutset,

(c) G is chordal,

(d) G is an inflated ℓ-hole,

(e) G contains a k-spine for some $k \geq 3$ and in particular if ℓ is odd G contains a k-pyramid.

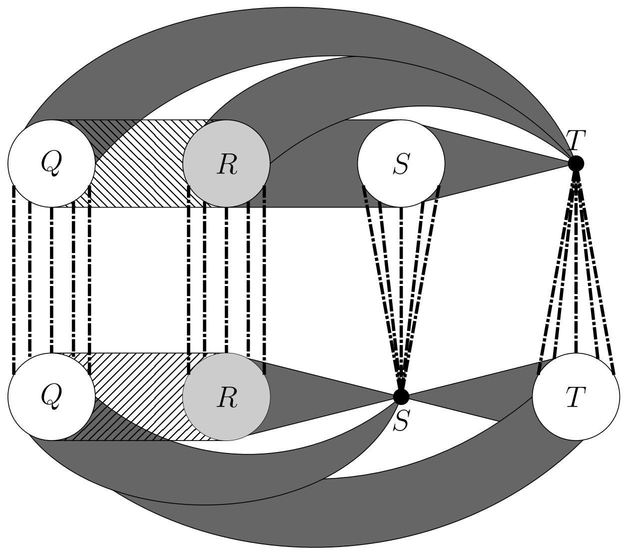

Figure 3.5: An illustration of a generalized k-pyramid. The large white circles represent stable sets and the large gray circles represent cliques. We add huge dark gray lines to indicate that sets are complete to each other. We use stripes between Q and R to illustrate that there is a half graph between them. We draw the stripes in opposite directions on top and bottom to indicate that the half graphs are complementary. Note some of the sets may be empty and the vertices drawn for S and T need not exist.

3.3.1 Proving Theorem 3.3.11

For the rest of this chapter G will be an ℓ-monoholed graph. We will now prove a series of lemmas about the structure of spiders and pairs of mated spiders in G in order to prove Theorem 3.3.11.

Lemma 3.3.2. *Let S be a 4-spider in G and let A be the body of S. If A has a cut-vertex then two of $d(1,2)+d(3,4)$, $d(1,3)+d(2,4)$ and $d(1,4)+d(2,3)$ are equal and at least two more than the third.*

Proof. Let a_1, a_2, a_3, a_4 be as in the definition of k-spider. Let v be a cut vertex of A and let A', A'' be two different components of $A \setminus \{v\}$. Let B', B'' denote $S[V(A') \cup \{v\}]$ and $S[V(A'') \cup \{v\}]$, respectively. Suppose none of $a_2, a_3, a_4 \in V(A')$. Then it follows from the minimality of H that B' is a a_1v-path, contradicting the definition of a leg. It follows that two of a_1, a_2, a_3, a_4 are in each of $V(A')$ and $V(A'')$. Hence, v does not equal a_1, a_2, a_3, a_4. We may assume $a_1, a_2 \in V(A')$ and $a_3, a_4 \in V(A'')$. Thus, $d(1,3)+d(2,4) = d(1,4)+d(2,3)$. Since B' is connected it contains a shortest a_1v-path Q_1 and a shortest a_2v-path Q_2. Since S is minimal, $V(B') = V(Q_1 \cup Q_2)$. Moreover, since A' is connected some vertex in $V(Q_1)$ is equal or adjacent to some vertex of $V(Q_2)$. Thus, $d(1,2) < d_S(a_1, v) + d_S(a_2, v)$. By symmetry, $d(3,4) < d_S(a_3, v) + d_S(a_4, v)$. Hence, $d(1,2)+$

$d(3,4) - 2 \leq d(1,3) + d(2,4) = d(1,4) + d(2,3)$. □

We make extensive use of the following easy observation:

Fact 3.3.3. *Let H be a k-spider for some $k \geq 3$. Let A, $a_1, \ldots, a_k$ be as in the definition of k-spider. Then for each $i \in [k]$ either a_i has degree at least two in A or there is some $a_i = a_j$ for some $j \in [k] \setminus \{i\}$.*

Proof. The proof follow immediately from the definition of leg. □

Lemma 3.3.4. *Let S be a 4-spider and let A be the body of S. Suppose A contains no cut-vertex, then $V(A) = \{a_1, a_2, a_3, a_4\}$, and either:*

(a) A consists of a single vertex,

(b) A consists of a single edge,

(c) A is a K_3,

(d) A is a K_4, or

(e) A is a diamond.

Proof. Suppose A has no cut-vertex. Then by minimality of A, $V(A) = \{a_1, a_2, a_3, a_4\}$. If $|V(A)| \leq 2$, (a) of (b) hold trivially. Suppose $|V(A)| = 3$. Without loss of generality, $a_1 = a_2$. By Fact 3.3.3, a_3 and a_4 have degree at least two in A. Thus, A is a K_3 and (c) holds. Suppose all of a_1, a_2, a_3, a_4 are distinct. Then since A is 2-connected, we may assume a_1-a_2-a_3-a_4-a_1 is a cycle. Since G contains no hole of length four, (d) or (e) holds. □

Lemma 3.3.5. *Let S be a mated 4-spider with the notation as in the definition of spider. Then A has no cut vertex $V(A) = \{a_1, a_2, a_3, a_4\}$ and one of the following holds:*

(a) A consists of a single vertex,

(c) A is a K_3,

(d) A is a K_4, or

(e) A is a diamond.

Proof. Let S' be a mate of S. Then by Lemma 3.3.2 and Lemma 3.3.4, two of $d_{S'}(1,2) + d_{S'}(3,4)$, $d_{S'}(1,3) + d_{S'}(2,4)$ and $d_{S'}(1,4) + d_{S'}(2,3)$ are equal and the third is at most one more than the other two. Thus, two of $d_S(1,2)+d_S(3,4)$, $d_S(1,3)+d_S(2,4)$ and $d_S(1,4)+d_S(2,3)$ are equal and the third is at most one less than the other two. Hence A cannot be an edge and by Lemma 3.3.2 A cannot contain a cut-vertex. The result now follows from applying Lemma 3.3.4 to S. ☐

Mated four structures have a well defined structure (see Figure 3.6 for an illustration).

Lemma 3.3.6. *Let S, S' be mated 4-spiders. Let A, A' be the body of S, S', respectively. Let the toes of A be $t_1, \ldots, t_4$ and for each $i \in [4]$ let a_i and a'_i be the t_i-joint of S, S', respectively. For each $i \in [k]$, let P_i be the union of the t_i-leg of S and S'. Then (by renumbering $t_1, \ldots, t_4$ or exchanging A and A' if necessary) either*

(a) A, A' are single vertices, ℓ is even, and P_i has length $\frac{\ell}{2}$ for each $i \in [k]$.

(b) A is a single vertex and A' is a K_4, ℓ is odd and P_i has length $\frac{\ell-1}{2}$ for each $i \in [k]$.

(c) A is a triangle, $a_1 = a_2$ and A' is a diamond with a'_1 non-adjacent to a'_2, ℓ is even and P_i has length $\frac{\ell}{2} - 1$ for each $i \in [k]$.

(d) A and A' are both equal to K_4, ℓ is even and P_i has length $\frac{\ell}{2} - 1$ for each $i \in [k]$.

Proof. Suppose that $d_S(1,2)+d_S(3,4) = d_S(1,3)+d_S(2,4)+d_S(1,4)+d_S(2,3)$. Then by Lemma 3.3.5, A must be a single vertex or a K_4. It follows from the definition of mated spiders that, $d_{S'}(1,2)+d_{S'}(3,4) = d_{S'}(1,3)+d_{S'}(2,4)+d_{S'}(1,4)+d_{S'}(2,3)$ so A' must be a single vertex or a K_4. Since $V(P_i \cup P_j)$ induces a hole in G for any two distinct $i, j \in [k]$, it follows that (a), (b) or (d) holds. Hence from Lemma 3.3.5, we may assume $d_S(1,2)+d_S(3,4) \neq d_S(1,3)+d_S(2,4) = d_S(1,4)+d_S(2,3)$. Moreover, $d_S(1,2)+d_S(3,4) = d_S(1,3)+d_S(2,4) \pm 1$ by Lemma 3.3.5. By exchanging S and S' if necessary, we may assume that $d_S(1,2)+d_S(3,4) = d_S(1,3)+d_S(2,4)-1$. Then by another application of Lemma 3.3.5 we obtain that A is a K_3 and $a_1 = a_2$ and that A' is a diamond with a'_1 non-adjacent to a'_2. Thus, (c) holds. ☐

Lemma 3.3.7. *Let S be a mated k-spider for some integer $k \geq 3$. Let A be the body of S and $a_1, \ldots, a_k$ be the joints of S. Then A has no cut-vertex and $V(A) = \{a_1, a_2, \ldots, a_k\}$.*

Proof. Suppose A has a cut-vertex v and let A_1, A_2 be two components of $A \setminus v$. Choose some $a_1 \in V(A_1)$. Suppose a_1 is not a t_i-joint for any $i \neq 1$. Let P be a shortest a_1v-path in A. By Fact 3.3.3, a_1 has degree at least two in A. Hence some $u \in V(A_1) \setminus V(P)$ is adjacent to a_1. Since A is minimally connected, there exists some $i \in [k]$ such that u belongs to every a_iv-path in A. Since $i \neq 1$, we may assume $i = 2$. It follows that $d_S(t_1,t_2) < d_S(t_1,v)+d_S(t_2,v)$.

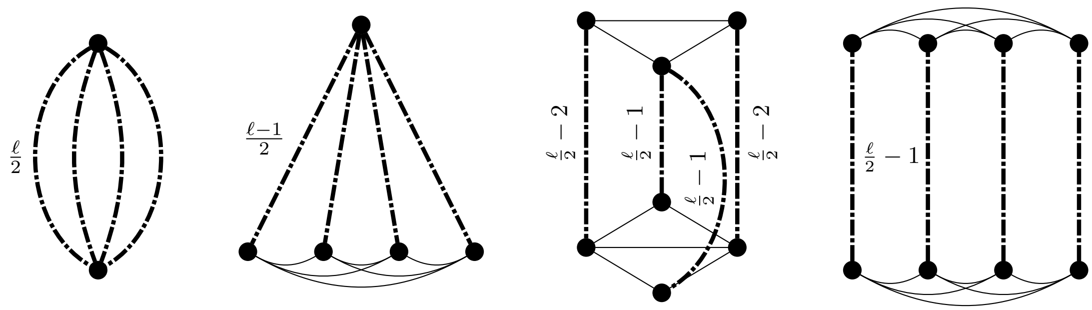

Figure 3.6: An illustration of Lemma 3.3.6

By symmetry we may assume that $a_3, a_4 \in A$ and $d_S(t_3, t_4) < d_S(t_3, v) + d_S(t_4, v)$. But then, $d_S(1,2) + d_S(3,4) - 2 \leq d_S(1,3) + d_S(2,4)$, contradicting Lemma 3.3.2. Hence Ahas no cut-vertex. Since A is minimally connected, $V(A) = \{a_1, a_2, \ldots, a_k\}$. □

We can now fully describe the structure of mated k spiders when ℓ is odd:

Theorem 3.3.8. *Suppose ℓ is odd and S, S' are mated k-spiders in G for some $k \geq 3$. Then $S \cup S'$ forms a k-pyramid where every constituent path has length $\frac{\ell-1}{2}$.*

Proof. Let A, A' be the body of S and S', respectively. Let the toes of S, S' be $t_1, t_2, t_3, \ldots, t_k$. For each $i \in [k]$, let a_i be the t_i-joint of S. Then by Lemma 3.3.7, $V(A) = \{a_1, a_2, \ldots, a_k\}$. Suppose A is not a clique. Then we may assume it contains an induced path a_1-a_2-a_3. By Fact 3.3.3, we may assume a_4 is equal or adjacent to a_3 and $a_4 \neq a_2$. Since $G[a_1, a_2, a_3, a_4]$ is connected, there is some 4-spider R contained in S such that $G[\{a_1, a_2, a_3, a_4\}]$ contains the body of R. Let B denote the body of R. S' contains a mate for R so by Lemma 3.3.6, B is a clique of size one or four. By the definition of spider, $a_2, a_3 \in B$, so B is a K_4 and $V(B) = \{a_1, a_2, a_3, a_4\}$. But, a_1 is not adjacent to a_3, a contradiction. Thus A is a clique and by symmetry A' is also a clique.

Suppose that $2 \leq |V(A)| \leq k - 1$. Then we may assume $a_1 = a_2 \neq a_3$. By Fact 3.3.3 we may assume a_4 is equal or adjacent to a_3. Then there is some 4-spider $R \subset S$ be with toes t_1, t_2, t_3, t_4 and body contained in $G[\{a_1, a_2, a_3, a_4\}]$. By definition of joint, $a_1 \in V(B)$ and at least one of $a_3, a_4 \in V(B)$. Hence B has cardinality two or three. But since R has a mate contained in S', B have cardinality one or four by Lemma 3.3.6, a contradiction.

Thus A and A' are both either a clique of size one or a clique of size k. If they have the same cardinality, G contains an even hole, contradicting that ℓ is odd. □

For the remainder of this section we will discuss the case when ℓ is even.

Lemma 3.3.9. *Suppose ℓ is even and A is the body of a mated k-spider in G for some integer $k \geq 3$. Then A is a threshold graph. In particular some vertex in $V(A)$ is adjacent to every other vertex in $V(A)$ and so A has diameter at most two.*

Proof. Suppose a_1-a_2-a_3-a_4 is a P_4 in A. Then using the notation from the definition of a spider, $d(1,2) + d(3,4) = d(1,4) + d(2,3) - 2$, contradicting Lemma 3.3.5. Suppose a_1a_2 and a_3a_4 are edges of A and $\{a_1, a_2\}$ is anticomplete to $\{a_3, a_4\}$. Then $d(1,2) + d(3,4) \leq d(1,4) + d(2,3) - 2$, contradicting Lemma 3.3.5. □

Theorem 3.3.10. *Suppose ℓ is even and S, S' are mated k-spiders for some $k \geq 3$. Then $S \cup S'$ is a k-theta or a k-spine.*

Proof. Since $\ell \geq 6$ there is some integer $n \geq 2$ such that $\ell = 2n + 2$. Let $t_1, t_2, \ldots, t_k$ be the toes of S, S'. For each $i \in [k]$ let a_i, a_i' be the t_i-joints of S and S', respectively. Let A, A' be the bodies of S and S', respectively. For each $i \in [k]$ let P_i be the union of the t_i-leg of S and S'. We begin with a series of observations about the structure of A, A' and its relationship to the lengths of the paths $P_1, P_2, \ldots, P_k$.

For every $i \in [k]$, P_i has length at most n or $S \cup S'$ is a k-theta. (3.4)

Suppose that P_1 has length at least $n + 1$. For every $i \in [2, k]$, there is a a_ia_i'-path R_i of $S \cup S'$ including P_i. Since $R_i \cup P_1$ is a hole, R_i has length $\ell - |E(P_1)| \leq n + 1$ for every $i \in [2, k]$. Since $G[V(R_1 \cup R_2)]$ includes a hole, $|E(R_1)|, |E(R_2)| \geq n + 1$. Hence, P_1 has length $n + 1$ and R_i has length $n + 1$ for every $i \in [2, k]$. It follows that every $V(R_i \cup R_j)$ induces a hole for any two distinct $i, j \in [2, k]$.

Let $a_1, \ldots, a_k$ be the joints of S. Suppose $S \cup S'$ is not a k-theta. Then we may assume the body of S is not a single vertex and $a_2 \neq a_1$. By Fact 3.3.3 we may assume $a_3 \neq a_1$ and is equal or adjacent to a_2. But then $V(R_2 \cup R_3)$ does not induce a hole, a contradiction. This proves (3.4).

Since the general construction includes k-thetas, we may assume that P_i has length at most n for every $i \in [k]$. We now prove a lower bound on the lengths of the paths P_i for $i \in [k]$.

For every $i \in [k]$, P_i has length at least $n - 1$. (3.5)

Consider the path P_1. By Fact 3.3.3, we may assume a_2 is equal or adjacent to a_1. Let Q be an shortest $a_1'a_2$-path in the body of S'. By 3.3.9, Q has length at most two. Since P_2 has length at most n, $G[V(P_1 \cup P_2 \cup Q)]$ includes a hole of length at most $|E(P_1)| + n + 3$. Hence, $|E(P_1)| \geq n - 1$. This proves (3.5).

We call a vertex $v \in A \cup A'$ a *multi-purpose joint* if it is the t_i-joint for more than one distinct $i \in [k]$.

> *There is at most one multi-purpose joint in each of A, A'. If there is a multi-purpose joint $v \in V(A)$, then for every $i \in [k]$ for which v is the t_i-joint of S, P_i has length n and a_i' is not a multi-purpose joint of S'. If $a_i = a_j$ for distinct $i, j \in [k]$ then a_i' is not adjacent to a_j'. Moreover, the same statements hold after exchanging A and A'.* (3.6)

Suppose A contains more than one multi-purpose joint. Without loss of generality, $a_1 = a_2$ and $a_3 = a_4 \neq a_1$. Consider the spider R contained in S with toes t_1, t_2, t_3, t_4. Then the body of R is an a_1a_3 path. But since R has a mate contained in S', this contradicts Lemma 3.3.5. Hence A has at most one multi-purpose joint.

Suppose a_1 is a multi-purpose joint of S. Suppose a_1 has a non-neighbor, say a_3. Without loss of generality $a_1 = a_2$. We may assume a_4 is equal or adjacent to a_3 by Fact 3.3.3. Again, consider the spider R contained in S with toes t_1, t_2, t_3, t_4. Let B be the body of R. Since R has a mate in S', $B \subseteq \{a_1, a_3, a_4\}$ by Lemma 3.3.5. But then $2 \geq |V(B)| \leq 3$. Hence by Lemma 3.3.5, B must be a triangle and $V(B) = \{a_1, a_3, a_4\}$. But a_1 is not adjacent to a_3, a contradiction. Thus, a_1 must be adjacent to every other vertex in $V(A)$.

Suppose P_1 has length less than n. By Lemma 3.3.9, there is an $a_1'a_2'$-path Q in A' of length at most two. But then since P_2 has length at most n, there is a $G[V(P_1 \cup P_2 \cup Q)]$ contains a hole of length at most $2n + 1$, a contradiction. Hence, P_i has length n for every $i \in [k]$ such that $a_1 = a_i$. Since there are no holes of length $2n + 1$ it follows that a_i' and a_j' are non-adjacent for every two distinct $i, j \in [k]$ such that $a_1 = a_i, a_j$.

Finally, suppose a_1' is a multi-purpose joint of S'. Then a_1' is distinct from and non-adjacent to a_2', for otherwise $G[V(P_1 \cup P_2)]$ is a hole of length at most $2n + 1$, a contradiction. So we may assume $a_1' = a_3'$. Then, a_1 is distinct from and non-adjacent to a_3. But then P_2, P_3 and the union of shortest paths between a_1 and a_3 in A and a_2' and a_3' in A' form a hole of length at $2n + 4$, a contradiction. By the symmetry between S and S', this proves (3.6).

For any two distinct $i, j \in [k]$, if P_i and P_j both have length n, then one of the following holds:

- $a_i a_j$ *and* $a'_i a'_j$ *are edges or* (3.7)
- $a_i = a_j$ *and* a'_i *and* a'_j *are non-adjacent (or vice versa.)*

(3.7) follows immediately from the fact that there is a hole in $S \cup S'$ containing P_i and P_j.

For a set $J \subseteq [k]$, let $A(J)$ and $A'(J)$ denote the sets $\{a_j \mid j \in J\}$ and $\{a'_j \mid j \in J\}$, respectively.

A, A' have minimum degree at least two (3.8)

By Fact 3.3.3 we need only show that multi-purpose joints in A, A' have degree at least two in A and A', respectively. Suppose a_1 is a multi-purpose joint of degree at most one in A. Let J be the set of integers $i \in [k]$ such that $a_1 = a_i$. By (3.6), $A'(J)$ is a stable set. Since A' is connected, it follows that $J \neq [k]$. Then since A is connected a_1 has a neighbor in A. Without loss of generality $a_1 = a_2$ and a_1 is adjacent to a_3. By (3.6) and Fact 3.3.3, we may assume $a_4 \neq a_1$ and a_4 is adjacent to a_3. Consider the spider R contained in S with toes t_1, t_2, t_3, t_4. Let B be the body of R. Then $G[\{a_1, a_3, a_4\}]$ contains B. By definition of joint, a_1 is a joint of R and at least one of a_1, a_2 is a joint of R. So the B is a path of length one or two, contradicting Lemma 3.3.6. This proves (3.8).

We now have enough tools to complete the proof. Let Q be the set of indices $i \in [k]$ such that P_i has length $n-1$. $A(Q)$ is a stable set for if any two $a_i, a_j \in A(Q)$ are adjacent the hole in $S \cup S'$ containing $P_i \cup P_j$ has length at most $2n+1$, a contradiction. At least two vertices in A have no non-neighbors in $V(A)$ since A is a threshold graph of minimum degree at least two by Lemma 3.3.9 and (3.8). Hence, there is some vertex in $a_T \in V(A) \setminus A(Q)$ that is adjacent to every other vertex in $V(A)$. Choose a_T to multi-purpose if possible. Let $T = \{i \mid a_T = a_i\}$. By (3.6) none of the vertices in $A(Q) \cup A'(Q)$ are multi-purpose since the constituent paths ending in $A(Q), A(Q')$ all have length $n-1$. Thus for any $i \in [k] \setminus Q$, $a'_i \notin A'(Q)$ and in particular $A'(Q)$ and $A'(T)$ are disjoint. By symmetry, there is some vertex $a'_S \in V(A) \setminus A'(Q)$ that is adjacent to every other vertex in $V(A')$. Choose a'_S to be multi-purpose if possible. Let $S = \{i \| a'_S = a'_i\}$. By (3.6), Q, S, T are pairwise disjoint. $A'(T)$ and $A(S)$ are stable sets for otherwise there is a hole of length $2n+1$, a contradiction. $A'(T)$ is anticomplete to $A'(Q)$ for otherwise we obtain a hole of length less than

$2n+2$ since a_T is complete to $A(Q)$, a contradiction. Similarly, $A(S)$ is anticomplete to $A(Q)$. Let $R = [k] \setminus (Q \cup S \cup T)$. Then by (3.7), $A(R)$ is complete to $A(S)$ and $A'(R)$ is complete to $A'(S)$. The edges between $A(Q)$ and $A(R)$ form a half graph since A is a threshold graph by Lemma 3.3.9. Similarly, the edges between $A'(Q)$ and $A'(R)$ are a half-graph. Moreover since every hole must have length $2n+2$, for any $i \in Q$ and $j \in R$ we have that a_i is adjacent to a_j if and only if a'_i is non-adjacent to a'_j.

No joint in $A(R) \cup A(R')$ is multi-purpose. (3.9)

Suppose some $a_z \in A(R)$ is multipurpose. Then by 3.6, a_T is not multi-purpose. By choice of a_T it follows that a_z must have a non-neighbor in $a_i \in A$. Let $Z = \{j \in [k] \mid a_j = a_z\}$. Since no joint in $A'(Z)$ is multi-purpose by (3.6) and P_z has length n, P_i cannot have length n by (3.7). So $i \in Q$. Let $t \in A'(T)$ and let u, v be distinct vertices n $A'(Z)$. Since there are complementary half-graphs between $A(Q), A(R)$ and $A'(Q), A(R)$, it follows that u, v are adjacent to a'_i. But then since t is complete to $A'(R)$ and anticomplete to $A'(Q)$, the graph $G[\{u, v, a'_i, t\}]$ is a C_4, a contradiction. This proves (3.9).

It follows that $A(R)$ and $A(R')$ are vertex sets of cliques by (3.7). Hence, $S' \cup S$ is a k-spine. □

We are now ready to prove the Theorem 3.3.11 which we restate:

Theorem 3.3.11. *Let G be an ℓ-monoholed graph. Then one of the following conditions holds:*

(a) G contains a vertex that is adjacent to every other vertex in $V(G)$.

(b) G contains a clique cutset,

(c) G is chordal,

(d) G is an inflated ℓ-hole,

(e) G contains a k-theta with paths of length $\frac{\ell}{2}$ (and ℓ is even), or

(f) There is some $k \geq 3$ such that G contains k-spine H and in particular if ℓ is odd H is a k-pyramid with constituent paths of length $\frac{\ell-1}{2}$.

Proof. Suppose none of (a), (b), (c) or (d) holds. Then by Theorem 3.2.3 G contains a pair of mated k-spiders S, S'. If ℓ is odd, (e) holds by Theorem 3.3.8. If ℓ is even, (e) holds by Theorem 3.3.10. □

3.4 On Corpora and Crowns

For technical reasons, our analysis would be easier if we could assume assume any two constituent paths k-skeleton are vertex disjoint. However, this clearly is false; k-thetas and some other k-spines have multiple constituent paths ending at the same vertex. Our solution is to consider only the subpaths of k-thetas and k-spines that are vertex disjoint. For example, if v is the end of two distinct constituent paths P_1, P_2 of some k-spine, we will analyze the paths $P_1 \setminus \{v\}$ and $P_2 \setminus \{v\}$ instead of analyzing paths P_1 and P_2.

Let F be a k-skeleton for some $k \geq 3$. We call any vertex $v \in V(F)$ belonging to multiple constituent paths of F an apex. Note by definition of k-spine every apex belongs to every apex is an end of a constituent path of F. Let J be the set of apexes of F and let $P_1, P_2, \ldots, P_k$ denote the constituent paths of F. Then we call $P_1 \setminus J$, $P_2 \setminus J$, $\ldots$, $P_k \setminus J$ the *elemental paths* of F. We define an analogue of terminating sets for elemental paths as follows: Let A, B be the terminating sets of F. For any $i \in [k]$ and end v of $P_i \setminus J$, if $v \in A$ or if there exists a neighbor of v in $F \cap A$ we call v the A-end of P_i. Otherwise, v is the B-end of P_i. We call the set of A-ends of elemental paths of F and the set of B-ends of elemental paths of F the *elemental sides* of F.

We call the graph obtained from F by removing apexes in F the *core* of F. We call an inflated graph $\mathcal{F}$ a *k-corpus* if the graph underlying F is a k-spine. and the bags of $\mathcal{F}$ corresponding to vertices in an elemental side of F or apexes of F are either complete or anticomplete to each other. In this section we will consider a k-corpus $\mathcal{F}$ chosen to maximize k and with respect to that maximize the number of vertices in the core. The elemental sides of $\mathcal{F}$ are the sets of bags corresponding to the elemental sides of F. The apexes of $\mathcal{F}$ are the bags corresponding to apexes of F.

Note by definition any k-spine is also a k-corpus. By definition of k-theta and k-spine, for any two constituent paths of a k-corpus there is a inflated C_ℓ containing both of them. We will make repeated use of the following easy fact.

Fact 3.4.1. *Let $\mathcal{F}$ be a k-corpus. Suppose X_1 and X_2 are bags in the same elemental side of $\mathcal{F}$. Let $\mathcal{Q}_1, \mathcal{Q}_2$ be the elemental paths of $\mathcal{F}$ containing X_1 and X_2, respectively. Let Y be the vertex set of the elemental side of $\mathcal{F}$ not containing X_1, X_2. Suppose $x_1 \in X_1$ and $x_2 \in X_2$ are non-adjacent. Then $\mathcal{F}$ contains an x_1x_2-path P of length $\ell - 2$ such that $V(P) \subseteq V(\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup Y)$.*

Proof. For any bag B in an inflated graph there is a vertex $b \in B$ such that b is complete to every neighboring bag of B by definition of inflated graph. Hence, the result follows from the definition of k-theta and k-spine. □

3.4.1 Crowns

In the next section we will begin to consider the structure of vertices outside of maximal k-corpus $\mathcal{F}$ with k-maximum. that have that neighbors in an elemental side of a maximal k-body. In order to do this, we will introduce a new object we call a *crown* and prove some properties about it.

Suppose G is a graph where I, J is a partition of $V(G)$. Let P be a induced path of G with vertices p_1-p_2-p_3-p_4-p_5. We call P a *mean* P_5 if $p_1, p_3, p_4 \in I$ and $p_3, p_4 \in J$. Suppose $i_1, i_2, i_3, i_4, j \in V(G)$ such j is adjacent to i_1, i_2 and i_2 is adjacent to i_3, i_4 and $G[\{i_1, i_2, i_3, i_4, j\}]$ contains no further edges. Then if $i_1, i_2, i_3, i_4 \in I$ and $j \in J$ we call the graph induced by $\{i_1, i_2, i_3, i_4, j\}$ a *mean fork*. See Figure 3.7.

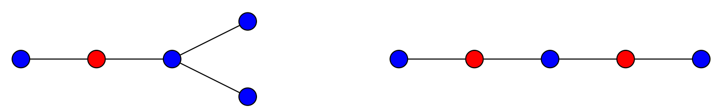

Figure 3.7: A mean fork is drawn at left and a mean P_5 is drawn at right. Blue vertices are in I and red vertices are in J.

We call a graph G a *crown* if there is a partition of $V(G)$ into non-empty sets I, J satisfying all of the following axioms:

(i) There is no clique Z such that $G \setminus Z$ has two parts X, Y such that Y is non-empty and $I \subseteq V(X \cup Z)$, .

(ii) For every induced $u, v \in I$, every uv-path P with $P^* \subseteq J$ has length two,

(iii) $H[I]$ does not contain a P_4,

(iv) H does not contain a mean P_5.

(v) H does not contain a mean fork.

Lemma 3.4.2. *Let G be a crown with partition I, J. Suppose G is ℓ-monoholed for some $\ell \geq 6$. Then for every $j \in J$, $N(j) \cap I$ contains two non-adjacent vertices.*

Proof. Let J' be the set of vertices $j \in J$ where $N(j) \cap I$ contains two non-adjacent vertices. Suppose $J \neq J'$ and let H be a component of $G[J \setminus J']$. By Axiom (i), the $N(V(H)) \cap (I \cup J')$ is not the vertex set of a clique. Hence there exist non-adjacent $u, v \in I \cup J$ such that u, v both have neighbors in $V(H)$. Let P be an induced uv-path with $P^* \subseteq J'$.

u, v are not both elements of I (3.10)

Suppose $u, v \in I$. Then by Axiom (ii), P has length two. But then P^* is a single vertex $j \in J'$. But then $N(j) \cap V(I)$ is not the vertex set of a clique, a contradiction. This proves (3.10).

u, v are not both elements of J'. (3.11)

Suppose $u, v \in J'$. Let I_{uv} denote the common neighbors of u and v in I. Let I_u and I_v denote the neighbors of u and v, respectively, in $I \setminus I_{uv}$. Suppose there are two non-adjacent vertices $i, i' \in I_{uv}$. Then v-i-u-i'-v is a hole of length four, a contradiction. Hence $G[I_{uv}]$ is a clique.

By definition of $V(H)$, I_u and I_v are not empty. Suppose there is some $i_u \in I_u$ and $i_v \in I_v$ such that i_u and i_v are not adjacent. By definition T does not contain a common neighbor of i_u and i_v. But then $G[V(T) \cup \{u, v\}]$ contains an $i_u i_v$-path of length greater than two, contradicting Axiom (ii). Hence I_u and I_v are complete to each other.

Suppose there exist non-adjacent $i, i' \in I_u$. Then there is a hole of length four induced by i, i', u and some vertex in I_v. Hence, $G[I_u]$ is a clique. Similarly, $G[I_v]$ is a clique. Since $N(u) \cap I$ contains two non-adjacent vertices, it follows that $I_{uv} \neq \emptyset$. Since $u, v \in J'$, there is exists $i_u \in I_u$, $i_{uv}, i'_{uv} \in I_{uv}$ and $i_v \in I_v$ such that i_u is not adjacent to i_{uv} and i_v is not adjacent to i'_{uv}. If $i_{uv} = i'_{uv}$ then u-i_{uv}-v-i_v-i_u-u is a hole of length five, a contradiction. It follows that every $i' \in I_{uv}$ is adjacent to one of i_u or i_v. Hence $i_{uv} \neq i'_{uv}$. Then i_u-i'_{uv}-i_{uv}-i_u is a P_4 in $G[I]$ contradicting Axiom (iii). This proves (3.11).

By (3.10) and (3.11), we may assume that $u \in I$ and $v \in J'$. Then there exist non-adjacent $i, i' \in N(v) \cap I$. By definition, $u \neq i, i'$. Then if u is adjacent to both i and i', u-i-v-i'-u is a hole of length four, a contradiction. Hence we may assume u is not adjacent to i'. Then the path uPv-i' must contain a common neighbor

u and i' by Axiom (ii). Hence, $V(P) \backslash \{u\}$ must contain a common neighbor of u and i', contradicting that $P^* \subseteq J \setminus J'$. □

Fact 3.4.3. *Let G be a crown with partition I, J. Suppose G is ℓ-monoholed for some $\ell \geq 6$. Then for every $j \in J$, the complement of the graph induced by $N(j) \cap I$ contains exactly one non-trivial component.*

Proof. Let $j \in J$ and let H denote $G[N(j) \cap I]$. By Axiom (i), H is not a clique so H^c contains at least one non-trivial component. Suppose H^c contains two nontrivial components C_1, C_2. Then there is some $w, x \in V(C_1)$ and $y, z \in V(C_2)$ such that $wx, yz \notin E(H)$. Since C_1 and C_2 are different components of H^c, it follows that w-y-x-z-w is a hole of length four in G, a contradiction. □

Suppose G is an ℓ-monoholed graph with partition I, J for some $\ell \geq 6$ and G is a crown. Then for every $j \in J$ let $g(j)$ denote the vertices of the nontrivial anticomponent of $N(j) \cap I$ guaranteed by Fact 3.4.3. We call the set $g(j)$ the *good children* of j. We call the set neighbors of j in $I \setminus g(j)$ the *bad children* of j and denote it by $b(j)$.

Lemma 3.4.4. *Let ℓ-monoholed G be a crown with partition I, J for some $\ell \geq 6$. Then for every two distinct $u, v \in J$ the following statements hold:*

(a) If u, v are adjacent then $N(u) \cap N(I) \subseteq N(v) \cap I$ and $g(u) \cap g(v)$ or $N(v) \cap N(I) \subseteq N(u) \cap I$ and $g(v) \cap g(u)$.

(b) If u, v are not adjacent then u is anticomplete to $g(v)$ and v is anticomplete to $g(u)$.

Proof.

(a) holds. (3.12)

Let $u, v \in J$ be adjacent vertices. Suppose there exists $i, i' \in I$ such that $iu, i'v \in E(G)$ and $iv, i'u \notin E(G)$. Then i, i' are not adjacent because otherwise u-i-i'-v-u is a hole of length four, a contradiction. But then i-u-v-i' is a path contradicting Axiom (ii). Hence we may assume $N(u) \cap I \subseteq N(v) \cap I$. But then by definition, $g(u) \cap I \subseteq g(v) \cap I$. This proves (3.12).

Thus, it only remains to show that statement (b) holds. Let $u, v \in J$ be adjacent vertices. Since G does not contain a hole of length four the set of common neighbors of u and v is either empty or the vertex set of a clique.

$g(u)$ and $g(v)$ are disjoint. (3.13)

Suppose there exists some $w \in g(u) \cap g(v)$. By definition, w is not adjacent to some $w_u \in g(u)$ and some $w_v \in g(v)$. It follows that v is not adjacent to w_u and u is not adjacent to w_v. If w_u and w_v are adjacent then w_u-u-w-v-w_v-w_u is a hole of length five, a contradiction. so $w_u w_v \notin E(G)$. But then w_u-u-w-v-w_v is a mean P_5, contradicting that G is a crown. This proves (3.13)

Suppose good child w of u is a neighbor of v. By definition w has a non-neighbor $x \in g(u)$. Then x is not adjacent to v because G does not contain a hole of length four. Choose non-adjacent vertices $r, s \in g(v)$. By (3.13), w is a bad child of v. Thus, w is adjacent to both r and s. Since

$G[\{r, s, w, x\}]$ does not induce a C_4, we may assume x is not adjacent to r. If x is adjacent to s the path x-s-w-r violates Axiom (iii) from the definition of crown. So x is not adjacent s. It follows that u is not adjacent to r, s because $r, s \notin g(u)$ by (3.13). But then $G[\{x, u, w, r, s\}]$ is a mean fork, contradicting that G is a crown. □

3.4.2 Defining a crowned k-corpus

Let G be an ℓ-monoholed graph. Let $\mathcal{F}$ be a k-corpus and let X, Y denote the vertex sets of the two-elemental sides of $\mathcal{F}$. Let A_X, A_Y contain all vertices of $\mathcal{F}$ in apexes adjacent to vertices in X and Y respectively. Possibly A_X, A_Y are empty. Let J_X, J_Y be sets of vertices satisying be a set of vertices in $V(G) \setminus V(\mathcal{F})$ with the property that $N(J_X) \cap V(\mathcal{F}) \subseteq X \cup A_X$ and for every $j \in J_X$ $N(j) \cap X$ contains two non-adjacent vertices. Define J_Y similarly for Y. Let $\mathcal{R}$ be the inflated graph by $G[V(\mathcal{F}) \cup J_X \cup J_Y]$ where bags of $\mathcal{F}$ are bags of $\mathcal{R}$ and $\{j\}$ is a bag for every $j \in J_X \cup J_Y$. We call any such graph $\mathcal{R}$ a *crowned k-corpus*. For simplicity, we will equate vertices i$J_X \cup J_Y$ with the bags containing them in our analysis. We refer to $\mathcal{F}$ as the k-corpus of $\mathcal{R}$. We call the apexes, elemental sets, and elemental paths of $\mathcal{F}$ the apexes, elemental sets and elemental path of $\mathcal{R}$, respectively.

The following lemma $G[X \cup J_X]$ and partition X, J_X and the graph $G[Y \cup J_Y]$ with partition Y, J_Y are both crowns. We will refer to them as the *crowns* of $\mathcal{R}$.

Lemma 3.4.5. *Let G be an ℓ-monoholed graph and $\mathcal{R}$ be a crowned k-corpus in G. Let X, Y, J_X, J_Y be as in the definition of crowned k-corpus. Then $G[X \cup J_X]$ is a crown with partition X, J_X and $G[Y \cup J_Y]$ is a crown with partition Y, J_Y.*

Proof. Let $\{X_1, X_2, \ldots, X_k\}$ and $\{Y_1, Y_2, \ldots Y_k\}$ denote the elemental sides of $\mathcal{R}$. For each $i \in [k]$ denote the elemental path of $\mathcal{R}$ with ends X_i, Y_i as $\mathcal{Q}_i$.

For every $u, v \in X$, every induced uv-path P with interior in J_X has length two. (3.14)

We may assume u is not adjacent to v. Then u, v are in different bags of $\mathcal{F}$. It follows that there is a uv-path M in $\mathcal{F}$ with interior anticomplete to J_X of length $\ell - 2$. Since G is ℓ-monoholed the statement follows. This proves (3.14).

$G[X]$ is P_4-free. (3.15)

Suppose v_1-v_2-v_3-v_4 is a path in $G[X]$. Since each bag of $\mathcal{F}$ induces a clique, we may assume $v_1 \in X_1$ and $v_4 \in X_2$. By definition of k-corpus, $v_2 \notin X_2$ and $v_3 \notin X_4$. By Fact 3.4.1 there is an v_2v_3-path

P such that $V(P) \subseteq V(\mathcal{Q}_1 \cup \mathcal{Q}_2) \cup Y$. But then the union of P and the path v_1-v_2-v_3-v_4 is a hole of length at least $\ell + 1$, a contradiction. This proves (3.15).

$G[X \cup J_X]$ does not contain a mean P_5 under the partition X, J_X. (3.16)

Suppose there exist v_1, v_2, v_3 and $z_1, z_2 \in J_X$ such that v_1-z_1-v_2-z_2-v_3 is an induced path. Then since v_1, v_2, v_3 are pairwise non-adjacent we may assume $v_1 \in X_1, v_2 \in X_2, v_3 \in X_3$. By Fact 3.4.1 v_1v_3-path P of length $\ell - 2$ with $V(P) \subseteq V(\mathcal{Q}_1 \cup \mathcal{Q}_3 \cup J_Y)$. Hence the union of v_1-z_1-v_2-z_2-v_3 and P is a hole and it is longer than ℓ, a contradiction. This proves (3.16).

$G[X \cup J_X]$ does not contain a mean fork under the partition X, J_X. (3.17)

Suppose there exist $v_1, v_2, v_3, v_4 \in X$ and $z \in J_X$ such that v_2 adjacent to each of z, v_3, v_4 and v_1 is adjacent to z and suppose there are no further edges in $G[\{v_1, v_2, v_3, v_4, z\}]$ does not contain any other edges. By definition of k-corpus none of v_1, v_2, v_3, v_4 are in the same bag. Hence, we may assume $v_i \in X_i$ for $i \in [4]$. By Fact 3.4.1 v_1v_4-path P of length $\ell-2$ such that $V(P) \subseteq V(\mathcal{Q}_1 \cup \mathcal{Q}_3 \cup J_Y)$. But then the union of P and the path v_1-z-v_2-v_4 is a hole and it has length greater than ℓ, a contradiction. This proves (3.16).

It follows that $G[X \cup J_X]$ is a crown under the partition X, J_X. By symmetry, $G[Y \cup J_Y]$ is a crown under the partition Y, J_Y. □

We will make repeated use of the following consequence of the definition of crowned k-corpus.

Fact 3.4.6. *Let G be an ℓ-monoholed graph for some $\ell \neq 5$. Suppose G contains a k-corpus $\mathcal{F}$ for some $k \geq 3$. The set of vertices H in $V(G) \setminus V(\mathcal{F})$ complete to $V(\mathcal{F})$ induces a clique. If $\mathcal{R}$ is a crowned k-corpus such that the corpus of $\mathcal{R}$ is $\mathcal{F}$, the vertex set $V(\mathcal{R}) \setminus V(\mathcal{F})$ is complete to H.*

Proof. By definition a k-corpus contains a stable set of size at least two and for every $j \in V(\mathcal{R}) \setminus V(\mathcal{F})$, $N(j) \cap V(\mathcal{F})$ contains two non-adjacent vertices. The result follows from the fact that G is C_4-free. □

3.5 Analyzing a maximal crowned corpus

3.5.1 Vertices with neighbors in a maximal crowned corpus

Theorem 3.5.1. *Let G be an ℓ-monoholed graph. Suppose G does not contain a clique cutset and suppose G contains a k-spine for some $k \geq 3$. Let $\mathcal{R}$ be a crowned k-corpus in G chosen to maximize k and with respect to that to maximize $V(\mathcal{R})$. Let $\mathcal{Z}$ be the core of $\mathcal{R}$. Let $v \in V(G) \setminus V(\mathcal{Z})$. Then either v is complete to $V(\mathcal{R})$, $N(v) \cap V(\mathcal{Z})$ is the vertex set of a clique or the neighbors of v in $V(\mathcal{Z})$ are contained in a single elemental side of $\mathcal{R}$.*

Proof. Let $\{X_1, X_2, \ldots, X_k\}$ and $\{Y_1, Y_2, \ldots, Y_k\}$ be the elemental side of $\mathcal{R}$. For each $i \in [k]$ let $\mathcal{Q}_i$ denote the elemental path of $\mathcal{R}$ with ends X_i, Y_i. Suppose for some $v \in V(G) \setminus V(\mathcal{Z})$, $N(v) \cap V(\mathcal{Z})$ contains two non-adjacent vertices and v is not complete to $V(\mathcal{R})$. Suppose for a contradiction v contains a neighbor in an interior bag of an elemental path of $\mathcal{R}$. Let $\mathcal{F}$ be the k-corpus of $\mathcal{R}$.

v is not complete to $V(\mathcal{F})$ (3.18)

Suppose v is complete to $V(\mathcal{F})$. Then v is not adjacent to some $r \in V(\mathcal{R}) \setminus V(\mathcal{F})$. By definiton of crowned k-corpus, we may assume r has a neighbors $x, x' \in X_1 \cup X_2 \cdots \cup X_k$ and x_1 and x_2 are non-adjacent. But then r-x-v-x'-r is a hole of length four, a contradiction. This prove (3.18).

The underlying distance of any two neighbors of v in $V(\mathcal{F})$ is at most two. (3.19)

Suppose x, y are neighbors of v in $V(\mathcal{F})$ and suppose that the underlying distance between x and y is greater than two. By definition of k-corpus, we may assume there is some inflated hole $\mathcal{C}$ of length ℓ containing x, y and the elemental paths $\mathcal{Q}_1, \mathcal{Q}_2$. Then by Lemma **??**, v is complete to $V(\mathcal{C})$. By definition of k-corpus for any $w \in V(\mathcal{F})$ there is an inflated hole $\mathcal{C}'$ of length ℓ containing $\mathcal{Q}_1$ and w. It follows from Lemma 3.2.2 that v is complete to $V(\mathcal{C}')$. Hence v is complete to $V(\mathcal{F})$, contradicting (3.18). This proves (3.19).

There is some bag J of $\mathcal{F}$ such that every neighbor of v in $V(\mathcal{F})$ is contained in J or a neighboring bag of $\mathcal{F}$. (3.20)

Suppose v has neighbors in three pairwise non-adjacent bags A, B, C such that no bag of $\mathcal{F}$ is adjacent to each of A, B, C. By (3.19) there are distinct bags D_{AB}, D_{BC}, D_{AC} in $\mathcal{F}$ such that A and B are adjacent to D_{AB}, B and C are adjacent to D_{BC} and A, C are adjacent to D_{AC}. Let H

be the graph with vertex set $\{A, B, C, D_{AB}, D_{BC}, D_{AC}\}$ where two vertices in $V(H)$ are adjacent if and only if they are adjacent bags in $\mathcal{F}$. Then by definition $\mathcal{F}$ contains H. Consider the cycle A-D_{AB}-B-D_{BC}-C-D_{CA}-A. Since H does not contain a hole of length at most 6, D_{AB}, D_{BC} and D_{AC} must be pairwise adjacent. See Figure 3.8.

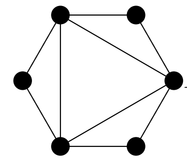

Figure 3.8: An illustration of the graph H from the proof of (3.20).

Since $D_{AB}.D_{BC}, D_{CA}$ have degree four in H they cannot be contained in the interior of any constituent path of $\mathcal{F}$. Hence D_{AB}, D_{BC}, D_{CA} are the ends bags of some distinct constituent paths $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ of $\mathcal{F}$. Then by definition of k-corpus, A, B, C cannot be interior bags of any constituent path of $\mathcal{F}$. Thus $A, B, C, D_{AB}, D_{BC}, D_{AC}$ are all contained in the same terminating set of $\mathcal{F}$. It follows that H is a threshold graph. But the only way to partition $V(H)$ into the vertex set of a clique and a stable set is $\{A, B, C\}$ and $\{D_{AB}, D_{BC}, D_{BA}\}$ and the graph between the two sets is not a half graph, a contradiction. This proves (3.20).

There is some bag J in a elemental side of $\mathcal{F}$ such that every neighbor of v in $\mathcal{F}$ is contained in J or a neighboring bag of J. (3.21)

By (3.20) there is some bag J of $\mathcal{F}$ such that every neighbor of v in $V(\mathcal{F})$ is contained in J or in a neighboring bag of J in $\mathcal{F}$.

Since v has neighbors in the interior of an elemental path of $\mathcal{F}$, J cannot be an apex of $\mathcal{F}$. Suppose J is an interior bag of some elemental path $\mathcal{Q}_i$ of $\mathcal{F}$. Then by definition, $N(v) \cap V(\mathcal{F}) \subseteq V(\mathcal{P}_i)$. Then by definition of k-corpus there is some inflated hole $\mathcal{C}$ contained in $\mathcal{F}$ such that $\mathcal{P}_i$ is a sub-inflated-graph of $\mathcal{C}$. By applying Lemma 3.2.2 we obtain that the graph obtained from $\mathcal{C}$ by adding v to J is another inflated C_ℓ. But then the graph obtained from $\mathcal{S}$ by adding v to J is another k-body, a contradiction. This proves (3.21).

Without loss of generality $J = X_1$. Let W_1 denote the neighbor of X_1 in $\mathcal{Q}_1$. By assumption, v has a neighbor in W_1 and a neighbor in one of $X_2 \cup X_3 \cup \cdots \cup X_k$. For each $i \in [k]$ let $\mathcal{P}_i$ be the constituent path of $\mathcal{F}$ containing $\mathcal{Q}_i$ and let T_i denote the end of $\mathcal{P}_i$ that is equal or adjacent to X_i. Note that $X_i = T_i$ unless T_i is an apex of $\mathcal{F}$. It follows that $X_1 = T_1$. From (3.21), for any $i \in [2, k]$ if T_i is an apex, X_1 and X_i are not adjacent bags.

The graph $\mathcal{F}'$ obtained from $\mathcal{F}$ by adding v to X_1 is an inflated graph and the underlying graphs of $\mathcal{F}'$ and $\mathcal{F}$ are isomorphic. (3.22)

Let T_i be a neighboring bag of X_1 for some $i \in [2, k]$. By definition of k-corpus there is an induced inflated hole $\mathcal{C}$ in $\mathcal{F}$ containing $\mathcal{Q}_1, \mathcal{Q}_i$. Hence by Lemma 3.2.2 and (3.19) the graph obtained from $\mathcal{C}$ by adding v to X_1 is an inflated hole. Hence we need only show for any two $i, j \in [k]$ if T_i and T_j are neighboring bags of X_1 then the graph between $X_1 \cup \{v\}$ and T_i and the graph between $X_1 \cup \{v\}$ and T_j are both half graphs and they are compatible with respect to $X_1 \cup \{v\}$.

If T_i and T_j are not adjacent bags, $\mathcal{F}$ contains some inflated hole $\mathcal{C}'$ such that $\mathcal{P}_i, \mathcal{P}_j, X_1 \subseteq \mathcal{C}'$ so the result follows from Lemma 3.2.2. Hence, we may assume T_i and T_j are adjacent bags. But X_1 is complete to $T_i \cup T_j$ and the result follows. This proves (3.22).

$X_1 \cup \{v\}$ is mixed on one of $T_2, T_3, \ldots, T_k$. (3.23)

Since $|V(\mathcal{F})|$ was chosen to be maximum, $\mathcal{F}'$ is not a k-corpus. By definition of k-corpus and (3.19), There exists some $i \in [2, k]$ such that $X_1 \cup \{v\}$ is mixed on T_i. This proves (3.23)

Suppose T_i is a neighboring bag of X_1 for some $i \in [2, k]$ and v has a non-neighbor in $x_i \in T_i$. for every $j \in [2, k]$ such that $j \neq i$, if T_j is a neighboring bag of T_i then v is anticomplete to X_j. (3.24)

Suppose for some $j \in [2, k]$, v has a neighbor $t_j \in T_j$ and T_j is a neighboring bag of T_i. Let $\mathcal{C}$ be an inflated C_ℓ contained in $\mathcal{F}$ as a sub-inflated-graph such that $\mathcal{P}_1, \mathcal{P}_i \subseteq \mathcal{C}$. Then by definition of inflated graph $G[V(\mathcal{P}_1 \cup \mathcal{P}_i) \cup \{v\}]$ contains a vt_i-path R of length $\ell - 1$. Since T_j is a neighboring bag of T_i they are complete to each other by definition of k-corpus. But then $V(R) \cup \{t_j\}$ induces a hole of length $\ell + 1$, a contradiction. This proves (3.24).

Let $\mathcal{T}$ denote the set $\{T_1, T_2, \ldots, T_k\}$. By (3.21), X_1 is not an apex and and since $T_1 = X_1$ this implies $|\mathcal{T}| > 1$. Without loss of generality v has a neighbor in T_2. Let F be the underlying graph of $\mathcal{F}$. Then since $|\mathcal{Z}| > 1$, the subgraph W of F induced by vertices corresponding to bags in $\mathcal{Z}$ is a 2-connected threshold graph by definition of k-skeleton. Hence there is some $m \in [2, k]$, such that

T_m is complete to every other bag in $\mathcal{Z}$. Then by (3.24), v must have a neighbor in T_m because v has a neighbor in T_2 and if $2 \neq m$ T_2 and T_m are neighboring bags. Thus, it follows from (3.24) that if T_i is a neighboring bag of X_1 for some $i \in [k] \setminus \{m\}$, then v is complete to $V(T_i)$. Thus by (3.23) and (3.21), v has a non-neighbor $t_m \in V(T_m)$. Since W is a two connected threshold graph, T_1 has a neighboring bag T_i for some $i \in [k] \setminus \{1, m\}$ and so v is complete to T_i. But then Z_m, T_i contradict (3.24). □

3.5.2 Paths with neighbors in a maximal crowned corpus

Theorem 3.5.2. *Let G be an ℓ-monoholed graph for some $\ell \geq 7$. Suppose that G contains a k-spine F for some $k \geq 3$. Choose F to maximize k. Let A, B be the terminating sets of S. Let W be an induced path w_1-w_2-$\dots$-w_n in $G \setminus S$ of length at least one satisfying:*

- *W^* is anticomplete to $V(F)$*
- *$N(q_1) \cap V(F)$ and $N(q_2) \cap V(F)$ are both vertex sets of cliques.*

Then q_1, q_n are both anticomplete to one of A, B.

Proof. Suppose neither A nor B is anticomplete $\{q_1, q_n\}$. By definition of k-skeleton, A and B are anticomplete to we may assume that q_1 has a neighbor in A and q_2 has a neighbor in B. For every two distinct $i, j \in [4]$ let A_{ij} be a shortest $a_i a_j$-path in $G[A]$ and let B_{ij} be a shortest $b_i b_j$-path in $G[B]$. Since we will only consider $i, j \leq 4$ in this proof this notation is not ambiguous.

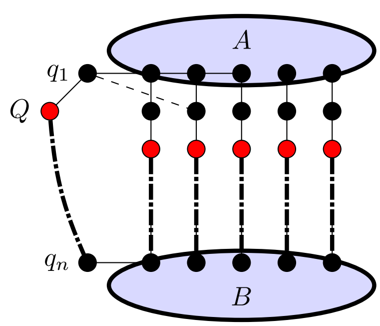

Figure 3.9: An illustration of the a case where (i) and (ii) from (3.25) both do not hold and $k = 5$. The figure contains a mated $(k+1)$-spider where the toes are the red vertices.

One of the following statement holds:

(i) There exists $i \in [k]$ and $j \in \{1, n\}$ such that P_i has length two and q_j is adjacent to the internal vertex of P_i or (3.25)

(ii) $n = 2$ and q_1, q_2 both have at least two neighbors in $V(S)$.

We may assume that q_1 is adjacent to a_1 and q_n is adjacent to b_j for some $j \in [k]$. Suppose, $G[(V(Q \cup S)]$ contains a pair of mated $(k+1)$-spiders. Then by Theorems 3.3.8 or Theorems 3.3.10 $G[V(Q \cup Q)]$ is a $(k+1)$-spine or a $(k+1)$-theta, so G contains a $(k+1)$-skeleton, a contradiction. Thus $G[V(Q \cup S)]$ does not contain a pair of mated $(k+1)$-spiders.

For each $i \in [k]$ choose $t_i \in P_i^*$. Let X_i, Y_i be the $a_i t_i$ and $t_i b_i$-paths of P_i, respectively. Choose t_i to be non-adjacent to both q_1, q_n, if possible. Let X_i, Y_i denote the $t_i a_i$ and $t_i b_i$-paths of P_i, respectively. Then $V(\cup_{i=1}^k X_i)$ induces a k-spider S_X and $V(\cup_{i=1}^k Y_i)$ induces a k-spider S_Y. Moreover, S_X and S_Y are mated to each other and have toes $t_1, t_2, \ldots, t_k$.

Suppose (i) does not hold. Then none of $t_1, t_2, \ldots, t_k$ have a neighbor in $V(Q)$. Suppose $n > 2$. Then $G[\{q_1, q_2\} \cup V(S_X)]$ contains a $(k+1)$-spider S'_X with toes $q_2, t_1, t_2, \ldots, t_k$ and $G[\{q_2, q_3, \ldots, q_n\} \cup V(S_Y)]$ contains a $(k+1)$-spider S'_Y with toes $q_2, t_1, t_2, \ldots t_k$. S'_X and S'_Y cannot be mated to each other so $V(S'_X) \setminus \{q_2, t_1, t_2, \ldots, t_k\}$ and $V(S'_Y) \setminus \{t_1, t_2, \ldots, t_k\}$ are not anticomplete to each other. Since Q^* is anticomplete to $V(S)$, we may assume q_1 has a neighbor in both $V(S'_X) \setminus \{t_1, t_2, \ldots, t_k\}$ and $V(S'_Y) \setminus \{t_1, t_2, \ldots, t_k\}$. But then $N(q_1) \cap V(S)$ contains two non-adjacent vertices, a contradiction. Hence $n = 2$. See Figure **??** for an illustration of this case.

Suppose q_1 has exactly one neighbor in $V(S)$. Then $G[V(S_X) \cup \{q_1\}]$ and $G[V(S_Y) \cup \{q_1, q_2\}]$ contain $(k+1)$-spider S''_X and S''_Y respectively. Moreover since $N(q_2) \cap V(S)$ is the vertex set of a clique S''_X and S''_Y are mates, a contradiction. This proves (3.25).

We will first show that statement (i) does not hold and then we will show (ii) cannot hold for a contradiction.

If some constituent path of S has length two then $\ell = 8$ and every constituent path has length two or three. (3.26)

By definition of k-skeleton every constituent path has a length equal to $\frac{\ell-1}{2}$ if ℓ is odd or $\frac{\ell}{2}, \frac{\ell}{2}-1, \frac{\ell}{2}-2$ if ℓ is even and the difference in length between any two constituent paths it at most one. Since

$\ell \geq 7$ it follows that $\ell = 8$. Hence, every constituent path of S has length 2 or 3. This proves (??).

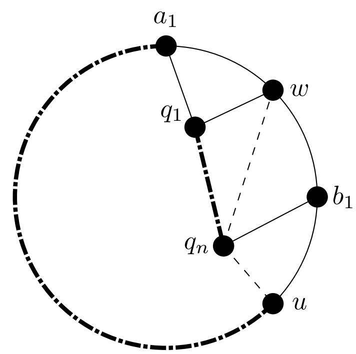

Figure 3.10: An illustration of Statement 3.27. C is drawn as the outer face. Every hole in the picture must have length ℓ, so we reach a contradiction.

If for some $i \in [k]$, P_i has length two and q_1 is adjacent to the interior vertex of P_i then q_n is anticomplete to $V(P_i)$. The same statement holds with q_1 and q_n exchanged.. (3.27)

Suppose P_1 has length two. Then by (3.26), $\ell = 8$. Let the vertices of P_1 be a_1-w-b_1, in order. Suppose q_1 is adjacent to a_1, w and q_n has a neighbor in $V(P_1)$. Then q_n is adjacent to b_1 since $N(q_n) \cap V(S)$ is the vertex set of a clique and q_n has a neighbor in B. Since $N(q_1) \cap V(S)$ is a vertex set of a clique, $N(q_1) \cap S = \{a_1, w\}$. Then q_1-$q_2 \dots$-q_n-b_1-w-q_1 is a cycle of length $|E(Q)| + 3$.

Suppose $G[V(Q) \cup \{w, b_1\}]$ contains a hole. $|E(Q)| \geq \ell - 3 > 3$ or $G[V(Q) \cup \{w, b_1\}]$ does not contain a hole. Let C be a hole of S containing P_1. Let u be the neighbor of b_1 in $C \setminus \{w\}$. Then if q_n is not adjacent to u, $G[V(C \cup Q) \setminus \{x\}]$ is a hole of length $\ell + |E(Q)| - 2$ and if b_1 is adjacent to u, $G[V(C \cup Q) \setminus \{x, u\}]$ is a hole of length $\ell + |E(Q)| - 3$. In either case G contains a hole of length greater than ℓ, a contradiction.

Hence $G[V(Q) \cup \{w, b_1\}]$ is chordal. Hence $n = 2$ and w is adjacent to q_2. Then q_1 is not adjacent to u. Then union of a_1-q_1-q_2-b_1 and $C \setminus w$ is a hole of length at least $\ell + 1$, a contradiction. See Figure 3.10 for an illustration of this argument. This proves (3.27).

(i) does not hold. (3.28)

Suppose for a contradiction that P_1 has length two and q_1 is adjacent to a_1 and the interior vertex of P_1. Then by (3.26), $\ell = 8$. Let the vertices of P_1 be a_1-w-b_1, in order. Then $N(q_1) \cap S = \{a_1, w\}$. By (3.27), q_n is not adjacent to b_1. So we may assume q_n is adjacent to b_2 and $a_2 \neq a_1$. Thus q_n is not adjacent to w. The union B_{12} and the path b_1-w-q_1-q_2-$\dots$-q_n-b_2 is a cycle C of length $|E(Q)| + |E(B_{12})| + 3$ and $G[V(C)]$ contains a hole. Since B_{12} is a shortest b_1b_2-path in $G[B]$,

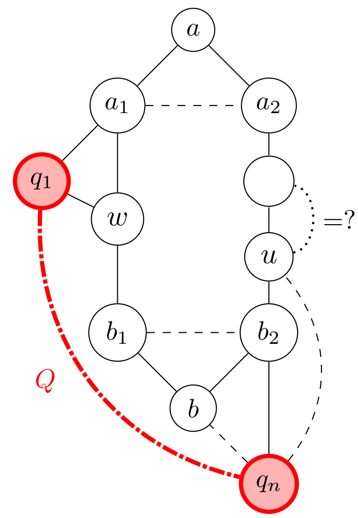

Figure 3.11: An illustration of the Statement 3.28 of Theorem 3.5.2. The vertices $a \in A$ and $b \in B$ exist because $G[A]$, $G[B]$ do not contain a cut vertex.

$|E(B_{12})| \leq 2$. Thus since $\ell = 8$ it follows that $|E(Q)| \geq 3$. See Figure 3.28.

Let C' be the cycle consisting of $P_2 \cup A_{12} \cup Q$ and the edges q_1a_1, q_nb_2. Then C has length $|E(Q)| + |E(P_2)| + |E(A_{12})| + 2$. Since $\ell = 8$, by definition of k-skeleton if P_2 has length two R' has length two. $|E(A_{12})| \geq 1$ because $a_1 \neq a_2$. Thus C' has length at least $|E(Q)| + 6 \geq 9 > \ell$, so it is not a hole. Let u denote the neighbor of b_2 in P_2. Since C' is not a hole, q_n is adjacent to u and so $N(q_n) \cap V(S) = \{b_2, u\}$. Then $G[V(C') \setminus \{u\}]$ is a hole of length $|E(C')| - 1$, so $|E(Q)| \leq 3$. Thus, $|E(Q)| = 3$.

The union of the path a_1-q_1-q_2-q_3-$\ldots$-q_n-b_2 and $B_{23} \cup P_3 \cup A_{13}$ is a hole C'' of length $|E(Q)| + |E(A_{23})| + |E(P_3)| + |E(A_{13})| + 2$. Since b_2 is the only neighbor of q_n in B, C is a hole so $\ell = E(Q)| + |E(B_{12})| + 3$. Thus $|E(B_{12}| = 2$ and in particular b_2 is not adjacent to b_1 so b_1, b_2 are not multipurpose from the definition of k-skeleton. Since P_1 has length two and $\ell = 8$, a_1 cannot be multipurpose from the definition of k-skeleton. Thus A_{13} and B_{23} both have length one or two. It follows that $|E(C'')| \geq |E(Q)| + 6 = 9 > \ell$, a contradiction. See Figure 3.12. This proves (3.28).

By (3.25), it follows that $n = 2$ and q_1, q_2 both have at least two neighbors in $V(S)$.

Every constituent path of S has length at most $\ell - 4$. (3.29)

Suppose P_1 has length $\ell - 3$. Then by defintion of k-skeleton every path of S has length at least $\ell - 4$. $P_1 \cup A_{12} \cup P_2 \cup B_{12}$ is a hole of length $\ell - 3 + |E(P_2)| + |E(A_{12})| + |E(B_{12})|$. So $\ell \geq$

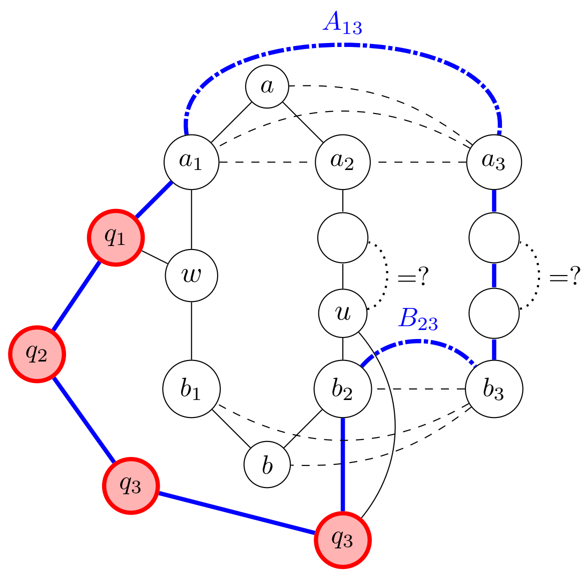

Figure 3.12: An illustration for the proof of statement 3.28 of Theorem 3.5.2. C'' is drawn with blue edges. Note a, a_2 maybe equal or adjacent to vertices in $V(A_{13})$ and b maybe equal or adjacent to vertices in $V(B_{23})$. A_{23} and B_{23} each have length at least one so C'' has length at least 9, a contradiction.

$2\ell - 7 + |E(A_{12})| + |E(B_{12})|$. So,

$$7 - |E(A_{12})| - |E(B_{12})| \geq \ell = \ell - 3 + |E(P_2)| + |E(A_{12})| + |E(B_{12})| \geq 7$$

Hence $|E(A_{12}| = |E(B_{12})| = 0$ and $|E(P_2)| = \ell - 4$. Since P_1 and P_2 have different lengths S is not a k-theta or a k-pyramid so it is a generalized k-prism and in particular for each $i \in [k]$ at most one of a_i, b_i is a multipurpose-vertex. But $|E(A_{12})| = |E(B_{12})| = 0$, so $a_1 = a_2$ and $b_1 = b_2$, a contradiction. This proves (3.29).

For every $i \in [k]$, q_1 is not adjacent to a_i or q_2 is not adjacent to b_i. (3.30)

Suppose a_1 is adjacent to q_1 and b_1 is adjacent to q_2. Then $G[V(Q \cup P_1)]$ contains a hole so $|E(P_1)| \leq \ell - 1$, contradicting (3.29). This proves (3.30).

Since q_1 has a neighbor in A and b_1 has a neighbor in B it follows from (3.30) that $|A|, |B| \geq 2$. By definition of k-pyramid if ℓ is odd one of A, B is a single vertex. Hence, ℓ is even and $\ell \geq 8$.

q_1, q_2 are both anticomplete to $P_1^* \cup P_2^* \cup \cdots \cup P_k^*$. (3.31)

Let a'_1 denote the neighbor of a_1 in P_1. Suppose q_1 is adjacent to both a_1 and a'_1. We may assume q_2 is adjacent to b_2 and thus $b_2 \neq b_1$ and $a_2 \neq a_1$ by (3.30). Suppose b_1 is adjacent to b_2. Let C denote the union of a'_1-q_1-q_2-b_2-b_1 and $P_1 \setminus a_1$. Then C is a hole of length $|E(P_1)| + 3$. So $|E(P_1)| = \ell - 3$, contradicting (3.29) . Hence b_1 and b_2 are not adjacent. Moreover by the same argument q_2 is not adjacent to any neighbor of b_1 in B.

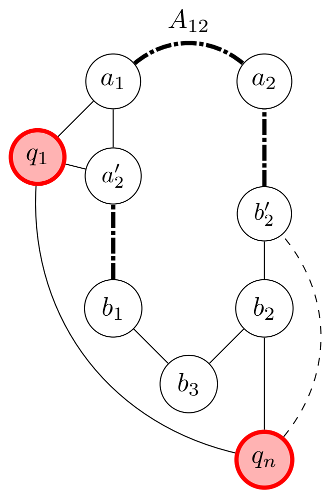

Figure 3.13: An illustration of $G[V(C' \cup Q)]$ from the the proof of Statement 3.31 of Theorem 3.5.2.

Since $|B| \geq 2$ and $G[B]$ is a 2-connected threshold graph we may assume b_3 is adjacent to both b_1 and b_2. Hence b_3 is not adjacent to q_2. Let C' denote the union of $P_1 \cup A_{12} \cup P_2$ and b_1-b_3-b_2. Then C' is a hole. Since $N(q_1) \cap V(S)$ and $N(q_2) \cap V(S)$ are vertex sets of cliques a'_1, a_1 are the only neighbors of q_1 in $V(C')$. Let b'_2 denote the neighbor of b_2 in P_2. Then b_2, b'_2 are the only possible neighbors of q_2 in $V(C')$.

Suppose q_2 is not adjacent to b'_2. $G[V(C \cup Q)]$ is a pyramid, so it contains an odd hole. But G is ℓ-monoholed and ℓ is even, a contradiction. Hence $q_2 b_2 \in E(G)$.

Then $G[V(C \cup Q)]$ is a prism. Since G is ℓ-monoholed for some $\ell \geq 8$ every constituent path of $G[V(C \cup q]$ has length n for some $n \geq 3$. But q_1-q_2 is a constituent path of $G[V(C \cup Q)]$, a contradiction. See Figure 3.13 for an illustration of this argument. This proves (3.31).

Since q_1, q_2 both have at least two neighbors in $V(S)$ we may assume $k \geq 4$, q_1 is adjacent to a_1, a_2, q_1 is not adjacent to a_3, a_4, q_2 is adjacent to b_3, b_4 and q_2 is not adjacent to b_1, b_2 and $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are pairwise distinct. Hence A_{ij} and B_{ij} both have length one or two for every two distinct $i, j \in [k]$. See Figure 3.14 for an illustration.

Let $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Then the graphs a_i-P_i-b_i-B_{ij}-b_j-q_2-q_1-a_i and b_j-P_j-a_j-A_{ij}-a_i-q_1-q_2-b_j

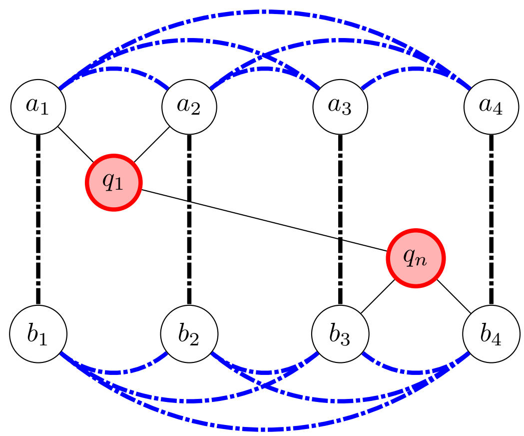

Figure 3.14: An illustration for the conclusion of the proof of Theorem 3.5.2. For distinct $i, j \in [4]$, A_{ij} and B_{ij} are drawn in blue. Note that for distinct $i', j' \in [4]$, vertices in $V(A_{ij})$ and $V(B_{ij})$) may be equal or adjacent to vertices in $V(A_{i'j'})$ or $V(B_{i'j'})$, respectively.

are holes of length $|E(P_i)| + |E(B_{ij})| + 3$ and $|E(P_j)| + |E(A_{ij})| + 3$ respectively. Hence, $\ell - 3 = |E(P_i)| + |E(B_{ij})| = |E(P_j)| + |E(A_{ij})|$. But then, $P_1 \cup A_{12} \cup P_2 \cup B_{12}$ is a hole of length $2\ell - 6$. So $2\ell - 6 \leq \ell$. But $\ell > 6$, a contradiction. $\square$

Corollary 3.5.3. *Let G be an ℓ-monoholed graph for some $\ell \geq 7$. Suppose G does not contain a clique cutset and suppose G contains a k-spine for some $k \geq 3$. Let $\mathcal{R}$ be a crowned k-corpus in G chosen to maximize k and with respect to that to maximize $V(\mathcal{R})$. Let H, I be the crowns of $\mathcal{R}$. Let W be an induced path w_1-w_2-$\dots$-w_n in $G \setminus V(\mathcal{R})$ of length at least one satisfying:*

- *W^* is anticomplete to $V(\mathcal{R})$*
- *$N(w_1) \cap V(\mathcal{R})$ and $N(w_n) \cap V(\mathcal{R})$ are both vertex sets of cliques.*

Then w_1, w_n are both anticomplete to one of $V(H), V(I)$.

Proof. Suppose neither $V(H)$, $V(I)$ both have neighbors in $\{w_1, w_n\}$. Then we may assume w_1 has a neighbor in $V(H)$ and w_n has a neighbor in $V(I)$. Let F be the k-corpus underlying $\mathcal{R}$. Let A, B be the terminating sets of F. Then by definition, we may assume $A \subseteq V(H)$ and $B \subseteq V(I)$. Let $P_1, P_2, \dots, P_k$ denote the constituent paths of F. For each $i \in [k]$, let the ends of P_i be a_i, b_i where $a_i \in A$ and $b_i \in B$. By Theorem 3.5.2, we may assume w_1 is anticomplete to A.Hence w is anticomplete to $V(F)$. $G[V(\mathcal{R} \cup W)]$ does not contain a mated $(k+1)$-spider. Thus by definition of crowned corpus we may assume w_n is complete to interior of $V(P_1)$. Hence $|E(P_1)| = 2$ and $\ell = 8$. Hence a_1, b_1 cannot be apexes of F. By definition w_1 has some neighbor $h \in V(H) \setminus A$ such that h has a neighbor in A. By Theorem 3.5.2, $N(h) \cap V(A)$ is not the vertex set of a clique. In particular, we may assume h is adjacent to a_2. Let C be a hole in F containing P_1, P_2. Consider the graph $G[V(W \cup C) \cup \{h\}]$ depicted in Figure 3.15. By definition $N(w_n) \cap V(C) = \{v_1, b_1\}$. Since a_1 is not an apex $a_1 \neq a_2$.

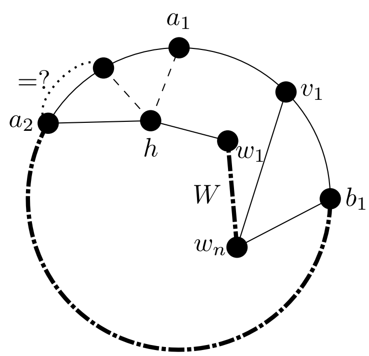

Figure 3.15: The graph $G[V(W \cup C) \cup \{h\}]$ from Corollary 3.5.3 is drawn with C as the outer face.

Suppose a_1 and a_2 are not adjacent. By definition of k-spine, a_1 and a_2 have a common neighbor in $a_i \in V(C)$ and $a_i \in A$. $N(h) \cap V(C) \subseteq \{a_1, a_2, a_i\}$ so either $G[V(W \cup C) \cup \{h\}]$ or $G[V(W \cup C) \cup \{h\}] \setminus a_i$ is a pyramid. But a pyramid contains an odd hole and $\ell = 8$, a contradiction.

Hence a_1 and $a + 2$ are adjacent and since $G[V(W \cup C) \cup \{h\}]$, h is adjacent to both a_1 and a_2. Hence, a_1-h-w_1-w_2-…-w_n-v_1-a_1 is a hole, so W has length 4 since $\ell = 8$. Let M be the a_2b_1-path of C not containing v_1. Then M has length 5 since $\ell = 8$. But then the union of M and a_2-h-w_1-$w_2 \dots w_n$-b_1 is a hole of length greater than 8, a contradiction. □

Theorem 3.5.4. *Let $\ell \geq 7$ be an integer. Let G be an ℓ-monoholed graph and suppose G contains a k-spine S for some $k \geq 3$. Choose S to maximize k. Suppose there is some path Q in $G \setminus V(S)$ such that there are two non-adjacent vertices in $N(Q) \cap V(S)$. Then there exists some $v \in V(Q)$ such that $N(q) \cap V(S)$ contains two non-adjacent vertices.*

Proof. Choose Q to be a minimal path in $G \setminus V(S)$ such that $V(Q)$ has two non-adjacent neighbors in $V(S)$. Let the vertices of Q be q_1-q_2-…-q_n, in order. Suppose for each $i \in [n]$ that $N(q_i) \cap V(S)$ is empty or the vertex set of a clique. We may assume $n > 1$. Let $P_1, P_2, \dots, P_k$ denote the constituent paths of S and let A, B denote the terminating sets of S. For each $i \in [k]$ let $a_i \in A$, $b_i \in B$ be the ends of P_i. By the minimality of P we may assume there exist non-adjacent $u, v \in V(S)$ such that u is adjacent to q_1 and no other vertex in $V(Q)$ and v is adjacent to q_n and no other vertex in $V(Q)$.

q_1 and q_n do not have a common neighbor in $V(S)$ and Q^* is anticomplete to $V(S)$. (3.32)

??

Let X be the set of common neighbors of q_1 and q_n in $V(S)$ and suppose $X \neq \emptyset$, then X is the vertex set of a clique. Let $x \in X$. Then u, v are both adjacent to x. By definition of k-skeleton there is some hole C of length ℓ containing the path u-x-v. It follows that C contains no vertex in $X \setminus \{x\}$. By minimality of Q, for every $i \in [2, n-1]$ $N(q_i) \cap V(S) \subseteq X$. But then $G[V(C \cup Q \setminus \{x\}]$ is a hole of length greater than ℓ, a contradiction. Thus $X =$ and it follows that Q^* is anticomplete to $V(S)$. This proves (3.32).

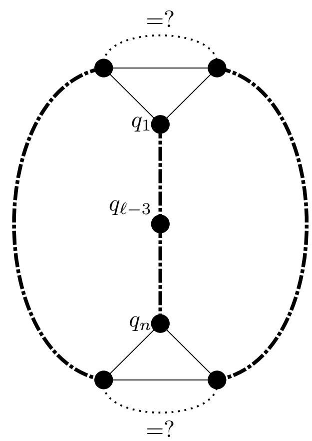

Figure 3.16: An illustration of proof (3.33). The graph induced by $G[V(Q \cup C)]$ is shown where C is the outer face in the drawing. Since every hole in the picture must have length ℓ we reach a contradiction.

$N(q_1) \cap V(S)$ and $N(q_n) \cap V(S)$ are anticomplete. (3.33)

By (3.32) we need only show that there is no edge between a neighbor of q_1 in $V(S)$ and a neighbor of q_n in $V(S)$. Suppose there are some adjacent $x, y \in N(S)$ with x adjacent to q_1 and y adjacent to q_n. Then by (**??**), $V(Q) \cup \{x, y\}$ induces a hole. Hence Q has length $\ell - 3$. Let C be a hole of S containing u and v. Let R_1, R_2 be the two u, v-paths of C. By (**??**) for $i \in \{1, 2\}$ $G[V(Q \cup R_i)]$ contains a hole C_i of length ℓ. Since q_1, q_2 each do not have two non-adjacent neighbors in $V(C)$, $|E(C_1) \setminus E(R_1)| + |E(C_2) \setminus E(R_2) \leq \ell - 2$. Then: $2\ell = |E(C_1)| + |E(C_2)| \geq 2|E(Q)| + |E(R_1)| + |E(R_2)| + 2 = 2|E(Q)| + \ell + 2 = 3\ell - 4$. But $\ell > 4$, a contradiction. This proves (3.33).

Thus, we may assume q_1 is anticomplete to $A \cup B$. Without loss of generality, $N(q_1) \cap V(S) \subseteq P_1^*$. Let α_1, β_1 denote the neighbors of q_1 in $V(P_1)$ with minimum P_1-distance to a_1 and b_1, respectively.

q_n *is anticomplete to* $V(P_1)$ (3.34)

Suppose q_n has a neighbor in $V(P_1)$. Let $x, y \in V(P_1)$ such that q_1 is adjacent to x, q_n adjacent to y. Choose x, y to maximize the P_1-distance between x and y. Let R be the xy-path contained in P_1. Let C be the ℓ-hole in S containing P_1 and P_2. Then $P_1 \setminus R^* \cup Q$ is a hole of length $\ell - |E(R)| + |E(Q)| + 2$. By definition of k-skeleton $|E(P_1) \leq \frac{\ell}{2}$, so $|E(Q)| < \frac{\ell}{2}$. But $G[V(R \cup Q)]$ contains a hole of length at most $|E(R)| + |E(Q)|$ which is less that ℓ, a contradiction. This proves (3.34).

Without loss of generality q_n has a neighbor in $V(P_2)$. et α_2, β_2 denote the neighbors of q_n in $V(P_2)$ with minimum P_2-distance to a_2 and b_2, respectively. For $i \in \{1, 2\}$, let A_i, B_i denote the paths of P_i with ends a_i, α_i and ends β_i, b_i, respectively. Hence, $|E(A_i)| + |E(B_i)|$ is equal to $|E(P_i)| - 1$ or $|E(P_i)$.

Let C be a hole in S containing P_1 and P_2 and let J be the graph induced by $V(C \cup Q)$. Then J is a theta, prism or pyramid and the constituent paths of J have length $\frac{\ell}{2}$, $\frac{\ell-1}{2}$ or $\frac{ell}{2} - 1$. Since there is a constituent path of J consisting of Q and at most two more edges, $|E(Q)| \geq \frac{\ell}{2} - 3$.

Let X be the path $A_1 \cup Q \cup B_2 \cup \{\alpha_1 q_1, \beta_2 q_n\}$ and let Y be the path $A_2 \cup Q \cup B_1 \cup \{\beta_1 q_1, \alpha_1 q_1\}$. For $i \in \{1, 2\}$, $|E(A_i)| + |E(B_i)|$ is equal to $|E(P_i)| - 1$ or $|E(P_i)$ and by definition of k-spine P_1, P_2 each have length at least $\frac{\ell}{2} - 1$. Thus, one of $|E(A_1 \cup B_2)|$ or $|E(A_2 \cup B_1)|$ is at least $\frac{\ell}{2} - 2$. Thus we may assume $|E(X)| \geq \ell - 3$.

q_n *is adjacent to* a_i *for every* $i \in [2, k]$ (3.35)

Suppose q_n is not adjacent to a_3. Since $G[A]$ is a connected threshold graph there is a $a_1 a_3$ path M of length at most two in $G[A]$. Since $G[B]$ is a connected threshold graph there is a $b_2 b_3$ path M' of length at most two in $G[A]$. Then $Q \cup X \cup P_3$ is a hole of length greater than ℓ, since $\ell > 7$, a contradiction. It follows from the fact that q_n does not have two non-adjacent neighbors in $V(S)$ and the fact that q has a neighbor in $V(P_2)$ that q_n is adjacent to a_1. This proves (3.35).

Since q_n does not have two non-adjacent neighbors in $V(S)$, $N(q_n) \cap V(S) = \{a_2, a_3, \ldots, a_k\}$ and $\{a_2, a_3, \ldots, a_k\}$ is the vertex set of a clique. Moreover since q_n is not adjacent to a_1, $|A| \geq 2$ so S is not a k-theta. Thus S is a generalized k-prism of a k-pyramid. Let C be a hole in S containing

P_2 and P_3. Let L be the graph induced by $V(C \cup B_1 \cup Q)$ and the vertices of a shortest path from b_1 to $\{b_2, b_3\}$ in $G[B]$. Then L is a pyramid. It follows that ℓ is odd and so S is a k-pyramid and $|B| = 1$. Moreover, $Q \cup \{q_1\beta_1\}$ is a constituent path of L and so Q has length $\frac{\ell-1}{2} - 1$. But then $Q \cup P_2 \cup B_1 \cup \{q_n a_2, q_1\beta_1\}$ is a hole of length greater than ℓ since $|E(B_1)| > 1$. □

Theorem 3.5.5. *Let G be an ℓ-monoholed graph for some $\ell \geq 7$ and suppose G contains a k-spine. Let $\mathcal{R}$ be a crowned k-corpus, chosen to maximize k and with respect to that maximize $|V(\mathcal{R})|$. Suppose there is some path W in $G \setminus V(\mathcal{R})$ such that there are vertices two non-adjacent bags in $N(W) \cap V(\mathcal{R})$. Then there exists some $w \in V(W)$ such that $N(w) \cap V(\mathcal{R})$ contains vertices in two non-adjacent bags.*

Proof. Choose W to be a minimal path in $G \setminus V(\mathcal{R})$ such that $V(W)$ has neighbors in two non-adjacent bags of $\mathcal{R}$. Let the vertices of W be w_1-w_2-…-w_n, in order. Suppose for each $i \in [n]$ that $N(w_i) \cap V(\mathcal{R})$ is empty or the vertex set of a clique. Then, $n > 1$. Let X, Y be the elemental sides of $\mathcal{R}$ and let C_X, C_Y be the crowns of $\mathcal{R}$ where $X \subseteq C_X$, $Y \subseteq C_Y$. Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_k$ be the elemental paths of $\mathcal{R}$ and lets X_i, Y_i be the end bags of $\mathcal{Q}_i$ where $X_i \in X, Y_i \in Y$.

The neighbors of $V(W)$ in $V(\mathcal{R})$ are not contained in $V(C_X)$ and they are not contained in $V(C_Y)$. (3.36)

Suppose may assume the neighbors of $V(W)$ in $V(\mathcal{R})$ are contained in $V(C_X)$. There exist some two nonadjacent vertices $c_1, c_2 \in C_X$ such that w_1 is adjacent to c_1 and w_n is adjacent to c_n. By definition of crowned k-corpus for each $i \in 1, 2$, c_i is equal or adjacent to some vertex x_i in a bag of X. Then, x_1 and x_2 are non-adjacent. Then the graph induced by $V(W) \cup \{c_1, c_2, x_1, x_2\}$ contains a x_1x_2-path Z of length at least three. By 3.4.1, $\mathcal{R}$ contains an x_1x_2-path P such that P^* is anticomplete to $V(C_X)$. But then $M \cup P$ is a hole of length greater than ℓ, a contradiction. This proves (3.36).

W^* is anticomplete to $V(\mathcal{R})$ (3.37)

By minimality of W, the set $N(W^*) \cap V(\mathcal{R})$ is complete to $N(w_1) \cap V(\mathcal{R})$ and $N(w_n) \cap V(\mathcal{R})$. Suppose some $w_i \in W^*$ has a neighbor $r \in V(\mathcal{R})$.

By definition, there are some two vertices $u, v \in N(r) \cap V(\mathcal{R})$ such that u and v are not adjacent and w_1 is adjacent to u and w_n is adjacent to v. Suppose there is some hole C contained in $\mathcal{R}$ such that $u, v, r \in V(C)$. Then since $\ell \geq 7$, W^* is anticomplete to $V(C \setminus v)$. Thus $W \cup C \setminus v$ is a hole of length greater than ℓ, a contradiction. Thus u, v, r are not all contained in any hole of $\mathcal{R}$.

Let $\mathcal{F}$ be the k-corpus of $\mathcal{R}$. Then by definition $r \notin V(\mathcal{F})$. Thus we may assume $r \in V(C_X) \setminus X$. But then by minimality of W, $N(V(W)) \subseteq V(\mathcal{R})$ is contained in a single crown of $\mathcal{R}$, contradicting (3.36). This proves (3.37).

$V(W)$ is anticomplete to one of C_X, C_Y. (3.38)

Suppose $N(V(W))$ contains vertices in both C_X, C_Y. Then by minimality of W we may assume, w_1 has a neighbor in $V(C_X)$ and w_n has a neighbor in $V(C_Y)$. Then by Corollary 3.5.3, for some $i \in [2, n-1]$, w_i has a neighbor in $v \in V(\mathcal{R})$, contradicting (3.37). This proves (3.38).

Let $\mathcal{F}$ be the corpus of $\mathcal{R}$. Then $N(V(W)) \cap V(\mathcal{F})$ is the vertex set of a clique. (3.39)

Suppose $N(V(W)) \cap V(\mathcal{F})$ contains two non-adjacent vertices u, v . Then by definition of inflated graph there is some graph F underlying $\mathcal{F}$ with $u, v \in V(F)$. But then W, F contradicts Theorem 3.5.4. This proves (3.39).

By (3.39), we may assume $N(w_1) \cap V(\mathcal{R}) \subseteq V(C_X) \setminus X$. Then by (3.38), w_n is anticomplete to $V(C_Y)$. Then by (3.36), we may assume w_n has a neighbor j in an interior bag of $\mathcal{Q}_1$. Let h be a neighbor of w_1 in $V(C_X) \setminus X$. By definition of crowned k-corpus, we may assume h has a neighbor $x_2 \in X_2$. Let $\mathcal{C}$ be an inflated hole in $\mathcal{R}$ containing $\mathcal{Q}_1, \mathcal{Q}_2$. Then by definition of inflated graph there is a hole C underlying $\mathcal{C}$ such that $x_2, j \in V(C)$. Let x_1 be the vertex in $V(C)$ corresponding to the bag X_1 and let y_1, y_2 be the vertices in $V(C)$ corresponding to the bags Y_1, Y_2. Then there is an x_1x_2-path P_{12} of length at most two contained in $V(C)$. Moreover, by definition of crowned k-corpus $V(C) \cap V(C_X) \subseteq V(P_{12})$. Hence the neighbors of h in $V(C)$ are contained in $V(P_{12})$. By definition w_n has at most two neighbors in $V(C)$. Let j, j' denote the neighbors of w_n in $V(C)$. Then either $j = j'$ or j and j' are adjacent. Without loss of generality $x_1, j, j', y_1, y_2, x_2, v, x_1$ occur in order in $V(C)$. Let L_1 be the x_1j-path of C not containing y_1. Let L_2 be the x_2j' path of C not containing x_1. Then the graph induced by $V(L_1 \cup W \cup P_{12})$ includes a hole of length $|E(W)| + |E(L_1)| + 3, |E(W)| + |E(L_1)| + 4, |E(W)| + |E(L_1)| + 5$ depending on the length of P_{12} and the neighbors of h in $V(P_{12})$. The union of L_2 and the path x_2-h-w_1-w_2-$\ldots$-w_n-j' is a hole of length $|E(W)| + |E(L_2)| + 3$.

Hence $|(L_1)| + \zeta = |E(L_2)|$ for some $\zeta \in \{0, 1, 2\}$. Let Q_1, Q_2 be underlying paths of $\mathcal{Q}_1, \mathcal{Q}_2$ contained in C. Then, $|E(L_2)| = |E(Q_2)| + d_{L_2}(y_1, y_2) + d_{L_2}(y_2, j')$ and $|E(L_1)| = |E(Q_1)| -$

$d_{L_2}(y_2, j') - d_C(j, j')$. It follows that, $|E(Q_1)| - d_{L_2}(y_2, j') - d_C(j, j') + \zeta = |E(Q_2)| + d_{L_2}(y_1, y_2) + d_{L_2}(y_2, j')$. So, $\zeta = |E(Q_2)| - |E(Q_1)| + 2d_{L_2}(y_2, j') + d_{L_2}(y_1, y_2) - d_C(j, j')$ By definition of elemental path, Q_1, Q_2 differ in length by at most one. Hence $\zeta \geq 2$ so $\zeta = 2$.

It follows that $j = j'$, y_2 and j are adjacent and y_1 and y_2. Also, P_{12} has length two and the only neighbor of h in $V(C)$ is x_2. Then, $G[V(C \cup W) \cup \{h\}]$ is a theta and each of paths h_1-w_1-w_2-$\ldots$-w_n-j, $G[V(Q_1 \cup P_{12}) \setminus y_1]$ and j-y_1-y_2-Q_2 each have length $\frac{\ell}{2}$. Thus $|E(Q_2)| = \frac{\ell}{2} - 3$.

Let v be the central vertex of P_{12}. Since h is not adjacent to v and h and v have a common neighbor in a X, $v \notin V(C_X) \setminus X$ by definition of crown. Hence, v is in a bag X_3 of X. Let Q_3 be a path underlying $\mathcal{Q}_3$ such that $v \in V(Q_3)$. Let y_3 be the end of Q_3 not equal to v. Let Z_{23} be a shortest y_2y_3-path in C_Y. Then the union of Q_2, Q_3, Z_{23}, and the edge x_2x_3 is a hole. So $|E(Q_3)| + |E(Z_{23})| = \frac{\ell}{2} + 2$. Let Z_{13} be a shortest y_1y_2-path in C_Y. Then the union of Z_{13}, Q_3, and the path x_3-x_2-h-w_1-$\ldots$-w_n-j-y_1 is a hole. So $|E(Q_3)| + |Z_{13}| = \frac{\ell}{2}$. But by definition of C_Y, the lengths of Z_{13}, Z_{23} differ by at most one. □

3.5.3 Everything is a crowned k-corpus

Lemma 3.5.6. *Let G be an ℓ-monoholed graph. Suppose G does not contain a clique cutset and suppose G contains a k-spine for some $k \geq 3$. Let $\mathcal{R}$ be a crowned k-corpus in G chosen to maximize k and with respect to that to maximize $V(\mathcal{R})$. Let $\mathcal{F}$ be the k-corpus of $\mathcal{R}$. Suppose for some vertex v in $V(G) \setminus V(\mathcal{R})$, $N(v) \cap V(\mathcal{R})$ contains two non-adjacent vertices. Then v contains two non-adjacent vertices in $V(\mathcal{F})$ and v has a neighbor in the core of $\mathcal{R}$.*

Proof. Let X, Y be the elemental sides of R and let C_X, C_Y be the crowns of R where $X \subseteq C_X, Y \subseteq C_Y$. Let $C_X \setminus X$ denote the graph formed from C_X by deleting any vertices in a bag in X. Suppose $N(v) \cap V(\mathcal{R})$ contains two non-adjacent vertices in $V(\mathcal{R}) \cap V(\mathcal{F})$. Suppose v is anticomplete to $V(\mathcal{F})$ or $N(v) \cap V(\mathcal{F})$ is the vertex set of a clique.

$N(v) \cap V(\mathcal{R})$ is not contained in $V(C_X)$ or $V(C_Y)$. (3.40)

Suppose every neighbor of v in $V(\mathcal{R})$ is contained in $V(C_X)$. Then v has a neighbor $u \in V(C_X \setminus X)$ and a neighbor w in $V(C_X)$ and u and w are non-adjacent. By definition of crown there are some two non-adjacent vertices in $x_1, x_2 \in X$ such that u is adjacent to x_1 and w is equal or adjacent to x_2. It follows that $G[\{v, u, w, x_1, x_2\}]$ is an x_1x_2-path of length at least three. But by Fact 3.4.1 $\mathcal{F}$ contains an x_1x_2-path M of length $\ell - 2$ such that M^* is anticomplete to $V(C_X)$. Hence the union of $G[\{v, u, w, x_1, x_2\}]$ and M is a hole of length greater than ℓ, a contradiction. This proves (3.40).

v does not have a neighbor in the interior of an elemental path of $\mathcal{F}$. (3.41)

Suppose v has a neighbor in the interior of some elemental path of $\mathcal{F}$. Then for some graph R underlying $\mathcal{R}$, $N(v) \cap V(R)$ contains two non-adjacent vertices and v has a neighbor in the interior of an elemental path of R. By definition, $C_X \setminus X \subseteq R$. Without loss of generality, v has a neighbor $h \in V(C_X \setminus X)$. Let the elemental paths of R be $Q_1, Q_2, \ldots, Q_k$ where for each $i \in [k]$, $Q_i \subseteq \mathcal{Q}_i$ and v has a neighbor $j \in Q_1^*$. For each $i \in [k]$ let x_i, y_i denote the ends of Q_i contained in X, Y, respectively. Let Q_1^x be the jx_1-path of Q_1 and let Q_1^y be the jy_1-path of Q_1.

Let M be a shortest path from x_1 to a vertex in $N(v) \cap V(C_X)$ in C_X. Then, M has length one or two. $G[V(Q_1^x \cup M) \cup \{v\}]$ is a hole of length $|E(Q_1^x)|+3$ of $|E(Q_1)^x|+4$. So $|E(Q_1^x)| \in \{\ell-3, \ell-4\}$ and so $|E(Q_1)| \geq \ell - 3$. By definition, every elemental path of R has length at most $\frac{\ell}{2} - 1$. Hence $\ell \leq 4$, a contradiction. This proves (3.41).

Thus we may assume v has a neighbor in $h \in V(C_X \setminus X)$ and a neighbor in C_Y. Let F be a graph underlying $\mathcal{F}$ and let $Q_1, Q_2, \ldots, Q_k$ be the elemental paths of F. For each $i \in [k]$ let x_i denote the end of Q_1 that lies in a bag of X and let y_i denote the end of Q_i that lies in bag of Y. We may assume h is adjacent to x_1 and x_2 and x_1 is not adjacent to x_2.

Suppose v is adjacent to y_1. Then $|E(Q_1)| = \ell - 3$ since Q_1 and the path x_1-h-v-y_1 would be a hole. But then by definition of k-corpus, $\ell < 6$, a contradiction. Similarly h is not adjacent to y_2.

Let $j_1 \in Q_1^*$ and $j_2 \in Q_2^*$. There is some 3-spider S with toes j_1, j_2, v and body equal to h contained in $G[V(Q_1 \cup Q_2 \cup \{h\}]$. Since v has some neighbor in $z \in V(C_Y)$ there is a spider S' with toes j_1, j_2, v contained in $G[V(Q_1 \cup Q_2 \cup C_Y)]$. The body of S' is contained in C_Y and S, S' are mated k-spiders.

By Theorems 3.3.8 and 3.3.10, $S \cup S'$ is a prism, pyramid or theta. Let M be the shortest path of $S \cup S'$ from h to the body of S' such that $v \in V(M)$. Then M is a constituent path of $S \cup S'$. But M has length at most three, so $\ell = 8$, $S \cup S'$ is a pyramid and M has length three. Hence, z is not equal or adjacent to y_1, y_2. F is a k-pyramid since ℓ is odd. But then C_Y must be a clique, a contradiction since z is not adjacent to y_1, y_2. □

Theorem 3.5.7. *Let G be an ℓ-monoholed graph. Suppose G does not contain a clique cutset and G does not contain a vertex v that is adjacent to every other vertex in $V(G)$. Suppose G contains a*

k-spine for some $k \geq 3$. Let $\mathcal{R}$ be a crowned k-corpus in G chosen to maximize k and with respect to that to maximize $|V(\mathcal{R})|$. Suppose G does not contain a clique cutset and suppose G contains a k-spine for some $k \geq 3$. Then $G = \mathcal{R}$

Proof. Suppose $G \neq \mathcal{R}$. Let H be the set of vertices that are complete to $V(\mathcal{R})$. Then H is a clique since G does not contain a C_4. By assumption $V(G) \setminus (V(\mathcal{R}) \cup H) \neq$.

Then since G does not contain a clique cutset there is connected induced subgraph W of $G \setminus (V(\mathcal{R}) \cup H)$ such that $V(W) \cap V(\mathcal{R})$ contains two non-adjacent vertices. Choose W to be minimal. Then W is a path. By Lemma 3.5.6 and Theorem 3.5.1, W does not consist of a single vertex. Then, by Theorem , $N(W) \cap V(\mathcal{F})$ does not contain vertices from two non-adjacent bags. Hence there are some two adjacent bags J, J' of $\mathcal{R}$ and $j \in J$ and $j \in J'$ such that $N(W)$ contains both j and j'. Let the vertices of W be w_1-w_w-$\ldots$-w_n, in order. Then we may assume w_1 is adjacent to j and w_n is adjacent to j'. By minimality W^* is anticomplete to j, j'.

There is no inflated hole $\mathcal{C}$ contained in $\mathcal{F}$ with $J, J' \subseteq \mathcal{C}$. (3.42)

Suppose $\mathcal{C}$ is an inflated hole in $\mathcal{F}$ containing both J and J' as bags. Then there is some jj'-path R contained in $\mathcal{C}$ of length $\ell - 1$. But then the union of R and path j-w_1-w_2-$\ldots$-w_n-j' is a hole of length greater than ℓ, a contradiction. This proves (3.42).

Let $\mathcal{F}$ be the k-corpus of $\mathcal{R}$. Then it follows from (3.42) and the definition of k-corpus that J, J' cannot both be in $\mathcal{F}$. Since the bags of $\mathcal{F} \setminus \mathcal{R}$ are single vertices, J and J' cannot both be in $\mathcal{R} \setminus \mathcal{F}$. Then we may assume J is a bag of an elemental side X of $\mathcal{R}$ and J' is in $\mathcal{R} \setminus \mathcal{F}$. By (**??**), J is complete to every neighboring bag of J' in X. By definition of crowned k-corpus J' consists of a single vertex j' and $N(j')$ contains nonadjacent vertices x_1, x_2 from bags in X. Then j-x_1-j'-x_2-j is a hole of length four, a contradiction. □

3.6 The main result for when $\ell \geq 7$ and odd.

In this section we use Theorems 3.5.7 to fully characterize ℓ-monoholed graphs when ℓ is at least seven and odd. Note this case is more simple than the case when ℓ is even because one elemental side of a corpus when ℓ is odd is a set of pairwise adjacent bags and the other is a set of a pairwise non-adjacent bags. This property restricts the structure of the crowns of a crowned k-corpus. When ℓ is even an elemental side of a corpus can contain both pairs of adjacent adjacent bags and pairs of non-adjacent bags. Thus when ℓ is even the relationship between an elemental side and the crown containing it in a crowned corpus can be complicated.

We need the following lemma about transitive closures of trees.

Lemma 3.6.1. *Let H be a connected graph such that $V(H)$ can be partitioned into a stable set S and a set X satisfying the following conditions:*

1. *Every $x \in X$ has a neighbor in S.*

2. *For any $x_1, x_2 \in X$ x_1 is adjacent to x_2 if and only if $N(x_1) \cap S \subseteq N(x_2) \cap S$ or $N(x_2) \cap S \subseteq N(x_1) \cap S$,*

3. *For any $x_1, x_2 \in X$, x_1 is not adjacent to x_2 if and only if $N(x_1) \cap S$ and $N(x_2) \cap S$ are disjoint.*

Then H is a transitive closure of a tree and S is the set of leaves of H.

Proof. We proceed by induction on $|X|$.

$G[X]$ is connected. (3.43)

Suppose C_1, C_2 are two components of $G[X]$. Then since H is connected, there is some vertex in H with a neighbor in both $V(C_1)$ and $V(C_2)$, contradicting (c). This proves (3.43).

If x_1-x_2-x_3 is a P_3 contained in X, then $N(x_i) \cap S \subsetneq N(x_2) \cap S$ for $i \in \{1, 3\}$. (3.44)

Suppose $N(x_1) \cap S$ is not a proper subset of $N(x_2) \cap S$. Then by (b), $N(x_2) \cap S \subseteq N(x_1) \cap S$. But then by (b), $N(x_1) \cap S$ and $N(x_3) \cap S$ are not disjoint, contradicting (c). This proves (3.44)

Choose $x \in X$ to maximize $|N(x) \cap S|$.

x is complete to $V(H) \setminus \{x\}$. (3.45)

Suppose x is not complete to $X \setminus \{x\}$. Then since $G[X]$ is connected for some $x', x'' \in X$, x-x'-x'' is a P_3. But then by (3.44), $|N(x') \cap S| > |N(x) \cap X|$, a contradiction. Hence x is adjacent to every other vertex in X. Since H is connected every $s \in S$ has a neighbor in X, so x is complete to S by (b) and our choice of x.

Let $H_1, \ldots, H_k$ be the components of $H \setminus \{x\}$. By induction for each $i \in [k]$, H_i is the transitive closure of some tree T_i with root $r_i \in V(T_i)$ and leaves $V(H_i) \cap S$. Let T be the tree obtained from the union of $T_1, T_2, \ldots, T_k$ by adding a new vertex r adjacent to $r_1, r_2, \ldots, r_k$. Then H is the transitive closure of T since x is adjacent to $V(H) \setminus \{x\}$. □

Lemma 3.6.2. *$\mathcal{R}$ be a crowned k-corpus for some $\ell \geq 7$ and $k \geq 3$. Let C_X be a crown of $\mathcal{R}$ and let $\mathcal{X}$ be the elemental side of $\mathcal{R}$ contained in C_X. Let $\mathcal{X}'$ be a non-empty subset of $\mathcal{X}$ chosen so that any two bags in $\mathcal{X}'$ are non-adjacent. Let Z be the vertices of $z \in V(C_X)$ such that z has neighbors in two bags of $\mathcal{X}'$. Let H be the graph induced $Z \cup \cup_{X' \in \mathcal{X}'} X'$. And let H' be the graph obtained from H by removing all edges between any two vertices . Then, H' is the transitive closure of some $\mathcal{X}$-friendly tree.*

Proof. By definition of crown and Lemma 3.6.1, H' is the transitive closure of some tree T where $\cup_{X' \in \mathcal{X}'} X'$ is the set of leaves of T. Suppose for some $X' \in \mathcal{X}'$ there are two non-adjacent vertices $t, t \in V(\mathcal{C}_{\mathcal{X}}) \setminus X'$ such that t and t' both have a neighbor in X'. Then by definition t, t'
not $\in \cup_{X' \in \mathcal{X}'} X'$. Thus, t, t' have each have neighbors in at least two bags of $\mathcal{X}$. Without loss of generality $X_1, X_2 \in \mathcal{X} \setminus X'$ and t has a neighbor $x_1 \in X_1$ and t' is adjacent to some $x_2 \in X_2$. Let $x, x' \in X'$ be adjacent to t, t', respectively. By definition, t is not adjacent to x', x_2 and t' is not adjacent to x, x_1.

Suppose $X_1 \neq X_2$. Then x_1-t-x-x'-t'-x_2 is an induced path of $\mathcal{R}$. By Fact 3.4.1 $\mathcal{R}$ contains an x_1x_2-path P of length $\ell - 2$ such that $P*$ is anticomplete to $V(C_X)$, Thus the union of x_1-t-x-x'-t'-x_2 and P is a hole of length greater than ℓ, a contradiction.

Thus $X_1 = X_2$ and so x_1 and x_2 are adjacent. But then x_1-t-x-x'-t'-x_2-x_1 is a hole of length five, contradicting that $\ell \geq 7$. Hence, T is $\mathcal{X}'$-friendly. □

3.6.1 k-pyramidoids

If G is ℓ-monoholed when ℓ is odd G may be a type of graph we call a "(k, ℓ)-pyramidoid" for some $k \geq 3$ which we define in this subsection.

Let T be a tree with root r. Let L denote the set of leaves of T and S denote the set of parents of vertices in L. Let $L_1, L_2, L_3, \ldots, L_n$ be a partition of L. We say T is $\{L_1, L_2, L_3, \ldots, L_k\}$-friendly if all of the following statements hold:

(i) Every vertex in S has neighbors in at least two of $X_1, X_2, \ldots, X_k$ and

(ii) For each $i \in [k]$ there is a path P of T from the root to a vertex in S such that $N(X_i) \subseteq V(P)$.

We define (k, ℓ)-*pyramidoid* for $k \geq 3$ and $\ell \geq 7$ and odd as follows:

Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots \mathcal{Q}_k$ be inflated paths of length $\frac{\ell-3}{2}$. For each $i \in [k]$ let X_i, Y_i be denote the end bags of $\mathcal{Q}_i$. For every two distinct $i, j \in [k]$ add edges to make Y_i, Y_j complete to each-other and call the resulting graph H. Let T be a $(X_1, X_2, \dots, X_k)$-friendly tree with root r. Let M be the graph obtained by taking the union of H and the transitive closure of T. We call $X_1, X_2, \dots, X_k$ the *elemental leaf bags* of M. We call any graph M obtained from this process a (k, ℓ)-pyramidoid. Note by definition, every hole in a k-pyramidoid has length ℓ.

Theorem 3.6.3. *Let G be an ℓ-monoholed graph for some $\ell \geq 7$ and odd. Then one of the following conditions holds:*

(a) G contains a vertex that is adjacent to every other vertex in $V(G)$.

(b) G contains a clique cutset,

(c) G is chordal,

(d) G is an inflated ℓ-hole,

(e) G is a k-pyramidoid for some $k \geq 3$.

Proof. Suppose G does not satisfy (a), (b), (c) or (d). Then by Theorem 3.3.8 G contains a k-pyramid for some $k \geq 3$. Then by definition G contains a crowned k-corpus $\mathcal{R}$. Choose $\mathcal{R}$ to maximize k and with respect to that maximize $|V(\mathcal{R})|$. Then by Theorem 3.5.7, $G = \mathcal{R}$. Let $\{X_1, X_2, \dots, X_k\}$ and $\{Y_1, Y_2, \dots, Y_k\}$ be the elemental sides of $\mathcal{R}$. For each $i \in [k]$ let $\mathcal{Q}_i$ denote the elemental path of $\mathcal{R}$ with ends X_i, Y_i. Since ℓ is odd, the graph underlying the corpus of $\mathcal{R}$ is a k-pyramid. Hence we may assume that $X_1, X_2, \dots, X_k$ are pairwise non-adjacent bags and $Y_1, Y_2, \dots, Y_k$ are pairwise adjacent bags. Let C_X be the crown of $\mathcal{R}$ containing $X_1, X_2, \dots, X_k$. Then by Lemma 3.6.1 and by definition of crown, C_X is the transitive closure of some tree T where $X_1 \cup X_2 \cup \dots \cup X_k$ is the set of leaves of T. By Lemma 3.6.2, T is $\{X_1, X_2, \dots, X_k\}$-friendly. It follows that $\mathcal{R}$ is a k-pyramidoid. □

3.7 Analyzing the structure of a maximal crowned k-corpus when ℓ is even

3.7.1 Helpful partitions of elemental sides

Let F be a generalized k-prism for some $k \geq 3$. Let Q, R, S, T be a defining partition of F. Let A, B be the terminating set of F Let $D_1, D_2, \dots, D_k$ be the elemental paths of F. Let X, Y be the

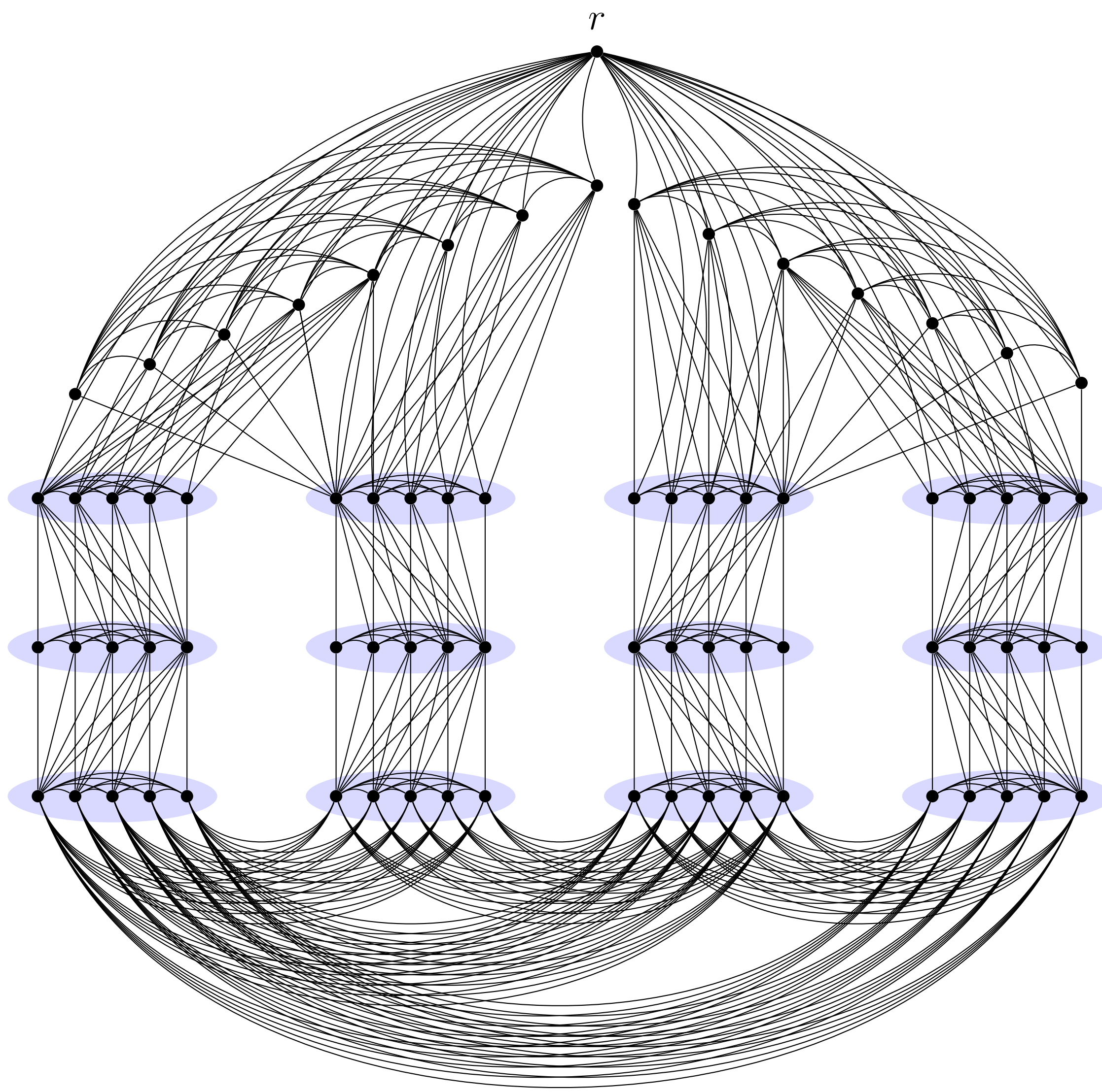

Figure 3.17: An example of a 4-pyramidoid. The vertices grouped in blue ovals are bags. The tree-top of this pyramidoid has root r and is a caterpillar. (A caterpillar is a type of tree with a single path containing every vertex of degree greater than one.) By definition of friendly tree, every 4-pyramidoid will have a transitive closure of a caterpillar as a crown

elemental sides of F where every vertex in X is equal or adjacent to some vertex in A and every vertex in Y is equal or adjacent to some vertex in B. For $L \subseteq [k]$ let $X(L), Y(L)$ denote the set of ends of $\{Q_i \mid i \in L\}$ that lie in X and Y, respectively. We call I, J a *helpful* partition of $[k]$ with respect X and the defining partition Q, R, S, T if all of the following conditions hold.

- $Q \subseteq I$,
- $X(I)$ is a stable set,
- $X(J)$ is a clique,
- Either, every $x \in X(J)$ has two neighbors in $X(I)$ or there is some $x \in X(J)$ such that x is anticomplete to $X(I)$.

We need the following lemma:

Lemma 3.7.1. *If F a generalized k-prism for some $k \geq 3$, X is an elemental side of F and Q, R, S, T is a defining partition of F, then there is a helpful partition of $[k]$ with respect to X and Q, R, S, T.*

Proof. Let A, B, X, Y be as in the definition of helpful partition. Suppose there is no helpful partition with respect to X. Let Q, R, S, T be a defining partition of F. By definition, $A([k] \setminus T) = X([k] \setminus T)$ and $A(T) = X(T)$ if and only if T consists of a single vertex.

We may assume $Q \neq \emptyset$ (3.46)

Suppose $Q = \emptyset$. If $|T| \leq 1$, $G[A]$ is a clique. If $|T| \geq 2$, $G[A]$ is a stable set. In either $\emptyset$, $R \cup S \cup T$ is a helpful partition with respect to $[k]$ and Q, R, S, T. This proves (3.46).

Some vertex in $X(R)$ is anticomplete to $X(Q)$. (3.47)

By definition $G[B]$ is a two-connected threshold graph. Thus there is some vertex $i \in [k] \setminus S$ such that $B(\{i\})$ is adjacent to every other vertex in B. Since $Q \neq \emptyset$ and by definition $B(T)$ is anticomplete to $B(Q)$ by definition. So, $i \in R$. By definition of generalized k-prism, $A(\{i\})$ is anticomplete to $A(Q)$ and $A(\{i\}) = X(\{i\})$. This proves (3.47).

We may assume $|T| \leq 1$. (3.48)

Suppose $|T| \geq 2$. Then $X(T)$ is a stable set and every $x \in X(T)$ is anticomplete to $X \setminus \{x\}$. Thus, $X(Q \cup S \cup T)$ is a stable set. It follows from (3.47), that $Q \cup S \cup T$, R is a helpful partition of $[k]$ with respect to X. This proves (3.48).

We may assume $|S| \leq 1$. (3.49)

Since $|T| \leq 1$, $X(T \cup R)$ is the vertex set of a clique. By definition $X(R)$ is complete to $X(S)$ and $X(S \cup Q)$ is a stable set. Thus, if $|S| \geq 2$ then $S \cup Q$, $T \cup R$ is a helpful partition of $[k]$ with respect to X and Q, R, S, T. This proves (3.49).

By (3.48), (3.49), $A(R \cup S \cup T)$ is the vertex set of a clique. Thus it follows from (3.47) that Q, $R \cup S \cup T$ is a helpful partition of $[k]$ with respect to X and Q, R, S, T. □

Let $\mathcal{R}$ be a k-corpus for some $k \geq 3$. Let $\mathcal{X}$ be an elemental side of $\mathcal{R}$. Let C_X be the crown of $\mathcal{R}$ that contains all vertices in $\mathcal{X}$. We call $\mathcal{I}, \mathcal{J}$ a *helpful division* of C_X if all of the following properties hold:

- $\mathcal{I}$ is a a set of pairwise non-adjacent bags of $\mathcal{X}$,
- $\mathcal{J}$ the set of vertices of $V(C_X)$ that are not in any bag of $\mathcal{I}$,
- $\mathcal{X} \setminus \mathcal{I}$ is a set of pairwise adjacent bags,
- Every vertex in $\mathcal{J}$ has neighbors in two bags of $\mathcal{X}$.

Lemma 3.7.2. *Let $\mathcal{R}$ be a k-corpus for some $k \geq 3$ and let C be a crown of $\mathcal{R}$. Then there is a helpful division of C.*

Proof. Let $\mathcal{X}$ be the elemental side of $\mathcal{R}$ whose bags are contained in $V(C)$. Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_k$ be the elemental paths of $\mathcal{R}$. For each $L \subset [k]$ we denote subset of bags of $\mathcal{X}$ contained $\cup_{i \in L} V(\mathcal{Q}_i)$ by $\mathcal{X}(L)$. Let $\mathcal{F}$ be the k-corpus of $\mathcal{R}$ and let F be a graph underlying $\mathcal{F}$. Let X be the elemental side of F corresponding to $\mathcal{X}$. Let U, W be a helpful partition of $[k]$ with respect to X and Q, R, S, T.

Suppose every vertex in $X(W)$ has two neighbors in $X(U)$. Let J denote the set of vertices in $V(C)$ not contained in any bag of $X(U)$. Then by definition of crowned k-corpus it follows that $\mathcal{X}(U)$, Z is a helpful division of C. Hence we may assume for some $i \in W$, the vertex $X(i)$ is anticomplete to $X(U)$.

Let J' denote the set of vertices in $V(C)$ not contained in any bag of $X(U \cup \{i\})$. Then by definition of crowned k-corpus, it follows that $\mathcal{X}(U \cup \{i\})$, J' is a helpful division of C. □

3.7.2 Crowns and transitive closures of trees

Theorem 3.7.3. *Let $\mathcal{R}$ be a crowned k-corpus. Let $\mathcal{X}$ be an elemental side of $\mathcal{R}$ and let C_X be the crown of $\mathcal{R}$ containing vertices in bags $\mathcal{X}$. Let $\mathcal{I}, \mathcal{J}$ be a helpful division of C_X. Then the graph obtained from C_X by removing all edges between vertices in bags of $\mathcal{I}$ is a transitive closure of some $\mathcal{I}$-friendly tree T.*

Proof. Let $C_X \setminus \mathcal{X}$ denote the graph obtained from C_X by removing all vertices in bags of $\mathcal{X}$. Let $\mathcal{J}_X$ denote $\mathcal{J} \setminus V(C_X \setminus \mathcal{X})$ and let $\mathcal{J}_C$ denote $\mathcal{J} \cap V(C_X \setminus \mathcal{X})$. $V(\mathcal{I})$ denote the union of bags in $\mathcal{I}$.

Suppose u, v are non-adjacent vertices in $V(C_X)$. Then there are no two non-adjacent bags of X_1, X_2 of $\mathcal{X}$, such that u, v both have neighbors in X_1, X_2. (3.50)

Suppose there exists some bag $X_2 \in \mathcal{X}$ such that x, v both have a neighbor in X_2 and $X_2 \neq X_1$. Then, $G[X_1 \cup X_2 \cup \{x, v\}]$ contains a hole of length at most six, a contradiction. This proves (3.50).

Suppose $x \in \mathcal{J}_X$ and $v \in \mathcal{J}_C$ are non-adjacent. Then there is no bag of $X_1 \in \mathcal{I}$ such that x, v both have neighbors in X_1. (3.51)

Suppose x, v both have a neighbor in X_1. By (3.50), v is anticomplete to $V(\mathcal{J})$. Then by definition of crown v has a neighbor x_2 in some bag $X_2 \in \mathcal{X}$ where $X_2 \neq X_1$. By definition of helpful partition, x has a neighbor x_3 in some bag $X_3 \in \mathcal{X}$ where $X_3 \neq X_1$. By (3.50), $X_2 \neq X_3$. Hence $G[X_1 \cup X_2 \cup X_3 \cup \{x, v\}]$ contains an x_3x_4-path R of length greater than two. By Fact 3.4.1, $\mathcal{R}$ contains a x_1x_2-path P such that P^* is anticomplete to $V(C_X)$. But then $R \cup P$ is a hole of length greater than ℓ, a contradiction. This proves (3.51).

Suppose $x \in \mathcal{J}_X$ and $v \in \mathcal{J}_C$ are adjacent. Then $N(x) \cap V(\mathcal{I}) \subseteq N(v) \cap V(\mathcal{I})$ or $N(v) \cap V(\mathcal{I}) \subseteq N(s) \cap V(\mathcal{I})$. (3.52)

Suppose there exist $x_1, x_2 \in V(\mathcal{I})$ such that $xx_1, x_2v \in E(G)$ and $xx_2, vx_1 \notin E(G)$. Then, x_1 and x_2 are not adjacent for otherwise x_1-x-x_2-v-x_1 is a hole of length four, a contradiction. It follows that x_1, x_2 are two different bags of $\mathcal{X}$. But then x_1-x-v-x_2 is a path of length three so by Fact 3.4.1, $\mathcal{F}$ contains a hole of length $\ell + 1$, a contradiction. This proves (3.52). It follows from (3.51), (3.52), Lemma 3.6.2, Lemma 3.6.1 and the definition of crowned k-corpus that the graph obtained from C_X by removing all edges between vertices in bags of $\mathcal{I}$ is the transitive closure of some $\mathcal{I}$-friendly tree. □

Corollary 3.7.4. *The crowns of a crowned k-corpus for any $k \geq 3$ are the transitive closure of a tree.*

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